Topic 7: Hamiltonian Floer flow categories

With the understanding of how to regularize moduli spaces of stable J-holomorphic maps in the Gromov–Witten setting, we move on to Hamiltonian Floer theory.

1. Flow categories: stratification

We introduce the notion of flow categories and flow bimodules, which captures the natural stratifications of moduli spaces of Morse–Floer trajectories and continuation maps.

Here is the motivation for the following setup. For closed symplectic manifold (M, ω) and a non-degenerate Hamiltonian $H_t : S^1 \times M \to \mathbb{R}$, to set up the Hamiltonian Floer theory, we need to consider *capped* 1-periodic orbits of X_{H_t}. The action functional assigns to each capped orbit its energy, and the Conley–Zehnder index assgins to each capped orbit an integer. If we shift a capped orbit by an element in $A \in \mathrm{im}(\pi_2(M) \to H_2(M; \mathbb{Z}))$, the action is shifted by $\omega(A)$ while the index is shifted by $2c_1(A)$.

Setup 1.1. *Let N be a nonnegative integer, Π be an infinite cyclic group, and $\omega : \Pi \to \mathbb{R}$ be a group injection.*

Let $\mathcal{P}$ be a countable poset equipped with the following extra data: a free Π-action and two functions (called the **action** *and the* **index***)*

$$\mathcal{A}^{\mathcal{P}} : \mathcal{P} \to \mathbb{R}, \qquad \mathrm{ind}^{\mathcal{P}} : \mathcal{P} \to \mathbb{Z}/2N.$$

Assume the following conditions.

(1) *The Π-action is order-preserving. Namely, for all $p, q \in \mathcal{P}$ and $a \in \Pi$*
$$p \leq q \Longleftrightarrow a \cdot p \leq a \cdot q.$$
(2) *For all $p \in \mathcal{P}$ and $a \in \Pi$,*
$$\mathcal{A}^{\mathcal{P}}(a \cdot p) = \mathcal{A}^{\mathcal{P}}(p) + \omega(a)$$
and
$$\mathrm{ind}^{\mathcal{P}}(a \cdot p) = \mathrm{ind}^{\mathcal{P}}(\mathrm{p}).$$
(3) *For all $p, q \in \mathcal{P}$,*
$$p < q \Longrightarrow \mathcal{A}^{\mathcal{P}}(p) < \mathcal{A}^{\mathcal{P}}(q).$$
(4) *The quotient set $\underline{\mathcal{P}} := \mathcal{P}/\Pi$ is finite.*

The conditions that $\omega : \Pi \to \mathbb{R}$ is injective and that $\underline{\mathcal{P}}$ is finite imply that $\mathcal{P}$ is "locally finite-dimensional," namely, for any pair of elements $p < q$ of $\mathcal{P}$, there are at most finitely many elements lying between them.

Notation 1.2. *Given a pair of elements $p < q$ in $\mathcal{P}$, we define a poset*

$$\mathbb{A}_{pq}^{\mathcal{P}} := \left\{\alpha = pr_1 \cdots r_l q \mid p < r_1 < \cdots < r_l < q,\ r_1, \ldots, r_l \in \mathcal{P}\right\}$$

whose partial order is induced by inclusion

$$ps_1 \cdots s_m q \leq pr_1 \cdots r_l q \Longleftrightarrow \{r_1 \cdots r_l\} \subseteq \{s_1, \ldots, s_m\}.$$

The poset $\mathbb{A}_{pq}^{\mathcal{P}}$ has a unique maximal element pq. We can define its depth function to be

$$\mathrm{depth}(pr_1 \cdots r_l q) = l.$$

Notation 1.3. *Given the poset $\mathbb{A}_{pq}^{\mathcal{P}}$, for an element $\alpha \in \mathbb{A}_{pq}^{\mathcal{P}}$, define the poset*

$$\partial^{\alpha}\mathbb{A}_{pq}^{\mathcal{P}}$$

consisting of all $\beta \in \mathbb{A}_{pq}^{\mathcal{P}}$ such that $\beta \leq \alpha$ with the induced partial order.

The following exercise formalizes the observation that the boundary strata of moduli spaces of Morse–Floer trajectories can be written as a product, namely, broken trajectories.

Exercise 1.4. *For the triple $prq \in \mathbb{A}_{pq}^{\mathcal{P}}$, show that there is an isomorphism of posets*

$$\mathbb{A}_{pr}^{\mathcal{P}} \times \mathbb{A}_{rq}^{\mathcal{P}} \cong \partial^{prq}\mathbb{A}_{pq}^{\mathcal{P}} \tag{1.1}$$

under the concatenation of strings.

Now we introduce the notion of flow categories under the setting of Setup 1.1.

Definition 1.5. *Let $\mathcal{P}$ be as in Setup 1.1. A* ***flow category*** *$T^{\mathcal{P}}$ over $\mathcal{P}$ is a topologically enriched category*[1] *with the set of objects given by $\mathcal{P}$, with morphism spaces T_{pq} satisfying the following conditions.*

1. *$T_{pq} \neq \emptyset$ only if $p \leq q$ in $\mathcal{P}$.*[2]
2. *T_{pp} is the singleton.*
3. *T_{pq} is a compact topological space equipped with stratification indexed by $\mathbb{A}_{pq}^{\mathcal{P}}$.*[3]
4. *Given a triple $p < r < q$ in $\mathcal{P}$, the composition map factors through a stratified homeomorphism*[4]

$$\begin{array}{ccc} T_{pr} \times T_{rq} & \longrightarrow & \partial^{prq}T_{pq} \\ \downarrow & & \downarrow \\ \mathbb{A}_{pr}^{\mathcal{P}} \times \mathbb{A}_{rq}^{\mathcal{P}} & \longrightarrow & \partial^{prq}\mathbb{A}_{pq}^{\mathcal{P}} \end{array}$$

[1]Namely, the set of morphisms are topological spaces and composition maps are continuous.

[2]In the Morse or Floer case, it is indeed true that $T_{pq} \neq \emptyset$ if and only if $p \leq q$.

[3]This means that the natural map which assigns a point in T_{pq} to the stratum from $\mathbb{A}_{pq}^{\mathcal{P}}$ in which it lies is a continuous map, where we equip $\mathbb{A}_{pq}^{\mathcal{P}}$ with the topology that $\partial^{\alpha}\mathbb{A}_{pq}^{\mathcal{P}}$ generates the closed subsets.

[4]Just means a homeomorphism respecting the stratification.

where the underlying poset isomorphism is the map (1.1). *We require that whenever* $p < r < s < q$, *the following associativity diagram holds:*

$$\begin{array}{ccc} T_{pr} \times T_{rs} \times T_{sq} & \longrightarrow & T_{ps} \times T_{sq} \\ \downarrow & & \downarrow \\ T_{pr} \times T_{rq} & \longrightarrow & T_{pq}. \end{array}$$

(5) Π *defines a strict action on* $T^{\mathcal{P}}$: *for any* $a \in \Pi$ *and* $p, q \in \mathcal{P}$, *there is a stratified homeomorphism*

$$\begin{array}{ccc} T_{pq} & \xrightarrow{\phi_a} & T_{a\cdot p\ a\cdot q} \\ \downarrow & & \downarrow \\ \mathbb{A}^{\mathcal{P}}_{pq} & \longrightarrow & \mathbb{A}^{\mathcal{P}}_{a\cdot p\ a\cdot q} \end{array}$$

where the underlying poset map is the natural isomorphism. Moreover, when $a_1, a_2 \in \Pi$, *we require that the equation* $\phi_{a_1 \cdot a_2} = \phi_{a_1} \circ \phi_{a_2}$ *holds and* ϕ_0 *is the identity map for* $a = 0 \in \Pi$.

