Topic 6: Abouzaid–McLean–Smith

1. Framings as stabilization data

Let (M^{2n}, J) be an almost complex manifold, and suppose Ω is a closed 2-form such that $\Omega^n \neq 0$ everywhere and $[\Omega] \in H^2(M, \mathbb{R})$ lifts to $H^2(M, \mathbb{Z})$ which tames J. In reality, such datum arises from:

- (M, ω) is a compact symplectic manifold and J is compatible with or tamed by ω;
- Ω is a (large) integral multiple of a symplectic form ω' such that $[\omega'] \in H^2(M, \mathbb{Q})$ and ω' is sufficiently close to ω in the C^∞-sense.

Fix $\beta \in H_2(M, \mathbb{Z})$, let $\overline{\mathcal{M}}_{0,k}(M, J, \beta)$ be the moduli space of stable maps with prescribed data. We wish to explain the following statement.

Proposition 1.1. *There is a compact Lie group G and a compact topological space V such that there exists a homeomorphism*

$$V/G \xrightarrow{\sim} \overline{\mathcal{M}}_{0,k}(M, J, \beta)$$

such that for a given representative $u \in \overline{\mathcal{M}}_{0,k}(M, J, \beta)$, the stabilizer of the G-orbit in V corresponding to u is isomorphic to the automorphism group of the stable map u.

In other words, at the expense of introducing Lie groups, $\overline{\mathcal{M}}_{0,k}(M, J, \beta)$ can be presented as a global quotient.

Lemma 1.2. *Let $u : \Sigma \to M$ be a J-holomorphic stable map in $\overline{\mathcal{M}}_{0,k}(M, J, \beta)$. Then there exists a holomorphic Hermitian line bundle $\mathcal{L}_u \to \Sigma$ whose Chern connection has curvature 2-form given by $-2\pi i u^*\Omega$. Moreover, such a line bundle is unique up to isomorphism.*

Proof. Let $\mathcal{L} \to \Sigma$ be the complex line bundle with $c_1(\mathcal{L}) = [u^*\Omega]$. Because $g(\Sigma) = 0$, the line bundle $\mathcal{L}$ admits a unique holomorphic structure. Equipping $\mathcal{L}$ with an arbitrary Hermitian metric h, whose associated Chern connection has curvature $-2\pi i\alpha$. We wish to find $f : C^\infty(\Sigma, \mathbb{R})$ such that the Chern connection with respect to $e^{-f/2}h$ has curvature $-2\pi i u^*\Omega$. In other words, we should solve

$$\partial\overline{\partial} f = -2\pi i(u^*\Omega - \alpha).$$

Using the Kähler identity $[\partial^*, L] = -i\overline{\partial}$ where L is taking wedge with the Kähler form, it is equivalent to

$$\Delta f = *2\pi(u^*\Omega - \alpha).$$

Because $[u^*\Omega]$ and $[\alpha]$ cohomologous, the above equation admits a solution unique up to a constant. Note that this argument is fully rigorous when Σ is smooth; in general, just carry out the argument over the irreducible components. $\square$

Introduce the quantity $d = \Omega(\beta)$. Using the Riemann–Roch theorem, we know that for any $[u] \in \overline{\mathcal{M}}_{0,k}(M, J, \beta)$, we have $\dim_{\mathbb{C}} H^0(\Sigma, \mathcal{L}_u) = d + 1$ and $H^1(\Sigma, \mathcal{L}_u) = \{0\}$ by $\Omega(\beta) \geq 0$. Choosing a complex basis

$$\{f_0, \dots, f_d\} \subset H^0(\Sigma, \mathcal{L}_u),$$

the Kodaira-type embedding map

$$\begin{aligned} F : \Sigma &\to \mathbb{C}P^d \\ x &\mapsto [f_0(x) : \cdots : f_d(x)] \end{aligned}$$

has the property that the image of Σ under F is not contained in any hyperplane of $\mathbb{C}P^d$.

