Lecture 1: overview

Let (M, ω) be a compact symplectic manifold. Given a 1-periodic smooth Hamiltonian $H_t : S^1 \times M \to \mathbb{R}$, its Hamiltonian vector field X_{H_t} is defined by

$$dH_t = \omega(-, X_{H_t}). \tag{0.1}$$

Integrating along t, the time-one flow of X_{H_t} defines a diffeomorphism

$$\phi_{H_t} : M \to M. \tag{0.2}$$

Exercise 0.1. *Show that $\phi_{H_t}^* \omega = \omega$ using the Cartan formula $L_X = d\iota_X + \iota_X d$.*

There is a one-to-one correspondence

$$\mathrm{Fix}(\phi_{H_t}) := \{\text{fixed points of } \phi_{H_t}\} \leftrightarrow \{\gamma : S^1 \to M \mid \dot\gamma(t) = X_{H_t}(\gamma)(t)\}. \tag{0.3}$$

ϕ_{H_t} is called non-degenerate if graph(ϕ_{H_t}) intersects Δ transversely in $M \times M$.

Exercise 0.2. *Show that ϕ_{H_t} is non-degenerate iff for any $x \in \mathrm{Fix}(\phi_{H_t})$, the differential $d\phi_{H_t}(x)$ does not admit 1 as an eigenvalue.*

Exercise 0.3. *Show that for a generic choice of H_t, ϕ_{H_t} is non-degenerate.*

Conjecture 0.4 (Arnold, weak form)**.** *Let R be a commutative ring. If ϕ_{H_t} is non-degenerate, then $\#\mathrm{Fix}(\phi_{H_t})$ is bounded from below by the mimimal number of generators of any chain complex over R whose homology is isomorphic to $H_*(M; R)$.*

Much of symplectic topology evolves surrounding variants of and ramifications from this conjecture. The following exercise was one of the main motivations for Arnold to introduce the conjecture, except for an early analogous result due to Poincaré–Birkhoff for the 2-annulus.

Exercise 0.5. *If $f : M \to \mathbb{R}$ is a C^2-small Morse function, show that every 1-periodic orbit of X_f is a critical point of f and vice versa.*

Following Floer, let's see how the proof formally goes. Floer's fundamental insight is to develop a Morse theory in infinite dimensions. To be more specific, consider the free loop space $LM := C^\infty(S^1, M)$.

Exercise 0.6. *Consider the space consisting of pairs (γ, u) with $\gamma : S^1 \to M$ and $u : D^2 \to M$ such that $u|_{\partial D^2} = \gamma$, modulo the equivalence relation*

$$(\gamma, u) \sim (\gamma', u') \text{ iff } \gamma = \gamma' \text{ and } u\#(-u') = 0 \text{ in } \pi_2(M). \tag{0.4}$$

Show that this is a model for the universal covering space $\widetilde{LM}$ of LM.

Define the energy functional

$$\begin{aligned} A_{H_t} : \widetilde{LM} &\to \mathbb{R} \\ [\gamma, u] &\mapsto -\int_{D^2} u^*\omega + \int_0^1 H_t(\gamma(t))dt. \end{aligned} \tag{0.5}$$

Choosing a 1-parameter family of ω-compatibile almost complex structures J_t, write $g_{J_t} = \omega(-, J_t-)$. it defines a Riemannian metric on LM which takes $(v_1, v_2) \in \Gamma(S^1, \gamma^*TM)$ to $\int_0^1 g_{J_t}(\gamma_*(v_1), \gamma_*(v_2))dt$.