Lemma 1.6. *Given* $\alpha = pr_1 \cdots r_l q \in \mathbb{A}^{\mathcal{P}}_{pq}$, *the space* $\partial^\alpha T_{pq}$ *is homeomorphic to* $T_{pr_1} \times \cdots \times T_{r_l q}$ *as* $\mathbb{A}^{\mathcal{P}}_{pr_1} \times \cdots \times \mathbb{A}^{\mathcal{P}}_{r_l q}$*-spaces.*

Proof. We prove the statement by induction on $\mathrm{depth}(\alpha)$. For $\mathrm{depth}(\alpha) = 0$, this is tautology, and for $\mathrm{depth}(\alpha) = 1$, the assertion follows from Definition 1.5. Suppose the lemma holds for all $\mathrm{depth}(\alpha) \leq l - 1$. Now suppose $\alpha = pr_1 \cdots r_l q$. Consider the homeomorphism between $\mathbb{A}^{\mathcal{P}}_{pr_1} \times \mathbb{A}^{\mathcal{P}}_{r_1 q}$ spaces $T_{pr_1} \times T_{r_1 q} \to \partial^{pr_1 q} T_{pq}$. Restricting the homeomorphism along the closed stratum $\mathbb{A}^{\mathcal{P}}_{pr_1} \times \partial^{r_1 \cdots r_l q} \mathbb{A}^{\mathcal{P}}_{r_1 q}$ and using the induction hypothesis, we obtain a homeomorphism of $\mathbb{A}^{\mathcal{P}}_{pr_1} \times \cdots \times \mathbb{A}^{\mathcal{P}}_{r_l q}$-spaces

$$T_{pr_1} \times \cdots \times T_{r_l q} \to \partial^\alpha T_{pq}.$$

By associativity, if we construct such a homeomorphism by decomposing α as $pr_1 \cdots r_k$ and $r_k \cdots r_l q$ for some $1 \leq k \leq l$, the resulting homeomorphism between the stratified spaces is the same. □

Next we discuss flow bimodules, which originate from the moduli spaces of continuation maps.

Notation 1.7. *Suppose* $\mathcal{P}$ *and* $\mathcal{P}'$ *are two posets as in Setup 1.1 equipped with own action and index functions*

$$(\mathcal{A}^{\mathcal{P}}, \mathrm{ind}^{\mathcal{P}}) : \mathcal{P} \to \mathbb{R} \times (\mathbb{Z}/2N), \qquad (\mathcal{A}^{\mathcal{P}'}, \mathrm{ind}^{\mathcal{P}'}) : \mathcal{P}' \to \mathbb{R} \times (\mathbb{Z}/2N).$$

For $p \in \mathcal{P}$ *and* $p' \in \mathcal{P}'$, *define a poset*

$$\mathbb{A}_{pp'} := \left\{\alpha = pq_1 \cdots q_k q'_{k'} \cdots q'_1 p' \mid p < q_1 < \cdots < q_k,\ q'_{k'} < \cdots < q'_1 < p'\right\}$$

The partial order is again induced by inclusion:

$$pq_1 \cdots q_k q'_{k'} \cdots q'_1 p' \leq p\tilde{q}_1 \cdots \tilde{q}_{\tilde{k}} \tilde{q}'_{\tilde{k}'} \cdots \tilde{q}'_1 p' \iff \{\tilde{q}_1, \ldots, \tilde{q}_{\tilde{k}}\} \subseteq \{q_1, \ldots, q_k\} \text{ and } \{\tilde{q}'_1 \cdots \tilde{q}'_{\tilde{k}'}\} \subseteq \{q'_1 \cdots q'_{k'}\}.$$

Then $\mathbb{A}_{pp'}$ is poset with a unique maximal element pp' and we can define its depth function to be

$$\operatorname{depth}(pq_1 \cdots q_k q'_{k'} \cdots q'_1 p') = k + k'.$$

There are similar characterizations of "broken configurations."

Exercise 1.8. *If $p < q$ are elements in $\mathcal{P}$, show that there is a natural isomorphism of posets*

$$\mathbb{A}^{\mathcal{P}}_{pq} \times \mathbb{A}_{qp'} \cong \partial^{pqp'} \mathbb{A}_{pp'}$$

by concatenation of words. Moreover, prove that such an isomorphism makes the following diagram commute:

$$\begin{array}{ccc} & \mathbb{A}^{\mathcal{P}}_{pq_1} \times \mathbb{A}^{\mathcal{P}}_{q_1q_2} \times \mathbb{A}_{q_2p'} & \\ \swarrow & & \searrow \\ \mathbb{A}^{\mathcal{P}}_{pq_2} \times \mathbb{A}_{q_2p'} & & \mathbb{A}^{\mathcal{P}}_{pq_1} \times \mathbb{A}_{q_1p'} \\ \searrow & & \swarrow \\ & \mathbb{A}_{pp'} & \end{array}$$

Similarly, if $q' < p'$ in $\mathcal{P}'$, one has

$$\mathbb{A}_{pq'} \times \mathbb{A}^{\mathcal{P}'}_{q'p'} \cong \partial^{pq'p'} \mathbb{A}_{pp'},$$

which satisfies a similar commutative relation as above, and in this case the poset $\mathbb{A}^{\mathcal{P}'}$ acts on the right. Moreover, these two types of isomorphisms are compatible in the following sense. Namely, the following diagram is commutative for which the arrows are induced by the obvious concatenation of words.

$$\begin{array}{ccc} & \mathbb{A}^{\mathcal{P}}_{pq} \times \mathbb{A}_{qq'} \times \mathbb{A}^{\mathcal{P}'}_{q'p'} & \\ \swarrow & & \searrow \\ \mathbb{A}^{\mathcal{P}}_{pq} \times \mathbb{A}_{qp'} & & \mathbb{A}_{pq'} \times \mathbb{A}^{\mathcal{P}'}_{q'p'} \\ \searrow & & \swarrow \\ & \mathbb{A}_{pp'} & \end{array}$$

Definition 1.9. *Let $T^{\mathcal{P}}$ and $T^{\mathcal{P}'}$ be flow categories over $\mathcal{P}$ and $\mathcal{P}'$ respectively. A* ***flow bimodule*** *M from $T^{\mathcal{P}}$ to $T^{\mathcal{P}'}$ consists of the following data.*

1. *A compact $\mathbb{A}_{pp'}$-space $M_{pp'}$ (which can be empty) for all $p \in \mathcal{P}$ and $p' \in \mathcal{P}'$.*
2. *For $p < q$, a homeomorphism of stratified spaces*

$$\begin{array}{ccc} T^{\mathcal{P}}_{pq} \times M_{qp'} & \longrightarrow & \partial^{pqp'} M_{pp'} \\ \downarrow & & \downarrow \\ \mathbb{A}^{\mathcal{P}}_{pq} \times \mathbb{A}_{qp'} & \xrightarrow[\cong]{} & \partial^{pqp'} \mathbb{A}_{pp'} \end{array}.$$

(3) *For $q' < p'$, a homeomorphism of stratified spaces*

$$\begin{array}{ccc} M_{pq'} \times T^{\mathcal{P}'}_{q'p'} & \longrightarrow & \partial^{pq'p'} M_{pp'} \;. \\ \downarrow & & \downarrow \\ \mathbb{A}_{pq'} \times \mathbb{A}^{\mathcal{P}'}_{q'p'} & \xrightarrow[\cong]{} & \partial^{pq'p'} \mathbb{A}_{pp'} \end{array}$$

These data should be subject to the following conditions.

(1) *There is a constant $C > 0$ such that for all $p \in \mathcal{P}$, $p' \in \mathcal{P}'$,*

$$M_{pp'} \neq \emptyset \Longrightarrow \mathcal{A}^{\mathcal{P}}(p) < \mathcal{A}^{\mathcal{P}'}(p') + C.^{5} \tag{1.2}$$

(2) *For $p < q_1 < q_2$ in $\mathcal{P}$ and $p' \in \mathcal{P}'$, the following diagram commutes:*

$$\begin{array}{ccc} T^{\mathcal{P}}_{pq_1} \times T^{\mathcal{P}}_{q_1q_2} \times M_{q_2p'} & \longrightarrow & T^{\mathcal{P}}_{pq_1} \times M_{q_1p'} \\ \downarrow & & \downarrow \\ T^{\mathcal{P}}_{pq_2} \times M_{q_2p'} & \longrightarrow & M_{pp'} \end{array}$$

where $T^{\mathcal{P}}_{pq_1} \times M_{q_1p'} \to M_{pp'}$ is induced by the composition of the homeomorphism $T^{\mathcal{P}}_{pq_1} \times M_{q_1p'} \to \partial^{pq_1p'} M_{pp'}$ and the inclusion $\partial^{pq_1p'} M_{pp'} \hookrightarrow M_{pp'}$ and so forth.