Lemma 1.3. *Let $\mathcal{F}_{0,k}(d) \subset \overline{\mathcal{M}}_{0,k}(\mathbb{C}P^d, d[\text{line}])$ be the subspace consisting of stable maps whose image is not contained in any hyperplane. Then $\mathcal{F}_{0,k}(d)$ and the universal family of nodal curves $\mathcal{C}_{0,k}(d) \to \mathcal{F}_{0,k}(d)$ are smooth complex manifolds.*

Proof. Firstly, we show that for any $v : \Sigma \to \mathbb{C}P^d$ representing an element in $\mathcal{F}_{0,k}(d)$, the linearized $\overline{\partial}$-operator is surjective. The cokernel of such a $\overline{\partial}$-operator can be identified with the cohomology group

$$H^1(\Sigma, v^* T\mathbb{C}P^d).$$

Using the Euler short exact sequence

$$0 \longrightarrow \underline{\mathbb{C}} \longrightarrow \mathcal{O}(1)^{\oplus d+1} \longrightarrow T\mathbb{C}P^d \to 0,$$

we deduce from the long exact sequence of cohomology groups

$$H^1(\Sigma, v^*\mathcal{O}(1))^{\oplus d+1} \longrightarrow H^1(\Sigma, v^* T\mathbb{C}P^d) \longrightarrow 0.$$

As $H^1(\Sigma, v^*\mathcal{O}(1)) = \{0\}$ by positivity, we see that $H^1(\Sigma, v^* T\mathbb{C}P^d) = \{0\}$.

The above calculation works for all $\overline{\mathcal{M}}_{0,k}(\mathbb{C}P^d, d[\text{line}])$. To see that $\mathcal{F}_{0,k}(d)$ has a manifold structure, we need to argue that the automorphism groups are trivial. Using the Kodaira-type embedding described as above, any $[v] \in \mathcal{F}_{0,k}(d)$ corresponds to the data of a k-pointed genus 0 nodal Riemann surface Σ, a line bundle $\mathcal{L} \to \Sigma$, and a basis of $H^0(\Sigma, \mathcal{L})$. Accordingly, an automorphism of v corresponds to a line bundle isomorphism of $\mathcal{L}$ which fixes the given basis. This means that

- any automorphism must fix the effective components as a map, because it acts as identity on the restriction of the basis;
- any automorphism cannot act as a nontrivial tree automorphism: for nodal curves, a global section is extended by constant to other irreducible components;
- for uneffective componenets, stability means that the automorphism acts trivially.

This shows that $\mathcal{F}_{0,k}(d)$ is a smooth manifold, in fact, a smooth quasi-projective complex variety. As for $\mathcal{C}_{0,k}(d)$, we can identify it with $\mathcal{F}_{0,k+1}(d)$, which finishes the proof. ◻

Definition 1.4. *A* framed curve *is an isomorphism class of the triple (u, Σ, F) where $u : \Sigma \to M$ represents an element in $\overline{\mathcal{M}}_{0,k}(M, J, \beta)$ and F is the holomorphic map $F : \Sigma \to \mathbb{C}P^d$ defined by the Kodaira-type embedding from a complex basis of $H^0(\Sigma, \mathcal{L}_u)$. A framed curve (u, Σ, F) is called* unitary *if F is a unitary basis with respect to the Hermitian metric on $H^0(\Sigma, \mathcal{L}_u)$ up to a constant multiple.*

We can define the moduli space of framed curves in a different way. Consider the family of curves

$$\mathcal{C}_{0,k}(d) \to \mathcal{F}_{0,k}(d),$$

and we consider the space of maps

$$W^{k,p}(\mathcal{C}_{0,k}(d), M).$$

There exists a Banach vector bundle over $W^{k,p}(\mathcal{C}_{0,k}(d), M)$ whose fiber over $u : \Sigma \to M$ (here Σ should be thought of as a point in the base $\mathcal{F}_{0,k}(d)$, which parametrizes domains) is

$$W^{k,p}(\Sigma, \Omega^{0,1}_\Sigma \otimes u^*TM).$$

Then the $\overline{\partial}_J$-operator defines a section. If we restrict to the open subset of $W^{k,p}(\mathcal{C}_{0,k}(d), M)$ which consists of $u : \Sigma \to M$ such that

- $u^*\Omega \geq 0$;
- $u^*\Omega > 0$ on unstable components,

we can construct a unique holomorphic Hermitian line bundle $\mathcal{L}_u \to \Sigma$ as before. Then the moduli space of framed curves is described as $(u, \Sigma) \in W^{k,p}(\mathcal{C}_{0,k}(d), M)$ such that $u : \Sigma \to M$ defines a J-holomorphic stable map. This is becasue the "domain map" $(u, \Sigma) \mapsto \Sigma$ defines the basis of $H^0(\Sigma, \mathcal{L}_u)$ by pulling back the standard basis of $H^0(\mathbb{C}P^d, \mathcal{O}(1))$, thus the "framing" $F : \Sigma \to \mathbb{C}P^d$. Certainly, starting from $\overline{\mathcal{M}}_{0,k}(M, J, \beta)$, we know how to get a point in this parametrized moduli space: just pick a basis as before. We will denote the moduli space of framed curves by $\tilde{V}$. Note that $\tilde{V}$ admit a $U(d+1)$-action from permuting the framing by a unitary matrix.