Exercise 0.7. *Show that the formal gradient vector of A_{H_t} is given by*

$$[\gamma, u] \mapsto \big(t \mapsto J_t(\gamma(t))(\dot{\gamma} - X_{H_t}(\gamma(t)))\big). \tag{0.6}$$

For simplicity, let's assume that $\pi_2(M) = \{1\}$, which includes the important case $(T^{2n}, \omega_{\text{linear}})$. The Floer chain complex over R is defined to be

$$CF_*(M, H_t, J_t) := \bigoplus_{x \in \text{Fix}(\phi_{H_t})} R \cdot x, \tag{0.7}$$

with differential defined by counting rigid unparametrized gradient flow lines of A_{H_t}, which are represented by

$$u : S^1_t \times \mathbb{R}_s \to M \text{ such that } \partial_s u + J_t(\partial_t u - X_{H_t}(u)) = 0, \tag{0.8}$$

with asymptotic condition $\lim_{s \to \pm\infty} u(s,t) = x_\pm \in \text{Fix}(\phi_{H_t})$. For different choices of pairs (H_t, J_t) and (H'_t, J'_t), one can find a family $(H_{s,t}, J_{s,t})$ interpolating between them, such that the counts of solutions of

$$u : S^1_t \times \mathbb{R}_s \to M \text{ such that } \partial_s u + J_{s,t}(\partial_t u - X_{H_{s,t}}(u)) = 0 \tag{0.9}$$

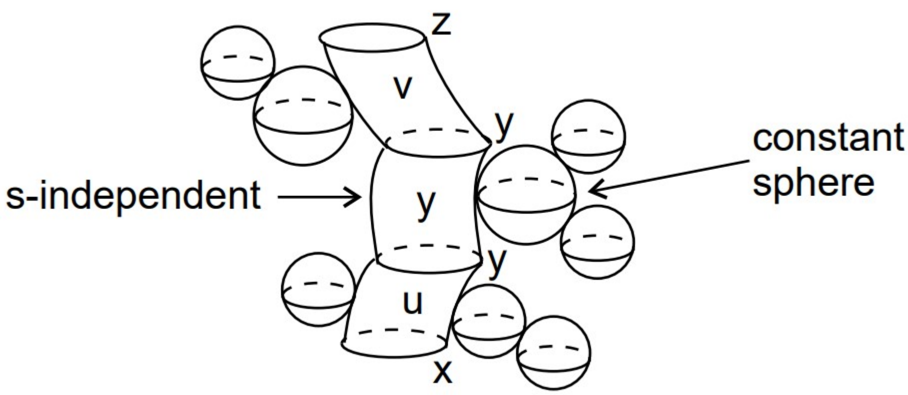
induces a chain map $CF_*(M, H_t, J_t) \to CF_*(M, H'_t, J'_t)$ which turns out to be a quasi-isomorphism. Finally, for a distinguished choice of (H_t, J_t), namely, $H_t = f$ is a C^2-small Morse function and $J_t = J$ is autonomous, there is a canonical isomorphism between $CF_*(M, f, J)$ and the Morse complex of the pair (f, g_J). Of course, the weak Arnold conjecture follows from the classical Morse inequality.

In the above sketch, numerous things are assumed. Most notably:

(1) For $x_\pm \in \text{Fix}(\phi_{H_t})$, one can produce well-defined counts from the moduli space $\overline{\mathcal{M}}(x_-, x_+)$ of solutions to the Floer equation. This requires $\overline{\mathcal{M}}(x_-, x_+)$ to be a 0-dimensional *compact* and *oriented* **manifold**. So, we need to *compactify* the moduli spaces in case that they are not compact; we need to ensure that such moduli spaces have enough regularity, which is a matter of *transversality*; for orientations, we need *index theory* for manifolds with cylindrical ends.

(2) The differential on $CF_*(M, H_t, J_t)$ is indeed a differential, i.e., $d^2 = 0$. This follows from the assertion that

$$\partial\overline{\mathcal{M}}(x_0, x_2) = \bigcup_{x_1} \overline{\mathcal{M}}(x_0, x_1) \times \overline{\mathcal{M}}(x_1, x_2) \tag{0.10}$$



with *coherence* on their orientations. This is concerned with the *gluing* construction of moduli spaces.

(3) The Floer chain complexes should exhibit certain TQFT aspects. We only encounter cylinders in the above story to address well-definedness and calculation issues. In general, one can use different Riemann surfaces to construct *algebraic structures* on the Floer homology, or even chain-level operations.

These technical problems are relatively easy to deal with under the assumption $\pi_2(M) = \{1\}$. We will cover some topics here depending on the audience's request.