(3) *Similarly, for $p \in \mathcal{P}$ and $q_2' < q_1' < p'$ in $\mathcal{P}'$, we have a commutative diagram*

$$\begin{array}{ccc} M_{pq_2'} \times T^{\mathcal{P}'}_{q_2'q_1'} \times T^{\mathcal{P}'}_{q_1'p'} & \longrightarrow & M_{pq_1'} \times T^{\mathcal{P}'}_{q_1'p} \\ \downarrow & & \downarrow \\ M_{pq_2'} \times T^{\mathcal{P}'}_{q_2'p'} & \longrightarrow & M_{pp'}. \end{array}$$

(4) *For $p < q$ in $\mathcal{P}$ and $q' < p'$ in $\mathcal{P}'$, we have a commutative diagram*

$$\begin{array}{ccc} T^{\mathcal{P}}_{pq} \times M_{qq'} \times T^{\mathcal{P}'}_{q'p'} & \longrightarrow & M_{pq'} \times T^{\mathcal{P}'}_{q'p'} \\ \downarrow & & \downarrow \\ T^{\mathcal{P}}_{pq} \times M_{qp'} & \longrightarrow & M_{pp'}. \end{array}$$

(5) *Strict Π-action: for any $a \in \Pi$, there is a stratified homeomorphism*

$$\begin{array}{ccc} M_{pp'} & \longrightarrow & M_{a\cdot p\ a\cdot p'} \\ \downarrow & & \downarrow \\ \mathbb{A}_{pp'} & \longrightarrow & \mathbb{A}_{a\cdot p\ a\cdot p'} \end{array}$$

such that for $a_1, a_2 \in \Pi$ the equation $\phi^M_{a_1\cdot a_2} = \phi^M_{a_1} \circ \phi^M_{a_2}$ holds, and such that ϕ^M_{id} is the identity map. Moreover, we require that the actions

$$T^{\mathcal{P}}_{pq_1} \times M_{q_1p'} \to M_{pp'}, M_{pq_1'} \times T^{\mathcal{P}'}_{q_1'p'} \to M_{pp'}$$

are Π-equivariant.

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[5] The constant is related to the Hofer-type norm of a given family of Hamiltonians.

The following statement is the analog of Lemma 1.6 for flow bimodules. The associativity conditions from Definition 1.9 guarantees that the maps between the stratified spaces are well-defined.

Lemma 1.10. *Suppose $M_{pp'}$ is nonempty. Given an element $\alpha = pq_1 \cdots q_k q'_{k'} \cdots q'_1 p' \in \mathbb{A}_{pp'}$, we have a stratified homeomorphism*

$$\begin{array}{ccc} T^{\mathcal{P}}_{pq_1} \times \cdots \times M_{q_k q'_{k'}} \times \cdots \times T^{\mathcal{P}'}_{q'_1 p'} & \longrightarrow & \partial^\alpha M_{pp'} \\ \downarrow & & \downarrow \\ \mathbb{A}^{\mathcal{P}}_{pq_1} \times \cdots \times \mathbb{A}_{q_k, q'_{k'}} \times \cdots \times \mathbb{A}^{\mathcal{P}'}_{q'_1, p'} & \xrightarrow{\cong} & \partial^\alpha \mathbb{A}_{pp'} \end{array} . \qquad \square$$

2. Orbifolds and vector bundles

Although we look at compact Lie group actions in the definition of global Kuanishi charts, it's more useful to look at the associated orbifold constructions for many purposes. This is crash course on orbifolds and related notions.

Let $\mathcal{U}$ be a Hausdorff and second countable topological space. We discuss how to equip it with an *effective* orbifold structure following Thurston's traditional approach.

Definition 2.1. *An n-dimensional orbifold chart of $\mathcal{U}$ is a triple*

$$C = (G, U, \psi)$$

where $U \subset \mathbb{R}^n$ is a nonempty open subset, G is a finite group acting effectively and smoothly on U, and $\psi : U \to \mathcal{U}$ is a G-invariant continuous map such that the induced map

$$\underline{\psi} : U/G \to \mathcal{U}$$

is a homeomorphism onto an open subset of $\mathcal{U}$. If $p \in \psi(U)$ we also say that x is contained in the chart C.

A *chart embedding* from another chart $C' = (G', U', \psi')$ to C is a smooth open embedding $\iota : U' \hookrightarrow U$ such that

$$\psi \circ \iota = \psi'.$$

The following statement allows us to include the group injection as part of the data of a chart embedding.

Lemma 2.2. *Given a chart embedding ι as above there exists a canonical group injection $G' \hookrightarrow G$ such that ι is equivariant.*

Proof. We need to use the following fact: given two embeddings of orbifold charts

$$\lambda, \mu : (G', U', \psi') \hookrightarrow (G, U, \psi),$$

there exists a unique $g \in G$ such that $\mu = g \cdot \lambda$. For a proof, see [MP97, Appendix] Assuming this, given any $g' \in G'$, we can regard it as a chart embedding of (G', U', ψ') to itself. Then for a chart embedding $\lambda : (G', U', \psi') \hookrightarrow (G, U, \psi)$, we can find $g \in G$ such that the chart embeddings λ and $\lambda \cdot g'$ differs by $g \in G$. We define $\lambda(g') := g$, which defines a monomorphism of groups. $\square$

As we are in the smooth category, we can always find "linear" charts around any point. An orbifold chart $C = (G, U, \psi)$ is called *linear* if G acts linearly on $\mathbb{R}^n$ and $U \subset \mathbb{R}^n$ is an invariant open subset. We say that a linear chart is *centered at* $p \in X$ if $0 \in U$ and $p = \psi(0)$. Going from a smooth chart to a linear done can be achieved by looking at the tangent space and compose with the (equivariant) exponential map.

Definition 2.3. *We say two charts $C_i = (G_i, U_i, \psi_i)$, $i = 1, 2$ are compatible if for each $p \in \psi_1(U_1) \cap \psi_2(U_2)$, there exists an orbifold chart $C_p = (G_p, U_p, \psi_p)$ containing p and chart embeddings into both C_1 and C_2.*

An *orbifold atlas* $\mathcal{A} = \{C_i \mid i \in I\}$ on X is a collection of mutually compatible charts C_i which cover $\mathcal{U}$. We say an atlas $\mathcal{A}' = \{C'_j \mid j \in J\}$ *refines* $\mathcal{A}$, equivalently, $\mathcal{A}'$ is a *refinement* of $\mathcal{A}$, if for each C'_j there exists a chart embedding $C'_j \hookrightarrow C_i$ for some $i \in I$. We say two orbifold atlases are *equivalent* if they have a common refinement.

Definition 2.4. *A Hausdorff and second countable topological space $\mathcal{U}$ together with an equivalence class of orbifold atlases is called a smooth effective orbifold.*

Every smooth effective orbifold has a unique maximal atlas; two atlases are equivalent if they are contained in the common maximal atlas. It is convenient to work with the maximal atlas. For our discussions, an orbifold chart of a smooth effective orbifold means a chart in the maximal atlas. We often use $|\mathcal{U}|$ to denote the underlying topological space (called the *coarse space*) of an effective orbifold $\mathcal{U}$ while forgetting the orbifold structure.

We do not define the general form of orbifold morphisms, whose definition would require the language of stacks. Below are a few special cases of "maps" between orbifolds.

(1) A continuous function on an effective orbifold is *smooth* if its pullback to each chart is a smooth function.
(2) An *isomorphism* of orbifolds from $\mathcal{U}$ to $\mathcal{U}'$ is a homeomorphism $f : |\mathcal{U}| \to |\mathcal{U}'|$ such that for each point $x \in \mathcal{U}$, there exist a chart $C = (G, U, \psi)$ of $\mathcal{U}$ containing x, chart $C' = (G', U', \psi')$ of $\mathcal{U}'$ containing $f(x)$, a group isomorphism $G' \cong G$, and an equivariant diffeomorphism $\tilde{f} : U' \to U$ which descends to $f|_{\psi(U)}$.
(3) An *open embedding* from $\mathcal{U}$ to $\mathcal{U}'$ is an isomorphism from $\mathcal{U}$ to an open subset of $\mathcal{U}'$.

Remark 2.5. *We also need the notion of orbifolds with boundary or corners. In that case, the domain of a chart $C = (G, U, \psi)$ is allowed to be a smooth manifold with boundary or corners such that the group G acts trivially on the normal directions to the boundary strata.*

The definition of orbifold vector bundles is very similar to that of effective orbifolds. Let $\mathcal{U}$ be an effective orbifold, $\mathcal{E}$ be a topological space, and $\pi_{\mathcal{E}} : \mathcal{E} \to \mathcal{U}$ be a continuous map.