Lemma 1.5. *The moduli space of* unitary *framed curves in $\tilde{V}$ can be identified with the zero locus of a section of the equivariant product bundle $\tilde{V} \times \mathcal{H}_{d+1}$, where $\mathcal{H}_n$ denotes the space of $n \times n$ Hermitian matrices.*

Proof. The induced Hermitian metric $\langle -, - \rangle_u$ on $H^0(\Sigma, \mathcal{L}_u)$ varies continuously in u. Then given a framed curve (u, Σ, F) where $F = \{f_0, \ldots, f_d\}$, we can consider the $(d+1) \times (d+1)$ Hermitian matrix $\mathcal{H}(u, \Sigma, F)$ whose (i, j)-entry is

$$\langle f_i, f_j \rangle_u.$$

We can define our section to be the inverse of the exponential map $\exp^{-1}$ on this matrix. The vanishing condition is equivalent to unitarity. □

Lemma 1.6. *The stabilizer of $(u, \Sigma, F) \in \tilde{V}$ under the $U(d+1)$-action agrees with the automorphism of the stable map $u : \Sigma \to M$, and there exists a homeomorphism* $(\exp^{-1} \mathcal{H})^{-1}(0)/u(d+1) \xrightarrow{\sim} \overline{\mathcal{M}}_{0,k}(M, J, \beta)$.

Proof. The automorphism group of $u : \Sigma \to M$ embeds in $U(d+1)$: any automorphism can be lifted to a unitary line bundle endomorphism of $\mathcal{L}_u$ which acts as unitary matrices on the space of basis, this is an embedding by the proof of Lemma 1.3. The action of $\mathrm{Aut}(u)$ on $\mathbb{C}^{d+1}$ induced from $\mathrm{Aut}(u) \hookrightarrow U(d+1)$ is faithful, thus taking unitary framings exactly records the automorphism group. □

As a consequence, we can take $V = (\exp^{-1}\mathcal{H})^{-1}(0)$ and $G = U(d+1)$ in Proposition 1.1. The proof also tells us that the map

$$\begin{aligned} V &\to \mathcal{F}_{0,k}(d) \\ (u, \Sigma, F) &\mapsto (F : \Sigma \to \mathbb{C}P^d) \end{aligned}$$

does not collapse the unstable components. Therefore, $\mathcal{F}_{0,k}(d)$ provides a good model for controlling the domains of stable maps.

To compare with the stabilization construction via intersecting with divisors, we need the following observation. Given any $[v] \in \mathcal{F}_{0,k}(d)$, we can take sufficiently many hypersurfaces in $\mathbb{C}P^d$ which intersects v transversely to define a map

$$\mathcal{F}_{0,k}(d) \to \overline{\mathcal{M}}_{0,k+d'}$$

near v by recording the marked points, which is a biholomorphism onto its image. We can "pull back" these divisors locally to reduce to the stabilization scheme we discussed before. However, this loses the advantage to have a single space to stabilize the moduli space.

2. Digression: K-theory

Recall that given a topological space Y, its complex K-theory in degree 0 is defined to be the group completion of the semi-group of isomorphism classes of vector bundles over Y, where addition is given by taking direct sum. In other words, an element of $K^0(X)$ is represented by a virtual vector bundle

$$V_1 - V_2$$

where $V_1 \to X$ and $V_2 \to X$ are genuinely complex vector bundles. Let's discuss two ways of constructing classes in $K^0(X)$, whose ideas will be used to address transversality problems for global Kuranishi charts.

(2a) Index for families.

Suppose Y is a compact and Hausdorff topological space.

