10/24 Microbundles

Ref AMSI Ch 4.2 – 4.5

Recall Global Kuranishi charts (G, T, E, s)

(1) G cpt Lie gp

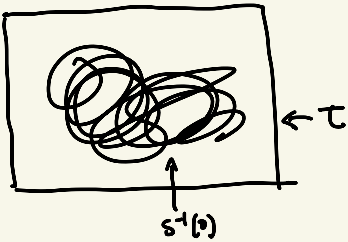
(2) T : topological mfd, G-action ctns + finite stabilizers

(3) $E \to T$: G- vector bundle

(4) s : G-section

$\Rightarrow s^{-1}(0)/G$

Ex



$\forall$ closed set = zero set of $\exists$ ctns ftn

(Urysohn lemma)

Def Stabilization of GKC

Given $\left[\begin{array}{l} \text{GKC } (G, T, E, s) \\ p : E' \to T \end{array}\right.$, define the stabilization

$$(G,\ E',\ p^*E \oplus p^*E',\ p^*s \oplus \Delta)$$

$\Delta(t, e') = (t, e', e')$

$\therefore$ $(t, e') \in \Delta^{-1}(0)$

$\Leftrightarrow e' = 0$

$(t, e') \in p^*s^{-1}(0)$

$\Leftrightarrow s(t) = 0$

$$\begin{tikzcd} E \arrow[d] \\ T \arrow[u, bend right, "s"'] \end{tikzcd} \qquad \begin{tikzcd} p^*E \arrow[dr] & & p^*E' \arrow[dl] \\ & E' \arrow[ul, bend left, "p^*s"] \arrow[ur, bend right, "\Delta"'] & \end{tikzcd}$$

$\therefore$ $p^*s \oplus \Delta^{-1}(0) = s^{-1}(0) \subset T$

Q What is the tangent bundle of a topological mfd?

(Rmk: not every top mfd admits smooth str → Donaldson
even a triangulation! → Manolescu)

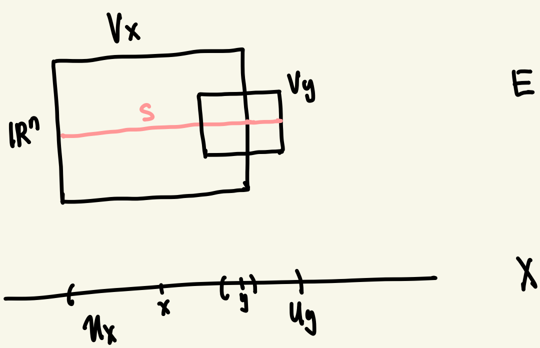
Def Microbundle of rank n / topological space X

= diagram $\{ X \xrightarrow[s]{} E \xrightarrow[p]{} X \}$, $E \in \mathrm{Top}$, s, p continuous s.t.

(1) $p \circ s = \mathrm{id}_X$

(2) $\forall x \in X$, $\exists\, U_x \subset X$, $V_x \subset E$ open, $\exists$ homeo $h_x : U_x \times \mathbb{R}^n \to V_x$ s.t.

$$\left[\begin{array}{l} p \circ h_x = pr_{U_x} \\ h_x|_{U_x \times \{0\}} = s \end{array} \right.$$



Rmk Only nbd of $s(X) \subset E$ is relevant

Def Morphism of microbundles

$U_1 = \mathrm{nbd}(s_1(X_1)) \subset E_1$

= equivalence class of $\phi : U_1 \to E_2$ commuting with s_i, p_i,

ϕ, ψ equivalent if agree on $\exists$ nbd of $s_1(X)$

$$\begin{tikzcd} & U_1 \arrow[dr, "p_1|_{U_1}"] \arrow[d, "\varphi"] & \\ X \arrow[ur, "s_1"] \arrow[r, "s_2"'] & E_2 \arrow[r, "p_2"'] & X \end{tikzcd}$$

Slogan: microbundle = germ of X embedded in larger topological space

Ex Vector bundle $V \to X$ of rank n $\Rightarrow$ associated microbundle V_μ

$$X \xrightarrow{0} V \longrightarrow X$$

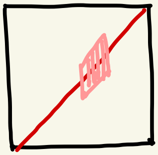
Def Vector bundle lift of microbundle E

= vector bundle V + isomorphism $V_\mu \cong E$

$$\begin{tikzcd}
X \arrow[r, "0"] \arrow[d, equal] & U_0 \subset V \arrow[r, "\pi"] \arrow[d, "\cong"] & X \arrow[d, equal] \\
X \arrow[r, "s"'] & U_1 \subset E \arrow[r, "p"'] & X
\end{tikzcd}$$