Definition 2.6. *A bundle chart of $\pi_{\mathcal{E}} : \mathcal{E} \to \mathcal{U}$ consists of an orbifold chart $C = (G, U, \psi)$ of $\mathcal{U}$, a G-equivariant smooth vector bundle $\pi_E : E \to U$, and a G-invariant continuous map $\hat{\psi} : E \to \mathcal{E}$ such that the induced map from E/G to $\mathcal{E}$ is a homeomorphism onto $\pi_{\mathcal{E}}^{-1}(\psi(U))$ and such that* *the following diagram commutes:*

$$\begin{array}{ccc} E & \xrightarrow{\hat{\psi}} & \mathcal{E} \\ {\scriptstyle \pi_E}\downarrow & & \downarrow{\scriptstyle \pi_{\mathcal{E}}} \\ U & \xrightarrow[\psi]{} & \mathcal{U} \end{array}.$$

In notation we will use a quadruple $\hat{C} = (G, U, E, \hat{\psi})$ to denote the bundle chart where the map $\psi : U \to \mathcal{U}$ is determined by the map $\hat{\psi} : E \to \mathcal{E}$.

Definition 2.7. *If $\hat{C}' = (G', U', E', \hat{\psi}')$ is another bundle chart, a bundle chart embedding from $\hat{C}'$ to $\hat{C}$ consists of an orbifold chart embedding $\iota : U' \hookrightarrow U$ (equivariant with respect to a group injection $G' \hookrightarrow G$) covered by a vector bundle embedding $\hat{\iota} : E' \to E$ such that*

$$\hat{\psi} \circ \hat{\iota} = \hat{\psi}'.$$

We can similarly define the notions of compatibility between bundle charts and bundle atlases. Then an *orbifold vector bundle structure* over $\pi_{\mathcal{E}} : \mathcal{E} \to \mathcal{U}$ is defined to be an equivalence class of bundle atlases as before. Similarly, an orbifold vector bundle has a unique maximal atlas and two atlases are equivalent if and only if they are contained in a common maximal atlas. A bundle chart then means a chart in the maximal atlas.

We spell out the definition of sections of an orbifold vector bundle because of their importance in the discussion of regularizations of moduli spaces.

Definition 2.8 (Sections)**.** *Let $\mathcal{E} \to \mathcal{U}$ be an orbifold vector bundle.*

(1) *Let $\hat{C}_i = (G_i, U_i, E_i, \hat{\psi}_i)$, $i = 1, 2$ be two bundle charts. We say that a G_1-equivariant section $S_1 : U_1 \to E_1$ and a G_2-equivariant section $S_2 : U_2 \to E_2$ are compatible if for any bundle chart $\hat{C}_0 = (G_0, U_0, E_0, \hat{\psi}_0)$ of E and chart embeddings $\hat{\iota}_i : \hat{C}_0 \hookrightarrow \hat{C}_i$, $i = 1, 2$ there holds*
$$\hat{\iota}_1^{-1} \circ S_1 \circ \iota_1 = \hat{\iota}_2^{-1} \circ S_2 \circ \iota_2$$
as sections of $E_0 \to U_0$.

(2) *A section of $\mathcal{E}$, denoted by $\mathcal{S} : \mathcal{U} \to \mathcal{E}$, is a collection of mutually compatible G_i-equivariant sections $S_i : U_i \to E_i$ for all bundle charts $\hat{C}_i$ belonging to the maximal atlas of $\mathcal{E}$.*

On each single chart $\hat{C} = (G, U, E, \hat{\psi})$, there are a lot of G-equivariant sections $S : U \to E$. The existence of partitions of unity implies that any orbifold vector bundle over an effective orbifold has a lot of smooth sections. In particular there is a zero section. Any section $\mathcal{S} : \mathcal{U} \to \mathcal{E}$ induces a continuous map $|\mathcal{S}| : |\mathcal{U}| \to |\mathcal{E}|$ between the coarse spaces.

Now we can combine the above definitions to talk about the orbifold counterpart of global Kuranishi charts.

Definition 2.9. *A derived orbifold (with or without boundary) is a triple $\mathcal{D} = (\mathcal{U}, \mathcal{E}, \mathcal{S})$ where $\mathcal{U}$ is an effective orbifold (with or without boundary), $\mathcal{E} \to \mathcal{U}$ is an orbifold vector bundle, and $\mathcal{S} : \mathcal{U} \to \mathcal{E}$ is a continuous section.*

(1) *$\mathcal{D}$ is called compact if $\mathcal{S}^{-1}(0)$ is compact.*
(2) *An orientation on $\mathcal{D}$ is an orientation on the orbifold real vector bundle $T\mathcal{U} \otimes \mathcal{E}^*$.*
(3) *The virtual dimension of $\mathcal{D}$ is the integer*

$$\dim^{\text{vir}}(\mathcal{D}) = \dim\mathcal{U} - \text{rank}\mathcal{E}.$$

There are a few standard and basic operations on derived orbifolds. First, when $\mathcal{D} = (\mathcal{U}, \mathcal{E}, \mathcal{S})$ has boundary, one can restrict to the boundary

$$\partial\mathcal{D} = (\partial\mathcal{U}, \mathcal{E}|_{\partial\mathcal{U}}, \mathcal{S}|_{\partial\mathcal{U}}).$$

If $\mathcal{D}$ is oriented and normally complex, so is $\partial\mathcal{D}$. On the other hand, one can reverse the orientation on $\mathcal{D}$ while keeping the normal complex structure. The corresponding object is denoted by $-\mathcal{D}$. One can also take the disjoint union $\mathcal{D}_1 \sqcup \mathcal{D}_2$ of two such D-charts.

Definition 2.10. *Let $\mathcal{M}$ be a compact topological space and $\partial\mathcal{M} \subset \mathcal{M}$ is a (possibly empty) closed subset such that the interior $\text{Int}\mathcal{M} := \mathcal{M} \setminus \partial\mathcal{M}$ is dense. A derived orbifold chart (D-chart for short) of $(\mathcal{M}, \partial\mathcal{M})$ consists of a derived orbifold $\mathcal{D} = (\mathcal{U}, \mathcal{E}, \mathcal{S})$ (possibly with boundary) and a homeomorphism $\mathcal{L} : (\mathcal{S}^{-1}(0), \mathcal{S}^{-1}(0) \cap \partial\mathcal{U}) \cong (\mathcal{M}, \partial\mathcal{M})$. When $\partial\mathcal{M} = \emptyset$, we also call $(\mathcal{D}, \mathcal{L})$ a D-chart of $\mathcal{M}$.*

We introduce the following moves, which are the counterparts of the moves we discussed for global Kuranishi charts in the orbifold setting.

Definition 2.11. *Let $\mathcal{M}$ be a compact topological space.*

(1) *A shrinking of a D-chart $(\mathcal{D}, \mathcal{L})$ is a D-chart $(\mathcal{D}', \mathcal{L}')$ where $\mathcal{D}'$ is the shrinking of $\mathcal{D}$ onto an open neighborhood $\mathcal{U}' \subset \mathcal{U}$ of $\mathcal{S}^{-1}(0)$ and $\mathcal{L}' = \mathcal{L}$.*
(2) *A stabilization of a D-chart $(\mathcal{D}, \mathcal{L})$ is a D-chart $(\hat{\mathcal{D}}, \hat{\mathcal{L}})$ where $\hat{\mathcal{D}}$ is the stabilization of $\mathcal{D}$ by a vector bundle $\mathcal{F} \to \mathcal{U}$ (with the zero section $\iota : \mathcal{U} \to \mathcal{F}$) and $\hat{\mathcal{L}} = \mathcal{L} \circ \iota^{-1}$.*
(3) *A cobordism of stably complex D-chart from $(\mathcal{D}_0, \mathcal{L}_0)$ to $(\mathcal{D}_1, \mathcal{L}_1)$ consists of a D-chart $(\tilde{\mathcal{D}}, \tilde{\mathcal{L}}) = ((\tilde{\mathcal{U}}, \tilde{\mathcal{E}}, \tilde{\mathcal{S}}), \tilde{\mathcal{L}})$ of the pair $([0,1] \times \mathcal{M}, \{0,1\} \times \mathcal{M})$, an isomorphism of derived orbifolds*

$$\phi_0 \sqcup \phi_1 : (-\mathcal{D}_0) \sqcup \mathcal{D}_1 \cong \partial\tilde{\mathcal{D}}$$

such that for $i = 0, 1$, the diagram

$$\begin{array}{ccc} \mathcal{S}_i^{-1}(0) & \xrightarrow{\mathcal{L}_i} & \mathcal{M} \\ \downarrow{\scriptstyle \phi_i} & & \downarrow{\scriptstyle \iota_i} \\ \tilde{\mathcal{S}}^{-1}(0) & \xrightarrow[\tilde{\mathcal{L}}]{} & [0,1] \times \mathcal{M} \end{array}$$

commutes. Here $\iota_i : \mathcal{M} \to \{i\} \times \mathcal{M} \to [0,1] \times \mathcal{M}$ is the natural inclusion map. If there exists a cobordism from $(\mathcal{D}_0, \mathcal{L}_0)$ to $(\mathcal{D}_1, \mathcal{L}_1)$, then we say that they are cobordant.