Definition 2.1. *Let Z be a topological space. It is said to be a* manifold over Y *if it is a topological fiber bundle over Y with fibers given by a smooth manifold X, such that:*

(1) *the transition maps take value in* $\mathrm{Diff}(X)$*;*
(2) *the transition maps depend continuously on the parameter of Y.*

Definition 2.2. *Suppose $\pi : Z \to Y$ is a manifold over Y with fiber X. Then a vector bundle $p : \tilde{E} \to Z$ is* smooth over Y *if the composition $\pi \circ p : \tilde{E} \to Y$ is a fiber bundle such that its structure group is reduced to* $\mathrm{Diff}(X, E)$*: here $E := \tilde{E}|_X$ is the vector bundle obtained by restricting to a fiber, and* $\mathrm{Diff}(X, E)$ *denotes the group of bundle isomorphisms of E covering diffeomorphisms of X.*

With the above two definitions, we can discuss bundles of pseudo-differential operators, which are the sources of families of Fredholm operators that will lead to K-theoretic classes. Given a smooth manifold X, suppose $E, F \to X$ are smooth vector bundles. Let $\mathcal{P}^m(X, E, F)$ be the space of order-m pseduo-differential operators from $\Gamma(E)$ to $\Gamma(F)$, and we can consider the Sobolev completion $\overline{\mathcal{P}}^m(X, E, F)$.

Suppose we have two vector bundles $\tilde{E}, \tilde{F} \to Z$ smooth over Y. Inside the group $\mathrm{Diff}(X, E) \times \mathrm{Diff}(X, F)$, we can consider the subgroup $\mathrm{Diff}(X, E, F)$ consisting of pairs covering the same diffeomorphism of X. Then an element $(\Phi, \Psi) \in \mathrm{Diff}(X, E, F)$ acts on $\mathcal{P}^m(X, E, F)$ by mapping a pseudo-differential opertor P to $\Psi^{-1} \circ P \circ \Phi$. This extends to an action of $\mathrm{Diff}(X, E, F)$ on the completion $\overline{\mathcal{P}}^m(X, E, F)$.

Exercise 2.3. *Show that the action of* $\mathrm{Diff}(X, E, F)$ *on* $\overline{\mathcal{P}}^m(X, E, F)$ *is continuous. (Hint: for a complete discussion, see the Appendix of Atiyah–Singer IV).*

With the above exercise, using the principal $\mathrm{Diff}(X, E, F)$-bundle over Y induced from $\tilde{E}, \tilde{F} \to Z$, we obtain a $\overline{\mathcal{P}}^m(X, E, F)$-bundle $\overline{\mathcal{P}}^m(Z, \tilde{E}, \tilde{F})$ by taking the associate bundle of the $\mathrm{Diff}(X, E, F)$-representation $\overline{\mathcal{P}}^m(X, E, F)$. On the other hand, for the bundle $\tilde{E} \to Z$, we can take fiberwisely the space of Sobolev sections of class $W^{2,s}$ to obtain the Hilbert space bundle $H_s(Z, \tilde{E})$, similarly the Hilbert space bundle $H_s(Z, \tilde{F})$. Then we have the natural action

$$\overline{\mathcal{P}}^m(Z, \tilde{E}, \tilde{F}) \times H_s(Z, \tilde{E}) \to H_{s-m}(Z, \tilde{F}).$$

Definition 2.4. *A family of pseudo-differential operators parametrized by Y is a continuous section P of the bundle $\overline{\mathcal{P}}^m(Z, \tilde{E}, \tilde{F}) \to Y$. The family is called* elliptic *if for each $y \in Y$, the operator P_y is elliptic.*

The most natural source for us is the following situation. Suppose $\mathcal{M}$ is a compact moduli space of pseudo-holomorphic maps to (M, J), over which we have the family of curves $\mathcal{C} \to \mathcal{M}$ equipped with the evaluation map $\mathrm{ev} : \mathcal{C} \to M$. Suppose for simplicity that the topological type of Riemann surfaces parametrized by $\mathcal{M}$ is unchanged in this family. Then there are two vector bundles

$$\mathrm{ev}^* TM \to \mathcal{C}, \qquad \Omega^{0,1}_{\mathcal{C}/\mathcal{M}} \otimes \mathrm{ev}^* TM$$

which are smooth over $\mathcal{M}$. Then the linearized Cauchy–Riemann operator $D_u \overline{\partial}_J$ at $u \in \mathcal{M}$ assembles to define a section of $\mathcal{P}^m(\mathcal{C}, \mathrm{ev}^* TM, \Omega^{0,1}_{\mathcal{C}/\mathcal{M}} \otimes \mathrm{ev}^* TM)$. It is elliptic because $D_u \overline{\partial}_J$ is elliptic.