Def Tangent microbundle $T_\mu X$ $\qquad X \xrightarrow[\Delta]{} X \times X \xrightarrow[p_1]{} X$

X



X

Prop X smooth(able) $\Rightarrow$ TX is a vector bundle lift of $T_\mu X$

Pf g: complete metric on X

$(x, v) \in TX \mapsto (x, \exp_x(v)) \in X \times X$

$$\begin{tikzcd}
X \arrow[r, "0"] \arrow[d, equal] & TX \arrow[r, "\pi_1"] \arrow[d] & X \arrow[d, equal] \\
X \arrow[r, "\Delta"'] & X \times X \arrow[r, "p_1"'] & X
\end{tikzcd}$$

Rmk X not smoothable $\not\Rightarrow$ $T_\mu X$ does not admit vector bundle lift!

($\Leftrightarrow$ Kirby-Siebenmann class $\in H^4(X; \mathbb{Z}/2) = 0$)

Prop

$$\begin{tikzcd}
\pi^*(E \oplus T_\mu M) \arrow[d] & T_\mu M \arrow[d] \\
E \arrow[r, "\pi"'] & M
\end{tikzcd}$$

honest vector bundle

vector bundle lift of $T_\mu M$ determines

" stabilization $\pi^*(E \oplus T_\mu M)$

Def G-microbundle : start with G-space X, $\exists$ G-space E, G-equivariant s, p

$$X \xrightarrow{s} E \xrightarrow{p} X$$

Rmk smooth X + cpt Lie gp $G \curvearrowright_{sm} X$

$\Rightarrow$ G-equivariant lift $TM \to T_\mu M$

Pf Complete G-equivariant metric + exponential map

(integrate along G-orbit)

Thm (Lashof, equivariant stabilized smoothing)

G: cpt Lie gp $\curvearrowright_{ctns}$ M: topological mfd

Assume (1) Only finitely many orbit types

(2) Microbundle $T_\mu M$ admits G-vector bundle lift $\ell: E \to T_\mu M$

Then $\exists$ G-rep V s.t. $V \times M$ admits G-equivariant smooth structure

Rmk (1) guarantees that E has an "stable inverse" E' s.t. $E \oplus E' \cong V$

E: G-v.b. lift of $T_\mu M$

V: orthogonal G-rep

$\Rightarrow$ Stabilize by E', reduce to case when $T_\mu M$ trivial

Pf Assume $E \simeq \underline{V}$ for some G-repn V

Since $\underbrace{\exists \text{ embedding } \eta : M \times \underline{V} \hookrightarrow M \times M}_{\text{microbundle}}$, and $\exists$ embedding $M \hookrightarrow \overset{\text{G-repn}}{\overset{\downarrow}{W}}$

Then $\exists$ G-invt nbd N of $M \subset W$, $r : N \to M$ G-equivariant retraction

Define $\theta' : M \times W \longrightarrow N \subset W$

$(x,y) \mapsto x + \underline{\delta(x)}\, y$ small enough s.t.

For δ small, $\underbrace{(\underbrace{r \circ \theta'(x,y)}_{\text{close to } x}, x)}_{\text{close to diagonal}} \subset \eta(M \times \underline{V})$

Define $\psi : M \times W \longrightarrow W \times V$

$(x, y) \mapsto (\theta'(x,y),\ pr_2\, \eta^{-1}(r \circ \varphi'(x,y), x))$

Check (1) ψ G-equivariant

(2) ψ injective

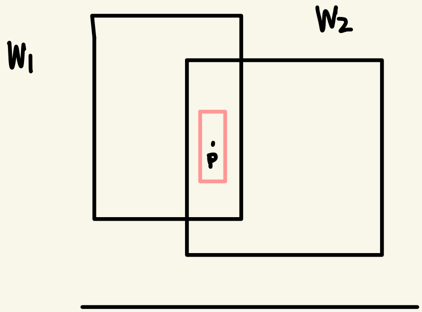
(3) $\psi(x,0) = (x,0)$

$\Rightarrow$ pulls back $TW \times TV$, thus G-smooth

Fix Lie grp G, ctns G-equivariant $\pi : M \to B$

Assume each $\pi^{-1}(b)$ equipped with sm structures.

Def C^1_{loc}-compatibility of charts



$\left\langle \begin{array}{l} W_1 \to W_1|_{b_1} \times \pi(W_1) \\ W_2 \to W_2|_{b_2} \times \pi(W_2) \end{array} \right.$ C^1_{loc}-compatible if

$\forall p \in W_1 \times W_2$, $\exists$ $W \to W|_b \times \pi(W)$ s.t.