1. Theme 1: regularizing the moduli spaces

Dropping this assumption, the moduli space $\overline{\mathcal{M}}(x_-, x_+)$ is rather nasty in nature, as it contains configurations shown above (cf. Dietmar Salamon's lecture notes). The spherical components may have nontrivial automorphism groups, making $\overline{\mathcal{M}}(x_-, x_+)$ locally only look like a quotient of a topological space (as opposed to open subsets in $\mathbb{R}^n$) by a finite group. This motivates the following central question which will be discussed thoroughly in this course.

Questions 1.1. *What's the most approporiate structure to put on* $\overline{\mathcal{M}}(x_-, x_+)$ *when it's not a manifold?*

Let's set things up in the appropriate context. Denote by j the standard complex structure on $\mathbb{R} \times S^1$. Over the manifold $\mathbb{R} \times S^1 \times M$, consider the endomorphism on its tangent bundle

$$\tilde{J} = \begin{pmatrix} j & 0 \\ J_t \circ X_{H_t} - X_{H_t} \circ j & J \end{pmatrix}. \tag{1.1}$$

Exercise 1.2. *Show that* $u : S^1 \times \mathbb{R} \to M$ *satisfies* $\partial_s u + J_t(\partial_t u - X_{H_t}(u)) = 0$ *iff the graph* $\tilde{u}(s,t) = (s, t, u(s,t))$ *satisfies the* $\tilde{J}$*-holomorphic curve equation*

$$d\tilde{u} \circ j = \tilde{J} \circ d\tilde{u}. \tag{1.2}$$

So the slightly more general question is:

Questions 1.3. *What's the most approporiate structure to put on the moduli spaces* $\overline{\mathcal{M}}$ *of pseudo-holomorphic curves?*

Pioneering work of Kuranishi on the moduli problem of complex sturctures on manifolds provides a uniform local solution to all nonlinear Fredholm problems, known as the *Kuranishi models*. The general assertion in our setting is the following.

Proposition 1.4. *For any point* $x \in \overline{\mathcal{M}}$, *there exists an open subset* $x \in U_x \subset \overline{\mathcal{M}}$ *and a tuple* (Γ, U, E, s, ψ) *such that* Γ *is a finite group which acts linearly on an open subset* U *in some Euclidean space* $\mathbb{R}^m$; E *is a finite-dimensional* Γ*-represenation;* $s : U \to E$ *is a* Γ*-equivariant map;* $\psi : s^{-1}(0)/\Gamma \xrightarrow{\sim} U_x$ *is a homeomorphism onto its image.*

In this course, we will see a *global* version of the above result. It asserts that for moduli spaces $\overline{\mathcal{M}}$ coming from Gromov–Witten theory or Hamiltonian Floer theory, one can find a tuple, known as *derived orbifold charts*,

$$\mathcal{D} = (\mathcal{U}, \mathcal{E}, \mathcal{S}, \Psi) \tag{1.3}$$

where $\mathcal{U}$ is an orbifold, $\mathcal{E} \to \mathcal{U}$ is an orbifold vector bundle, $\mathcal{S} : \mathcal{U} \to \mathcal{E}$ is section, and $\Psi : |\mathcal{S}^{-1}(0)| \xrightarrow{\sim} \mathcal{M}$ is a homeomorphism. Moreover, $\mathcal{D}$ has suitable orientation structures; in the setting of Floer theory, the derived orbifold charts are coherent under the description of the "boundary strata" of the moduli spaces $\partial\overline{\mathcal{M}} = \overline{\mathcal{M}}' \times \overline{\mathcal{M}}''$. This is the *global Kuranishi chart* approach to the regularization problem.

2. Theme 2: counting theory from derived orbifolds

Once the moduli spaces are regularized using derived orbifold charts, to define the Floer chain complex, we should ask ourselves the following.

Questions 2.1. *Given* $\mathcal{D} = (\mathcal{U}, \mathcal{E}, \mathcal{S}, \Psi)$ *with suitable orientation structures, how to produce a count or fundamental class of* $|\mathcal{S}^{-1}(0)|$, *such that when* $\mathcal{S}$ *is transverse to the* 0 *section and* $|\mathcal{S}^{-1}(0)|$ *is finite, the count agrees with the ordinary count of points?*

This question is most natural viewed from algebraic topology. The starting point is the following general result of Pardon.