The next statement, which is due to Atiyah–Singer, allows us to construct K-theory classes from families of elliptic operators.

Proposition 2.5. *Let $P \in \overline{\mathcal{P}}^m(Z, \tilde{E}, \tilde{F})$ be elliptic. Then there exists a finite collection of sections $s_1, \ldots, s_k$ of $C^\infty(Z, \tilde{F})$ such that the map given by*

$$\begin{aligned} Q_y : C^\infty(Z, \tilde{E}) \oplus \mathbb{C}^k &\to C^\infty(Z, \tilde{F}) \\ (u, \lambda_1, \ldots, \lambda_k) &\mapsto P_y(u) + \sum_j \lambda_j s_j \end{aligned}$$

is surjective for all y*. The spaces* $\ker(Q_y)$ *forms a vector bundle* $\ker(Q) \to Y$ *over* Y*. Furthermore,*

$$\ker(Q) - \underline{\mathbb{C}}^k$$

defines a well-defined K*-theory class only depending on (the symbol of)* P*.*

Proof. Locally, given a point $y_0 \in Y$, the family is a product family so the operator P comes from a continuous map

$$Y \to \overline{\mathcal{P}}^m(Z, \tilde{E}, \tilde{F})$$

near y_0, which gives rise to a continuous map

$$Y \to \mathrm{Fred}(H_s(Z, \tilde{E}), H_{s-m}(Z, \tilde{F}))$$

to the space of bounded Fredholm operators. Let $V_{y_0} = \ker P_{y_0}^*$. Then the extension

$$H_s(Z, \tilde{E}) \oplus V_{y_0} \to H_{s-m}(Z, \tilde{F})$$
$$(u, v) \mapsto P_y(u) + v$$

is surjective at y_0, and this property remains to hold over an open neighborhood of $y_0 \in U_{y_0} \subset Y$.

Because Y is compact and Hausdorff, we can take a finite cover $\{U_{y_1}, \dots, U_{y_N}\}$ from the open cover $\{U_y\}_{y \in Y}$ and a partition of unity subordinate to this cover. Then we can consider the space

$$V_{y_1} \oplus \cdots \oplus V_{y_N}$$

and use the partition of unity to extend the cokernel elements to be globally defined over Y. This constructs the sections $s_1, \dots, s_k$. The kernel spaces $\ker(Q)$ have the local triviality property: one can fix a right inverse at a reference point to construct right inverses for nearby points, which allow one to project. This also ensures the continuity of the transitions maps.

For two different constructions Q_1 and Q_2 with associated sections $s_1^{(1)}, \dots, s_{k_1}^{(1)}$ and $s_1^{(2)}, \dots, s_{k_2}^{(2)}$, we can consider the "double stabilization"

$$\tilde{Q} : C^\infty(Z, \tilde{E}) \oplus \mathbb{C}^{k_1+k_2} \to C^\infty(Z, \tilde{F})$$
$$(u, \lambda_1, \dots, \lambda_{k_1}, \mu_1, \dots, \mu_{k_2}) \mapsto P_y(u) + \sum_j \lambda_j s_j^{(1)} + \sum_i \mu_i s_i^{(2)}.$$

We can use such to construct to homotopies over the interval $[0, 1]$

$$P_y(u) + t \sum_j \lambda_j s_j^{(1)} + \sum_i \mu_i s_i^{(2)} \text{ and } P_y(u) + \sum_j \lambda_j s_j^{(1)} + t \sum_i \mu_i s_i^{(2)},$$

which shows that $\ker(\tilde{Q}) - \underline{\mathbb{C}^{k_1+k_2}}$ is equivalent to both $\ker(\tilde{Q}_1) - \underline{\mathbb{C}^{k_1}}$ and $\ker(\tilde{Q}_2) - \underline{\mathbb{C}^{k_2}}$. □

What we will see later for the moduli spaces of pseudo-holomorphic maps is nothing but just a slight generalization of the above arguments, with a slight extension to the nonlinear context.

(2b) Push-forward in algebraic geometry.

Taking family indices is just integrating along the fiber in K-theory. Let's see how it can be defined algebro-geometrically.