$W|_b \to W \hookrightarrow W_i \to W_i|_{b_i}$

smooth, vary ctnsly w.r.t C^1_{loc}-topology

Def Fiberwise smooth C^1_{loc} G-bundle is the datum of

(1) $\pi : M \to B$ G-equivariant

(2) C^1_{loc}-compatible G_{p_k}-invariant product nbds around $^\exists\{p_k\}$, covering M

(only need product nbds covering M)

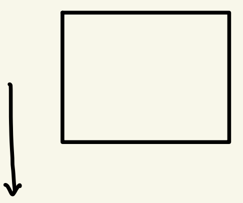
Def Vertical tangent microbundle

$M \xrightarrow{\Delta} M \times_B M \xrightarrow{pr_1} M$ $\qquad M \times_B M = \{(p,q) \mid \pi(p) = \pi(q)\}$

Def Vertical tangent bundle

Given fb sm C^1_{loc} G-bundle, define the vector bundle $T^{vt}M$ by

over $W \cong W|_b \times \pi(W)$ via



$$\begin{array}{ccc} pr_1^* T W|_b & \longrightarrow & T W|_b \\ \downarrow & & \downarrow \\ W|_b \times \pi(W) & \xrightarrow[pr_1]{} & W|_b \end{array}$$

Rmk C^1_{loc}-compatibility $\Rightarrow$ $T^{vt}M$ well-defined

Lemma 4.29

$\pi: M \to B$ fbsm C^1_{loc} G-bundle, B smooth G-mfd

$\Rightarrow$ $\exists$ G-equivariant lift of $T^{vt}_M M$ to $T^{vt} M$

Pf

$$\begin{tikzcd} M \arrow[r] \arrow[d, equal] & T^{vert} M \arrow[r] \arrow[d] & M \arrow[d, equal] \\ M \arrow[r] & M \times_B M \arrow[r] & M \end{tikzcd} \qquad \begin{array}{c} (p, v) \\ \downarrow \\ (p, \exp_p(v)) \end{array}$$

put metric on $T^{vt}(M)$ s.t. on each fiber $M|_b$, sm Riemannian metric g_b

v in fiber direction $\Rightarrow$ $\exp_p(v)$ evaluated in fibers

Prop 4.30

Spp $(G, \mathcal{T}, \mathcal{E}, s)$ GKC s.t.

(1) $\mathcal{T} \to B$ fiberwise smooth C^1_{loc} G-bundle, base B sm G-mfd

(2) $G \circlearrowright \mathcal{T}$ has only finitely many orbit types

Then, $\exists$ stabilization of $(G, \mathcal{T}, E, s)$ admits a smooth structure

Pf Lemma 4.29 ensures $T_M \mathcal{T}$ admits G-vector bundle lift

$\therefore$ $\exists$ G-rep V s.t. $V \times \mathcal{T}$ admits G-equivariant sm str

$\Rightarrow$ stabilize $(G, \mathcal{T}, \mathcal{E}, s)$ via $V \times \mathcal{T} \to \mathcal{T}$

Next wk

$\overline{\mathcal{M}}_{0,k}(X, J, \beta)$ admits a GKC with fiberwise smooth C^1_{loc} G-bundle str

AMS construction of GKC for genus 0 GW moduli

$G = U(d+1)$

$$\mathcal{T} = \{ (u, \Sigma, F, \eta) \}$$

$$\eta \in H^0(\overline{\mathrm{Hom}} (\nu_F^* T\mathcal{C}, u^* TX) \otimes \nu_F^* \mathcal{L}^k) \otimes_{\mathbb{C}} \overline{H^0(\nu_F^* \mathcal{L}^k)}$$

u : solves $\bar{\partial}_J u + \langle \eta \rangle \circ d\nu_F = 0$

$$\mathcal{T} \longrightarrow \overline{M}_{0, n+3+\dim \overline{M}_{0,0}(\mathbb{CP}^d, d)} \qquad [\phi_F]$$

$$\mathcal{E}_{(u, \Sigma, F, \eta)} = H^0(\overline{\mathrm{Hom}} (\nu_F^* T\mathcal{C}, u^* TX) \otimes \nu_F^* \mathcal{L}^k) \otimes_{\mathbb{C}} \overline{H^0(\nu_F^* \mathcal{L}^k)} \oplus \mathcal{H}_{d+1}$$

$$s : \quad (u, \Sigma, F, \eta) \longmapsto (\eta, \exp^{-1}(H)(u, \Sigma, F)))$$