It is straightforward to define (derived) orbifolds with corners, and, more specifically, (derived) orbifolds stratified by a given poset. We leave this as an exercise.

3. Stable complex structures and normal complex structures

The following is the analog of stable complex structures in the context of derived orbifolds.

Definition 3.1. *Let $\mathcal{D} = (\mathcal{U}, \mathcal{E}, \mathcal{S})$ be a derived orbifold.*

(1) *A stable complex strucutre on $\mathcal{E}$ is an equivalence class of quadruples*
$$(k, \mathcal{F}_0, \mathcal{F}_1, \psi)$$
where $k \geq 0$ is an integer, $\mathcal{F}_0, \mathcal{F}_1 \to \mathcal{U}$ are complex vector bundles, and
$$\psi : \underline{\mathbb{R}}^{\oplus k} \oplus \mathcal{E} \oplus \mathcal{F}_0 \cong \mathcal{F}_1$$
is an isomorphism of orbifold vector bundles. The equivalence relation is generated by the following two relations: 1) we require that
$$(k, \mathcal{F}_0, \mathcal{F}_1, \psi) \sim (k + 2, \mathcal{F}_0, \underline{\mathbb{C}} \oplus \mathcal{F}_1, \psi_0 \oplus \psi)$$
where $\psi_0 : \underline{\mathbb{R}}^{\oplus 2} \to \underline{\mathbb{C}}$ is the map $(x, y) \mapsto x + \mathbf{i}y$; 2) we require that
$$(k, \mathcal{F}_0, \mathcal{F}_1, \psi) \sim (k, \mathcal{F}_0 \oplus \mathcal{F}, \mathcal{F}_1 \oplus \mathcal{F}, \psi \oplus \mathrm{Id}_{\mathcal{F}})$$
where $\mathcal{F} \to \mathcal{U}$ is an arbitrary complex vector bundle.

(2) *A stable complex structure on $\mathcal{D}$ consists of a stable complex structure on $T\mathcal{U}$ and a stable complex structure on $\mathcal{E}$.*

Exercise 3.2. *Define the notion of a stably complex cobordism between a pair of stably complex derived orbifolds.*

For a lot of applications, the following notion is much more flexible. We will see that it covers the stable complex structure as a special case.

Let's introduce some notations that will be used all the time.

Definition 3.3. *Suppose W is a finite-dimensional real representation of a finite group G. Then W can be decomposed as the direct sum of irreducible representations. We call the (canonical) decomposition*
$$W = W_G \oplus \check{W}_G$$
where W_G is the direct sum of all trivial summands (i.e. G-fixed points) and $\check{W}_G$ is the direct sum of all nontrivial summands the basic decomposition *of W with respect to G. Notice that when W is a complex representation, the basic decomposition is complex linear.*

More generally, if M is a G-manifold and $E \to M$ is a G-equivariant vector bundle, then over the G-fixed points $M_G \subset M$, the fibrewise basic decomposition of $E|_{M_G}$ induces a decomposition of vector bundles
$$E = E_G \oplus \check{E}_G$$
where $E_G \subset E|_{M_G}$ coincides with the set of G-fixed points of E.

Going back to normal complex structures, we start by considering a single chart.

Definition 3.4. *Let G be a finite group and U be an effective G-manifold.*

(1) *Let $E \to U$ be a G-equivariant vector bundle. A normal complex structure (NC structure for short) on E, denoted by $\boldsymbol{I}^E$ consists, for each U-essential subgroup $H \subset G$, an H-invariant complex structure $I^{\check{E}_H}$ on the normal bundle $\check{E}_H \to U_H$ satisfying the following compatibility condition: for each pair of subgroups $H \subsetneq K$, one has the H-equivariant decomposition*

$$\check{E}_K = (E_H \cap \check{E}_K) \oplus \check{E}_H|_{U_K}.$$

We require that the restriction of $I^{\check{E}_K}$ on the second summand $\check{E}_H|_{U_K}$ coincides with $I^{\check{E}_H}$ restricted to U_K.

(2) *An NC structure on U is an NC structure on the tangent bundle $TU \to U$.*

Definition 3.5. *Let $\mathcal{U}$ be an effective orbifold.*

(1) *An NC structure on an orbifold vector bundle $\mathcal{E} \to \mathcal{U}$, denoted by $\mathcal{I}^{\mathcal{E}}$, consists, for each chart $\hat{C} = (G, U, E, \hat{\psi})$ of $\mathcal{E}$, an NC structure $\boldsymbol{I}^E = (I^{\check{E}_H})$ on E satisfying the following conditions. For each chart embedding from $\hat{C}' = (G', U', E', \hat{\phi}')$ to $\hat{C} = (G, U, E, \hat{\psi})$ given by a group injection $G' \hookrightarrow G$, an equivariant open embedding $\iota : U' \hookrightarrow U$ covered by an equivariant bundle isomorphism $\hat{\iota} : E' \to E$, for any subgroup $H' \subseteq G'$ mapped onto $H \subseteq G$, $\hat{\iota}$ maps $E'|_{U'_{H'}}$ into $E|_{U_H}$, we require that the induced bundle isomorphism*

$$\check{E}'|_{U'_{H'}} \to \check{E}|_{U_H}$$

is complex linear with respect to the complex structures $I^{\check{E}'_{H'}}$ and $I^{\check{E}_H}$.

(2) *An NC structure on $\mathcal{U}$ is an NC structure on the tangent bundle $T\mathcal{U}$.*

As examples, if $\mathcal{U}$ is almost complex, then the almost complex structure induces an NC structure. Similarly, if $\mathcal{E}$ is a complex vector bundle, then there is a naturally induced NC structure on $\mathcal{E}$.

Definition 3.6. *An NC vector bundle over an effective orbifold $\mathcal{U}$ is an orbifold vector bundle $\mathcal{E} \to \mathcal{U}$ together with an NC structure $\mathcal{I}^{\mathcal{E}}$ on $\mathcal{E}$. An NC orbifold is an effective orbifold $\mathcal{U}$ together with an NC structure $\mathcal{I}^{T\mathcal{U}}$ on $\mathcal{U}$.*

For *normally complex* a derived orbifold $(\mathcal{U}, \mathcal{E}, \mathcal{S})$, we mean that $(\mathcal{U}, \mathcal{E})$ is pair where $\mathcal{U}$ is an NC orbifold and $\mathcal{E}$ is an NC vector bundle over $\mathcal{U}$.

Below is another important model of normally complex orbifolds.

Lemma 3.7. *Let G be a finite group. Let X be a smooth manifold and $F \to X$ be a complex vector bundle equipped with a fiberwise effective linear G-action. Then the complex structure I^F on F induces a canonical NC structure on the total space of F.*

Proof. Indeed, for each subgroup $H \subset G$, the fixed point set of H is the total space of $F_H \subset F$ and the normal bundle is $\pi_{F_H}^* \check{F}_H \to F_H$. The restriction of I^F to $\check{F}_H$ is pulled back to an H-invariant complex structure on this normal bundle, which gives an NC structure on F. $\square$

Example 3.8. *In addition, if $E \to X$ is a real vector bundle equipped with a fiberwise linear G-action and a G-invariant complex structure $I^{\check{E}_G}$ on $\check{E}_G$, then the bundle $\pi_F^* E$ carries a canonical NC structure: for each subgroup $H \subset G$, over F_H, the complex structure on the bundle $\pi_F^* \check{E}_H \subset \pi_F^* E$ is the pullback of the restriction of $I^{\check{E}_G}$ to $\check{E}_H$.*

Exercise 3.9. *Let $(\mathcal{U}, \mathcal{E}, \mathcal{S})$ be a normally complex derived orbifold. Let $\pi_{\mathcal{F}} : \mathcal{F} \to \mathcal{U}$ be a complex vector bundle. Then the stabilization of the pair $(\mathcal{U}, \mathcal{E})$, $(\mathcal{F}, \pi_{\mathcal{F}}^* \mathcal{E} \oplus \pi_{\mathcal{F}}^* \mathcal{F})$, carries a naturally induced NC structure.*

Proposition 3.10. *Any stably complex $\mathcal{D} = (\mathcal{U}, \mathcal{E}, \mathcal{S})$ carries a natural normal complex structure.*