Proposition 2.6. *Suppose $\pi : C \to T$ is a proper family of algebraic curves, and suppose T is proper. Let $E \to C$ be a vector bundle. If C admits a relative ample line bundle $L \to C$ (i.e., it restricts to ample line bundles over the fibers), then the derived pushforward $R\pi_* E$ has global resolutions.*

Proof. Using the ampleness of L, we know that for N sufficiently large,

(1) $\pi^*\pi_*(E \otimes L^N) \to E \otimes L^N$ is surjective: this is because of the relative ampleness assumption;
(2) $R^1\pi_*(E \otimes L^N) = 0$: this follows from "Kodaira vanishing";
(3) for any $t \in T$, for the fiber C_t, we have that $H^0(C_t, L_t^{-N}) = 0$.

Consider the bundle $F := \pi^*\pi_*(L^N \otimes E) \otimes L^{-N}$, and let H be the kernel of $F \to E$ induced by the pairing $L \otimes L^{-N} \to \underline{k}$. Then we have a short exact sequence of vector bundles

$$0 \longrightarrow H \longrightarrow F \longrightarrow E \longrightarrow 0,$$

which induces the long exact sequence of cohomology groups

$$H^0(C_t, F) \longrightarrow H^0(C_t, E) \longrightarrow H^1(C_t, H) \longrightarrow H^1(C_t, F) \longrightarrow H^1(C_t, E) \longrightarrow 0.$$

Note that for any $t \in T$, we necessarily have $H^0(C_t, F) = \pi^*\pi_*(L^N \otimes E) \otimes H^0(C_t, L_t^{-N}) = \{0\}$. This also implies the vanishing of $H^0(C_t, H)$. Therefore, Riemann–Roch implies that $H^1(C, H)$ and $H^1(C, F)$ are vector bundles because $\pi_* H$ and $\pi_* F$ are trivial. This shows that

$$R\pi_* E = [R^1\pi_* H \to R^1\pi_* F],$$

a 2-step resolution by vector bundles. □

For us, $C \to T$ is the family of k-marked stable maps to some smooth projective variety X, which is endowed with the map $\mathrm{rm} : C \to X$. Given an ample line bundle $L \to X$, it is a classical fact that the line bundle

$$\omega_{C/T}(x_1 + \cdots x_k) \otimes \mathrm{ev}^* L^3$$

is a relatively ample line bundle over C. If we let $E = \mathrm{ev}^* TX$, we know that

$$[R\pi_* E] = H^0(C_t, \mathrm{ev}^* TX) - H^0(C_t, \mathrm{ev}^* TX),$$

which is isomorphic to the difference

$$\ker(D\overline{\partial}_J) - \mathrm{coker}(D\overline{\partial}_J).$$

The above discussion shows that we have a surjection

$$H^1(C_t, F) \longrightarrow\!\!\!\!\!\rightarrow \mathrm{coker}(D\overline{\partial}_J).$$

We can then thicken up our moduli space T by the bundle $H^1(C_t, F)$ to achieve transversality.

3. Thickening in global Kuranishi charts

(3a) Finite-dimensional approximations.

This is the analogue of Atiyah–Singer's thickening trick in the context of J-holomorphic curves. In the following definition, $\mathcal{C}^{\circ}_{0,k}(d)$ is the G-manifold which comes from the complement of images of marked point sections of $\mathcal{C}_{0,k}(d) \to \mathcal{F}_{0,k}(d)$ and nodal points.

Definition 3.1. *A finite dimensional approximation scheme for* $\Omega^{0,1}_{\mathcal{C}^{\circ}_{0,k}(d)/\mathcal{F}_{0,k}(d)} \boxtimes_{\mathbb{C}} TM$ *is a collection of tuples* $(V_\mu, \lambda_\mu)_{\mu\in\mathbb{N}}$ *indexed by the natural numbers satisfying the following conditions.*

(1) *For each* $\mu \in \mathbb{N}$*, the vector space* V_μ *is a* G*-representation and*

$$\lambda_\mu : V_\mu \to C^\infty_c(\Omega^{0,1}_{\mathcal{C}^{\circ}_{0,k}(d)/\mathcal{F}_{0,k}(d)} \boxtimes_{\mathbb{C}} TM) \tag{3.1}$$

is a G*-equivariant linear embedding.*

(2) *Each* V_μ *is a subrepresentation of* $V_{\mu+1}$ *and* $\lambda_{u+1}|_{V_\mu}$ *agrees with* λ_μ*.*