Theorem 2.2. *There is bordism theory of derived orbifolds* $d\Omega_*$ *which defines a homology theory for topological spaces.*

So, the question concerning counting can be alternatively formulated as:

Questions 2.3. *Suppose* E_* *is a (generalized) (co)homology theory, including* $H\mathbb{Q}$, $H\mathbb{Z}$, *complex* K*-theory,* $K(n)_*$*-local cohomology theories, and complex bordism* Ω_*^U, *how to construct natural transformations of functors*

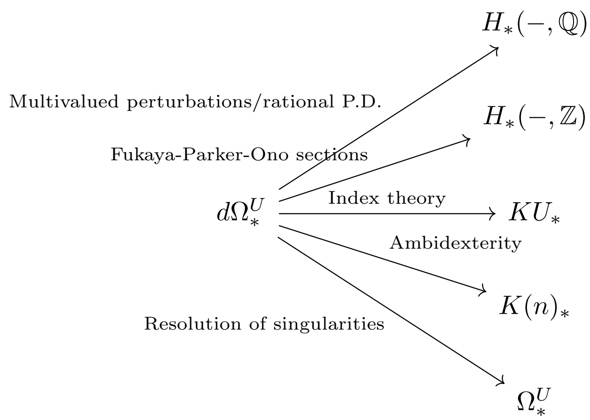
$$d\Omega_* \longrightarrow E_* \tag{2.1}$$

generalizing the map which assigns to an oriented vector bundle over a compact manifold $E \to M$ its Euler class?

Of course, for Floer theories, since we have to deal with chain-level objects, any answer to the above question is only an initial step.

The following diagram summarizes the existing answer to the above question. To be more precise, we need to specialize to the unitary version $d\Omega_*^U$, which consists of $\mathcal{D}$ such that the virtual vector bundle $T\mathcal{U} - \mathcal{E}$ admits a lift in the complex K-theory.

(2.2)



Some lectures will be devoted to the second and fifth arrow in the diagram. The natural transformations with target $H_*(-,\mathbb{Q})$, KU_*, and $K(n)_*$ are defined following the same scheme, namely, by exploring the Poincaré–Lefschetz duality of (complex-)oriented cohomology theories.

3. Theme 3: Applications to symplectic topology

Once the counting theory is set up for the integral homology theory, the weak Arnold conjecture follows as a consequence of the formal discussion we had before. Time permitting, we wish to cover some concrete applications.

Theorem 3.1 (Abouzaid–McLean–Smith, following Seidel, Lalonde–McDuff–Polterovich). *If $M \hookrightarrow P \to S^2$ is a Hamiltonian fibration, then*

$$H^*(P,\mathbb{Z}) \cong H^*(M,\mathbb{Z}) \otimes H^*(S^2,\mathbb{Z}). \tag{3.1}$$

This result has been used to address the integral formality problem.

Theorem 3.2 (Bai–Pomerleano). *Let M be acted on by the unitary group $U(k)$ in the Hamiltonian fashion. Then*

$$H^*_{U(k)}(M,\mathbb{Z}) \cong H^*(BU(k),\mathbb{Z}) \otimes H^*(M,\mathbb{Z}). \tag{3.2}$$

The above two results do not touch upon Floer theory in any serious way. Instead, they come from studying moduli spaces of pseudo-holomorphic maps with closed domains. As for applications based on Floer theory, the possibility of defining counts of Floer trajectories and variants thereof as integers enables applications in Hamiltonian dynamics.

Theorem 3.3 (Shelukhin, Sugimoto)**.** *A Hamiltonian diffeomorphism* $\phi_{H_t} : M \to M$ *admits infinitely many periodic points provided one of the following holds.*

- *M has semi-simple quantum cohomology and* $\#\mathrm{Fix}(\phi_{H_t})$ *exceeds the Arnold lower bound.*
- *M has a non-contractible periodic orbit which is "homologically visible".*

There are more to say under the broader slogan "Floer homotopy theory", and such a field is under rapid development. Hopefully, this course will serve as a useful introduction.