Proof. By definition, there exist two quadruples $(k, \mathcal{F}_0, \mathcal{F}_1, \psi)$ and $(l, \mathcal{E}_0, \mathcal{E}_1, \eta)$ where

$$\psi : \underline{\mathbb{R}}^{\oplus k} \oplus T\mathcal{U} \oplus \mathcal{F}_0 \cong \mathcal{F}_1$$

and

$$\eta : \underline{\mathbb{R}}^{\oplus l} \oplus \mathcal{E} \oplus \mathcal{E}_0 \cong \mathcal{E}_1$$

are isomorphisms. By taking equivalent quadruples, i.e,. finding a simultaneous stabilization, we can assume $\mathcal{F}_0 = \mathcal{E}_0$. The above two isomorphisms can be pulled back to the total space of $\mathcal{F}_0$ as

$$\pi_{\mathcal{F}_0}^* \psi : \underline{\mathbb{R}}^{\oplus k} \oplus \pi_{\mathcal{F}_0}^* T\mathcal{U} \oplus \pi_{\mathcal{F}_0}^* \mathcal{F}_0 \cong \pi_{\mathcal{F}_0}^* \mathcal{F}_1$$

and

$$\pi_{\mathcal{F}_0}^* \eta : \underline{\mathbb{R}}^{\oplus l} \oplus \pi_{\mathcal{F}_0}^* \mathcal{E} \oplus \pi_{\mathcal{F}_0}^* \mathcal{F}_0 \cong \pi_{\mathcal{F}_0}^* \mathcal{E}_1.$$

Upon choosing an isomorphism, induced from a connection (an affine space, therefore the choices of the isomorphisms are weakly contractible),

$$\Delta : \pi_{\mathcal{F}_0}^* T\mathcal{U} \oplus \pi_{\mathcal{F}_0}^* \mathcal{F}_0 \cong T\mathcal{F}_0$$

the pullback isomorphism $\pi_{\mathcal{F}_0}^* \psi$ becomes an isomorphism

$$\underline{\mathbb{R}}^{\oplus k} \oplus T\mathcal{F}_0 \cong \pi_{\mathcal{F}_0}^* \mathcal{F}_1.$$

As $\underline{\mathbb{R}}^{\oplus k}$ and $\underline{\mathbb{R}}^{\oplus l}$ do not affect the isotropy type, the above two isomorphism induce a normal complex structure on the stabilization of $\mathcal{D}$ by $\mathcal{F}_0$. □

Exercise 3.11. *Show that for any two pairs of quadruples which give rise to stably isomorphic stable complex structures on $\mathcal{D}$, there exists a common stabilization with compatible normal complex structure.*

4. D-chart lifts of flow categories

The goal is to describe what it means for a flow category or a flow bimodule to admit a coherent system of derived orbifold charts.

Definition 4.1. *A **system of D-chart presentation** of an $\mathbb{A}$-space X consists of the following objects.*

(1) *A collection of derived orbifold presentations*

$$\left\{C_\alpha = (\mathcal{U}_\alpha, \mathcal{E}_\alpha, \mathcal{S}_\alpha, \psi_\alpha)\right\}_{\alpha \in \mathbb{A}}$$

of the collection of boundary strata $\{\partial^\alpha X\}_{\alpha \in \mathbb{A}}$ *as stratified spaces.*

(2) *A collection of chart embeddings*

$$\left\{\iota_{\beta\alpha} : C_\alpha \to \partial^\alpha C_\beta\right\}_{\alpha \leq \beta}.$$

These objects need to satisfy the following conditions.

(A) *The collection of chart embeddings satisfy the cocycle condition. More precisely, for any triple of strata* $\alpha \leq \beta \leq \gamma$*, there holds*

$$\iota_{\gamma\beta} \circ \iota_{\beta\alpha} = \iota_{\gamma\alpha}.$$

(B) *Adjacent strata differ by a stabilization. More precisely, for any pair of strata* $\alpha \leq \beta$*, there exist an orbifold vector bundle* $\mathcal{F}_{\beta\alpha} \to \mathcal{U}_\alpha$ *and a germ equivalence*

$$\mathrm{Stab}_{\mathcal{F}_{\beta\alpha}}(C_\alpha) \simeq \partial^\alpha C_\beta$$

The following definition imposes certain regular structures on morphism spaces of flow categories and flow bimodules.

Definition 4.2. *Let* $T^{\mathcal{P}}$ *be a flow category over the poset* $\mathcal{P}$*. A* ***derived orbifold lift*** *of* $T^{\mathcal{P}}$*, denoted by* $\mathfrak{D}^{\mathcal{P}}$*, consists of the following objects.*

(1) *A collection*

$$\left\{C_{pq} = (\mathcal{U}_{pq}, \mathcal{E}_{pq}, \mathcal{S}_{pq}, \psi_{pq})\right\}_{p \leq q}$$

of derived orbifold presentations of the $\mathbb{A}^{\mathcal{P}}_{pq}$*-space* T_{pq} *such that for each connected component* $\mathcal{U}_{pq,j} \subset \mathcal{U}_{pq}$*, one has*

$$\dim_{\mathbb{R}} \mathcal{U}_{pq,j} - \mathrm{rank}_{\mathbb{R}} \mathcal{E}_{pq}|_{\mathcal{U}_{pq,j}} \equiv \mathrm{ind}^{\mathcal{P}}(p) - \mathrm{ind}^{\mathcal{P}}(q) - 1 \bmod 2N. \tag{4.1}$$

(2) *A collection of chart embeddings*

$$\left\{\boldsymbol{\iota}_{prq} : C_{pr} \times C_{rq} \hookrightarrow \partial^{prq} C_{pq}\right\}_{p \leq r \leq q}$$

(with the underlying poset identification $\mathbb{A}^{\mathcal{P}}_{pr} \times \mathbb{A}^{\mathcal{P}}_{rq} \cong \partial^{prq} \mathbb{A}^{\mathcal{P}}_{pq}$*). In particular, if* $\iota_{prq} : \mathcal{U}_{pr} \times \mathcal{U}_{rq} \hookrightarrow \mathcal{U}_{pq}$ *is the associated domain embedding and* $\widehat{\iota}_{prq} : \mathcal{E}_{pr} \boxplus \mathcal{E}_{rq} \hookrightarrow \mathcal{E}_{pq}$ *is the associated bundle embedding, then the following diagram commutes.*

$$\begin{array}{ccc} \mathcal{E}_{pr} \boxplus \mathcal{E}_{rq} & \xrightarrow{\widehat{\iota}_{prq}} & \mathcal{E}_{pq} \\ \mathcal{S}_{pr} \times \mathcal{S}_{rq} \uparrow\downarrow & & \downarrow\uparrow \mathcal{S}_{pq} \\ \mathcal{U}_{pr} \times \mathcal{U}_{rq} & \xrightarrow[\iota_{prq}]{} & \mathcal{U}_{pq} \end{array} \tag{4.2}$$

These objects need to satisfy the following conditions.

(A) *For* $p = q$*, the space* $\mathcal{U}_{pp}$ *is a singleton with trivial isotropy and* $\mathcal{E}_{pp} = \{0\}$ *is the trivial bundle.*

(B) *The chart embeddings satisfy the associativity. More precisely, whenever* $p < r < s < q$*, the following diagram commutes.*

$$\begin{array}{ccccc} & & C_{pr} \times C_{rs} \times C_{rq} & & \\ & \swarrow^{\boldsymbol{\iota}_{prs}\times \mathrm{id}} & & \searrow^{\mathrm{id}\times\boldsymbol{\iota}_{rsq}} & \\ \partial^{prs} C_{ps} \times C_{rq} & & & & C_{pr} \times \partial^{rsq} C_{rq} \\ & \searrow_{\boldsymbol{\iota}_{psq}} & & \swarrow_{\boldsymbol{\iota}_{prq}} & \\ & & \partial^{prsq} C_{pq} & & \end{array}$$

(C) *For each* pq *and all* $\alpha = pr_1 \cdots r_l q \in \mathbb{A}^{\mathcal{P}}_{pq}$*, define*

$$C_\alpha := C_{pr_1} \times \cdots \times C_{r_l q}.$$

Then condition (B) implies that for each pair of elements $\alpha \leq \beta$ *in* $\mathbb{A}^{\mathcal{P}}_{pq}$*, there is a well-defined chart embedding*

$$\boldsymbol{\iota}_{\beta\alpha} : C_\alpha \hookrightarrow \partial^\alpha C_\beta.$$

Then we require that the collection $\{C_\alpha\}_{\alpha\in\mathbb{A}^{\mathcal{P}}_{pq}}$ *of derived orbifold presentations and the collection* $\{\boldsymbol{\iota}_{\beta\alpha}\}_{\alpha\leq\beta}$ *of chart embeddings constitute a system derived orbifold presentations of* T_{pq}*.*

(D) *The strict* Π*-equivariance condition: for any* $a \in \Pi$ *and* $p < q$*, there is an isomorphism between derived orbifold charts (in the obvious sense)* $\tilde{\phi}_a : C_{pq} \to C_{a\cdot p\ a\cdot q}$ *satisfying* $\tilde{\phi}_{a_1\cdot a_2} = \tilde{\phi}_{a_1} \circ \tilde{\phi}_{a_2}$*, and* $\tilde{\phi}_a$ *restricts to* ϕ_a *along the zero locus* $\mathcal{S}_{pq}^{-1}(0)$ *to the map* ϕ_a *from Definition 1.5. Furthermore,* $\tilde{\phi}_0 = \mathrm{Id}$ *for* $a = 0 \in \Pi$ *should be the identity map.*

Now consider derived orbifold lifts of bimodules.