(3) *Given each element* $\phi \in \mathcal{F}_{0,k}(d)$*, for the curve* $\mathcal{C}^{\circ}_{0,k}(d)|_\phi$*, the restriction of the image* $\cup_{\mu\in\mathbb{N}}\lambda_\mu(V_\mu)$ *to* $\mathcal{C}^{\circ}_{0,k}(d)|_\phi$ *is dense in* $C^\infty(\Omega^{0,1}_{\mathcal{C}^{\circ}_{0,k}(d)|_\phi} \boxtimes_{\mathbb{C}} TM)$ *under the* C^∞*-topology.*

Lemma 3.2. $\Omega^{0,1}_{\mathcal{C}^{\circ}_{0,k}(d)/\mathcal{F}_{0,k}(d)} \boxtimes_{\mathbb{C}} TM$ *admits a finite dimensional approximation scheme.*

Proof. Let's introduce the notation $B := \mathcal{F}_{0,k}(d) \times M$ and let $E := \Omega^{0,1}_{\mathcal{C}^{\circ}_{0,k}(d)/\mathcal{F}_{0,k}(d)} \boxtimes_{\mathbb{C}} TM$. Then B is a smooth G-manifold and $E \to B$ is a G-equivariant vector bundle. The existence of finite dimensional approximation schemes follows from the following sequence of observations.

(1) If $E \to B$ is a G-equivariant vector bundle over a closed manifold, then after choosing a connection on E, a G-invariant Riemannian metric on B, and a G-invariant bundle metric on E, we can define the Laplacian operator Δ on $\Gamma(E)$. Using the eigenspace decomposition of $L^2(\Gamma(E))$, we can simply define V_μ to be the direct sum of the first μ-eigenspaces of Δ.

(2) If B is instead a manifold with boundary, we can close it up by taking the doubling construction to obtain a $G\times\mathbb{Z}/2$-manifold $\tilde{B}$ and a bundle thereon $\tilde{E}$ equivariant under the $G\times\mathbb{Z}/2$-action. Then we can take the $\mathbb{Z}/2$-invariant subspaces from the finite dimensional approximation scheme discussed in (1).

(3) In general, we can find an exhaustion of B by compact codimension 0 submanifolds with boundary $B = \cup_k B_k$, then a suitable cut-off of the λ_μ constructed in (2) works.

This procedure does not really ensure the fiberwise approximation requirement: one can do this using some covering argument. □

Note that for two different finite dimensional approximation schemes, one can take their direct sum to obtain a new one.

(3b) The global chart.

Let's go back to the moduli space $\overline{\mathcal{M}}_{0,k}(M,J,\beta)$. Recall the following definition.

Definition 3.3. *A* global Kuranishi chart *for* $\overline{\mathcal{M}}_{0,k}(M,J,\beta)$ *is a tuple*

$$(G,V,E,S,\Psi)$$

where G is a compact Lie group, V is a topological manifold admitting a G-action, $E\to V$ is a G-equivariant vector bundle, $S:V\to E$ is a continuous G-equivariant section such that

$$\Psi : S^{-1}(0)/G \xrightarrow{\sim} \overline{\mathcal{M}}_{0,k}(M,J,\beta)$$

is a homeomorphism respecting the stabilizers.

There are certain moves among global Kuranishi charts.

(1) Group enlargement: let G' be another compact Lie group and suppose $\pi_P : P\to V$ is a principal G'-bundle. This procedure replaces (G,V,E,S,Ψ) by $(G\times G',P,\pi_P^*E,\pi_P^*S,\Psi)$.
(2) Shrinking: suppose $U\subset V$ is a G-invariant open subset of V containing $S^{-1}(0)$. Shrinking means replacing (G,V,E,S,Ψ) by $(G,U,E|_U,S|_U,\Psi)$.
(3) Stabilization: let $\pi_F : F\to V$ be a G-equivariant vector bundle. The stabilization of (G,V,E,S,Ψ) by F is $(G,F,\pi_F^*F\oplus\pi_F^*E,\tau_F\oplus\pi_F^*S,\Psi)$, where $\tau_F : F\to\pi_F^*F$ is the tautological section.

Theorem 3.4. *The moduli space $\overline{\mathcal{M}}_{0,k}(M,J,\beta)$ admits a global Kuranishi chart. Moreover, different constructions are related via the three moves described above.*

Proof. We first describe the chart. The choices to make involve the following.