Definition 4.3. *Let* M *be a flow bimodule from a flow category* $T^{\mathcal{P}}$ *to* $T^{\mathcal{P}'}$ *as in Definition 1.9. Suppose* $T^{\mathcal{P}}$ *resp.* $T^{\mathcal{P}'}$ *is endowed with a derived orbifold lift*

$$\mathfrak{D}^{\mathcal{P}} = \Big(\big\{C^{\mathcal{P}}_{pq} = (\mathcal{U}^{\mathcal{P}}_{pq}, \mathcal{E}^{\mathcal{P}}_{pq}, \mathcal{S}^{\mathcal{P}}_{pq}, \psi^{\mathcal{P}}_{pq})\big\}_{p<q}, \big\{\boldsymbol{\iota}^{\mathcal{P}}_{\beta\alpha}\big\}_{\alpha\leq\beta}\Big) \text{ resp.}$$

$$\mathfrak{D}^{\mathcal{P}'} = \Big(\big\{C^{\mathcal{P}'}_{p'q'} = (\mathcal{U}^{\mathcal{P}'}_{p'q'}, \mathcal{E}^{\mathcal{P}'}_{p'q'}, \mathcal{S}^{\mathcal{P}'}_{p'q'}, \psi^{\mathcal{P}'}_{p'q'})\big\}_{p'<q'}, \big\{\boldsymbol{\iota}^{\mathcal{P}'}_{\beta'\alpha'}\big\}_{\alpha'\leq\beta'}\Big).$$

A ***derived orbifold lift*** *of* M ***compatible with (or extending)*** $\mathfrak{D}^{\mathcal{P}}$ *and* $\mathfrak{D}^{\mathcal{P}'}$ *consists of the following objects.*

(1) *A collection*

$$\big\{C^M_{pp'} = (\mathcal{U}^M_{pp'}, \mathcal{E}^M_{pp'}, \mathcal{S}^M_{pp'}, \psi^M_{pp'})\big\}_{p\in\mathcal{P}, p'\in\mathcal{P}'}$$

of derived orbifold presentations of $M_{pp'}$ *for* $M_{pp'}$ *as an* $\mathbb{A}_{pp'}$*-space such that for each connected component* $\mathcal{U}^M_{pp',j} \subset \mathcal{U}^M_{pp'}$*, one has*

$$\dim_{\mathbb{R}} \mathcal{U}^M_{pp',j} - \mathrm{rank}_{\mathbb{R}} \mathcal{E}^M_{pp'}|_{\mathcal{U}^M_{pp',j}} \equiv \mathrm{ind}^{\mathcal{P}}(p) - \mathrm{ind}^{\mathcal{P}'}(p') \bmod 2N. \tag{4.3}$$

(2) *Given $p \in \mathcal{P}$ and $p' \in \mathcal{P}'$ for $M_{pp'} \neq \emptyset$, for any $p \le q$, a chart embedding*

$$\boldsymbol{\iota}^M_{pqp'} : C^{\mathcal{P}}_{pq} \times C^M_{qp'} \hookrightarrow \partial^{pqp'} C^M_{pp'},$$

and for any $q' \le p'$, a chart embedding

$$\boldsymbol{\iota}^M_{pq'p'} : C^M_{pq'} \times C^{\mathcal{P}'}_{q'p'} \hookrightarrow \partial^{pq'p'} C^M_{pp'}.$$

The precise meaning can be spelled out as in Equation (4.2).

These objects are required to satisfy the following conditions.

(A) *For $p = q$, the chart embedding $\boldsymbol{\iota}^M_{pqp'}$ is the identity map after identifying $C^{\mathcal{P}}_{pq}$ with the trivial chart for the singleton. Similarly, for $q' = p'$, the chart embedding $\boldsymbol{\iota}^M_{pq'p'}$ is also the identity map.*

(B) *The chart embeddings satisfy the associativity. Namely, given $M_{pp'} \neq \emptyset$, the following three diagrams commute if the relevant topological spaces are nonempty.*

For $p < q_1 < q_2$, we have

$$\begin{array}{ccc} & C^{\mathcal{P}}_{pq_1} \times C^{\mathcal{P}}_{q_1q_2} \times C^M_{q_2p'} & \\ {\scriptstyle \boldsymbol{\iota}^{\mathcal{P}}_{pq_1q_2}\times \mathrm{id}}\swarrow & & \searrow{\scriptstyle \mathrm{id}\times\boldsymbol{\iota}^M_{q_1q_2p'}} \\ \partial^{pq_1q_2} C^{\mathcal{P}}_{pq_2} \times C^M_{q_2p'} & & C^{\mathcal{P}}_{pq_1} \times \partial^{q_1q_2p'} C^M_{q_1p'} \\ {\scriptstyle \boldsymbol{\iota}^M_{pq_2p'}}\searrow & & \swarrow{\scriptstyle \boldsymbol{\iota}^M_{pq_1p'}} \\ & \partial^{pq_1q_2p'} C^M_{pp'}. & \end{array}$$

For $q_2' < q_1' < p'$, we have

$$\begin{array}{ccc} & C^M_{pq_2'} \times C^{\mathcal{P}'}_{q_2'q_1'} \times C^{\mathcal{P}'}_{q_1'p'} & \\ {\scriptstyle \boldsymbol{\iota}^M_{pq_2'q_1'}\times \mathrm{id}}\swarrow & & \searrow{\scriptstyle \mathrm{id}\times\boldsymbol{\iota}^{\mathcal{P}'}_{q_1'q_2'p'}} \\ \partial^{pq_2'q_1'} C^M_{pq_1'} \times C^{\mathcal{P}'}_{q_1'p'} & & C^M_{pq_2'} \times \partial^{q_2'q_1'p'} C^{\mathcal{P}'}_{q_2'p'} \\ {\scriptstyle \boldsymbol{\iota}^M_{pq_1'p'}}\searrow & & \swarrow{\scriptstyle \boldsymbol{\iota}^M_{pq_2'p'}} \\ & \partial^{pq_2'q_1'p'} C^M_{pp'}. & \end{array}$$

Finally, for $p < q$ and $q' < p'$, we have

$$\begin{array}{ccc} & C^{\mathcal{P}}_{pq} \times C^M_{qq'} \times C^{\mathcal{P}'}_{q'p'} & \\ {\scriptstyle \boldsymbol{\iota}^M_{pqq'}\times \mathrm{id}}\swarrow & & \searrow{\scriptstyle \mathrm{id}\times\boldsymbol{\iota}^M_{qq'p'}} \\ \partial^{pqq'} C^M_{pq'} \times C^{\mathcal{P}'}_{q'p'} & & C^{\mathcal{P}}_{pq} \times \partial^{qq'p'} C^M_{qp'} \\ {\scriptstyle \boldsymbol{\iota}^M_{pq'p'}}\searrow & & \swarrow{\scriptstyle \boldsymbol{\iota}^M_{pqp'}} \\ & \partial^{pqq'p'} C^M_{pp'}. & \end{array}$$

(C) *For each p, p' and all $\alpha = pr_1 \cdots r_l r'_l \cdots r'_1 p'$, define*

$$C^M_\alpha = C^{\mathcal{P}}_{pr_1} \times \cdots \times C^M_{r_l r'_l} \times \cdots \times C^{\mathcal{P}'}_{r'_1 p'}.$$

As in Definition 4.2, for each pair $\alpha \le \beta$ in $\mathbb{A}_{pp'}$, there is a well-defined chart embedding

$$\boldsymbol{\iota}^M_{\beta\alpha} : C^M_\alpha \hookrightarrow C^M_\beta.$$

It is required that the derived orbifold presentations $\{C^M_\alpha\}_{\alpha \in \mathbb{A}_{pp'}}$ and the collection of chart embeddings $\{\boldsymbol{\iota}_{\beta\alpha}\}_{\alpha \le \beta}$ constitute a system of derived orbifold presentations of $M_{pp'}$.