(1) The 2-form Ω, which defines $d=\Omega(\beta)$ and the moduli spaces $\mathcal{F}_{0,k}(d)$ and $\mathcal{C}_{0,k}(d)$.
(2) A finite-dimensional approximation scheme $(V_\mu,\lambda_\mu)_{\mu\in\mathbb{N}}$ for $\Omega^{0,1}_{\mathcal{C}^\circ_{0,k}(d)/\mathcal{F}_{0,k}(d)}\boxtimes_{\mathbb{C}} TM$.

With these choices, we claim that the following defines a global Kuranishi chart for $\overline{\mathcal{M}}_{0,k}(M,J,\beta)$.

- We set the compact Lie group $G=U(d+1)$.
- We define V to be the following moduli space: it consists of pairs

$$([\Sigma],u,e)\in C^\infty(\mathcal{C}_{0,k}(d),M)\times V_\mu$$

  for a sufficiently large μ, where Σ represents a "domain" in $\mathcal{F}_{0,k}(d)$ and $u:\Sigma\to M$ satisfying the perturbed Cauchy–Riemann equation

$$\overline{\partial}_J u|_{\mathcal{C}^\circ|_\Sigma}+\lambda_\mu(e)\circ\Gamma_u=0$$

  in which Γ_u denotes the graph of $u:\Sigma\to M$ in $\mathcal{C}^\circ_{0,k}(d)\times M$.
- Let the bundle $E\to V$ be the product bundle $V\times\mathcal{H}_{d+1}\times V_\mu$ endowed with the natural G-action.
- As for the section S, for an element $([\Sigma],u,e)$ with the corresponding framing F, we map it to $(\exp^{-1}(F),e)$.

- Based on the description of S, it is immediate that we have a homeomorphism $S^{-1}(0)/G \xrightarrow{\sim} \overline{\mathcal{M}}_{0,k}(M, J, \beta)$ respecting the stabilizers: this is what we discussed above. The topology on both sides can be compared because the Gromov topology can be understood as the Hausdorff topology of graphs provided that the bubbling phenomena have been taken care of.

To show that the tuple (G, V, E, S, Ψ) is qualified to be called a global Kuranishi chart, we only need to show that with the assumption that μ is sufficiently large, V is a topological *manifold*. This straightforward: upon restricting to an open neighborhood of $S^{-1}(0)$, the compactness of the moduli space means that for μ large enough, the image of V_μ can annihilate the cokernels of all the linearized $\overline{\partial}_J$-operators.

For a different chart (G', V', E', S', Ψ') produced from $d' := \Omega'(\beta)$ and $V'_{\mu'}$, we can consider the following doubling construction.

- We set the Lie group $\tilde{G}$ to be $G \times G'$.
- Let $\tilde{V}$ be the space consisting of $(u, [\Sigma], [\Sigma'], e, e')$ where $[\Sigma] \in \mathcal{F}_{0,k}(d)$ and $[\Sigma'] \in \mathcal{F}_{0,k}(d')$ represent framed curves with the same domain curve, $u : \Sigma \to M$ satisfies
$$\overline{\partial}_J u|_{\mathcal{C}^\circ|_\Sigma} + \lambda_\mu(e) \circ \Gamma_u + \lambda_\mu(e') \circ \Gamma_u = 0$$
for $e \in V_\mu$ and $e' \in V'_{\mu'}$.
- Let $\tilde{E} = V \times \mathcal{H}_{d+1} \times V_\mu \times \mathcal{H}_{d'+1} \times V'_{\mu'}$ with the product $\tilde{G} = G \times G'$-action.
- Let $\tilde{S}$ be the map $(\exp^{-1}(F), \exp^{-1}(F'), e, e')$ where F and F' are the framings from Σ_1 and Σ_2. The map $\tilde{\Psi}$ is defined in the natural way.

Then one can see that $(\tilde{G}, \tilde{V}, \tilde{E}, \tilde{S}, \tilde{\Psi})$ serves as the "roof" for (G, V, E, S, Ψ) and (G', V', E', S', Ψ'). □

To sum up, other than the global quotient presentation of $\overline{\mathcal{M}}_{0,k}(M, J, \beta)$ using framed curves, the additional step for constructing the global Kuranishi chart is to "thicken up" the Cauchy–Riemann equation. The very existence of such a thickening is a nonlinear generalization of Atiyah–Singer's construction of K-theory classes from families of elliptic operators. The proof of invariance, which is based on a doubling discussion, already appeared in the linear index theory.

Next time: relative smooth structures from standard gluing, complex orientations.