(D) *The strict Π-equivariance condition: for any $a \in \Pi$ and $M_{pp'} \neq \emptyset$, there is an isomorphism between derived orbifold charts $\tilde{\phi}^M_a : C^M_{pp'} \to C^M_{a\cdot p\ a\cdot p'}$ satisfying $\tilde{\phi}^M_{a_1 \cdot a_2} = \tilde{\phi}^M_{a_1} \circ \tilde{\phi}^M_{a_2}$, and $\tilde{\phi}^M_a$ restricts to the map ϕ^M_a from Definition 1.9 along the zero locus $(\mathcal{S}^M_{pp'})^{-1}(0)$. Moreover, $\tilde{\phi}^M_{id}$ for $a = id \in \Pi$ is the identity map. Moreover, the left and right actions of the charts from $T^{\mathcal{P}}$ and $T^{\mathcal{P}'}$ on the charts of M should be Π-equivariant.*

We introduce the following notion to rigidify the stabilization bundle required for the discussion of compatibility of derived orbifold charts.

Definition 4.4. *A* **scaffolding** *of a system of D-chart presentations $((C_\alpha)_{\alpha\in\mathbb{A}}, (\boldsymbol{\iota}_{\beta\alpha})_{\alpha\le\beta})$ of an $\mathbb{A}$-space X is a collection of data*

$$\left(\mathcal{F}_{\beta\alpha}, \boldsymbol{\theta}_{\beta\alpha}\right)_{\alpha\le\beta}$$

where for each pair $\alpha \le \beta$

(1) *the* **difference bundle** *$\mathcal{F}_{\beta\alpha} \to \mathcal{U}_\alpha$ is an orbifold vector bundle $\mathcal{F}_{\beta\alpha} \to \mathcal{U}_\alpha$. In notation, when $\mathbb{A} = \mathbb{A}^{\mathcal{P}}_{pq}$ resp. $\mathbb{A}_{pp'}$ and β is the maximal element pq resp. pp', denote $\mathcal{F}_{\beta\alpha}$ by $\mathcal{F}_{pq,\alpha}$ resp. $\mathcal{F}_{pp',\alpha}$.*

(2) *the* **stabilization map** *$\boldsymbol{\theta}_{\beta\alpha}$ is a germ equivalence*

$$\boldsymbol{\theta}_{\beta\alpha} = (\theta_{\beta\alpha}, \widehat{\theta}_{\beta\alpha}) : \mathrm{Stab}_{\mathcal{F}_{\beta\alpha}}(C_\alpha) \simeq \partial^\alpha C_\beta \tag{4.4}$$

which extends the chart embedding $\boldsymbol{\iota}_{\beta\alpha} : C_\alpha \hookrightarrow \partial^\alpha C_\beta$. This germ equivalence induces a projection map

$$\pi_{\beta\alpha} : \partial^\alpha C_\beta \to C_\alpha$$

as well as a bundle splitting

$$\mathcal{E}_\beta|_{\mathcal{U}_\alpha} = \mathcal{E}_\alpha \oplus \mathcal{F}_{\beta\alpha} \tag{4.5}$$

where $\mathcal{E}_\beta|_{\mathcal{U}_\alpha} = \boldsymbol{\iota}^*_{\beta\alpha}\mathcal{E}_\beta$ and a bundle isomorphism

$$\vartheta_{\beta\alpha} : \pi^*_{\beta\alpha}(\mathcal{E}_\beta|_{\mathcal{U}_\alpha}) \cong \mathcal{E}_\beta|_{\partial^\alpha\mathcal{U}_\beta}. \tag{4.6}$$

These objects need to satisfy the following conditions.

(A) *For any triple of stratum $\alpha \le \beta \le \gamma$, as subbundles of $\mathcal{E}_\gamma|_{\mathcal{U}_\alpha}$ there holds*

$$\mathcal{F}_{\gamma\alpha} = \mathcal{F}_{\beta\alpha} \oplus \mathcal{F}_{\gamma\beta}|_{\mathcal{U}_\alpha}. \tag{4.7}$$

(B) *The stabilization map preserves stratum. More precisely, the following diagram commutes*

$$\begin{array}{ccc} \mathrm{Stab}_{\mathcal{F}_{\beta\alpha}}(C_\alpha) & \xrightarrow{\boldsymbol{\theta}_{\beta\alpha}} & \partial^\alpha C_\beta \\ \big\downarrow & & \big\downarrow \boldsymbol{\iota}_{\gamma\beta} \\ \mathrm{Stab}_{\mathcal{F}_{\gamma\alpha}}(C_\alpha) & \xrightarrow[\boldsymbol{\theta}_{\gamma\alpha}]{} & \partial^\alpha C_\gamma \end{array} .$$

It follows that

$$\pi_{\gamma\alpha} \circ (\iota_{\gamma\beta}|_{\partial^\alpha U_\beta}) = \pi_{\beta\alpha}. \tag{4.8}$$

(C) *The bundle isomorphism* (4.6) *preserves stratum. More precisely, for any triple of strata $\alpha \le \beta \le \gamma$, consider the following diagram*

$$\begin{array}{ccc} \pi^*_{\gamma\alpha}\left(\mathcal{E}_\gamma|_{\mathcal{U}_\alpha}\right)|_{\partial^\alpha \mathcal{U}_\beta} & \xrightarrow{\vartheta_{\gamma\alpha}|_{\partial^\alpha \mathcal{U}_\beta}} & \mathcal{E}_\gamma|_{\partial^\alpha \mathcal{U}_\beta} \\ \Big\| & & \Big\| \\ \pi^*_{\beta\alpha}\left(\mathcal{E}_\gamma|_{\mathcal{U}_\alpha}\right) & & \\ \Big\| & & \\ \pi^*_{\beta\alpha}\left(\mathcal{E}_\beta|_{\mathcal{U}_\alpha} \oplus \mathcal{F}_{\gamma\beta}|_{\mathcal{U}_\alpha}\right) & \xrightarrow[\vartheta_{\beta\alpha}\oplus\vartheta_{\gamma\beta\alpha}]{} & \mathcal{E}_\beta|_{\partial^\alpha \mathcal{U}_\beta} \oplus \mathcal{F}_{\gamma\beta}|_{\partial^\alpha \mathcal{U}_\beta} \end{array} .$$

We explain the notations here. The first vertical equal arrow on the left is due to (4.8) *and the second one is due to* (4.5)*. Here the requirement is that, we require*

$$\vartheta_{\gamma\alpha}\left(\pi^*_{\beta\alpha}(\mathcal{F}_{\gamma\beta}|_{\mathcal{U}_\alpha})\right) = \mathcal{F}_{\gamma\beta}|_{\partial^\alpha \mathcal{U}_\beta}$$

and the restriction to $\vartheta_{\gamma\alpha}$ to $\pi^*_{\beta\alpha}(\mathcal{F}_{\gamma\beta}|_{\mathcal{U}_\alpha})$ is equal to a linear isomorphism $\vartheta_{\gamma\beta\alpha}$ (which is in the above commutative diagram). And we require that the above diagram commutes.

(D) *The stabilization maps satisfy the cocycle condition. Namely, for each triple of strata $\alpha \le \beta \le \gamma$, the following diagram commutes.*

$$\begin{array}{ccc} \mathrm{Stab}_{\mathcal{F}_{\gamma\alpha}}(C_\alpha) & = & \mathrm{Stab}_{\mathcal{F}_{\gamma\beta}|_{\mathcal{U}_\alpha}\oplus\mathcal{F}_{\beta\alpha}}(C_\alpha) \\ & & \big\downarrow \boldsymbol{\theta}_{\beta\alpha} \\ \boldsymbol{\theta}_{\gamma\alpha} \big\downarrow & & \mathrm{Stab}_{\pi^*_{\beta\alpha}(\mathcal{F}_{\gamma\beta}|_{\mathcal{U}_\alpha})}(\partial^\alpha C_\beta) \\ & & \big\downarrow \vartheta_{\gamma\alpha} \\ \partial^\alpha C_\gamma & \xleftarrow[\boldsymbol{\theta}_{\gamma\beta}]{} & \mathrm{Stab}_{\mathcal{F}_{\gamma\beta}}(\partial^\alpha C_\beta) \end{array} . \tag{4.9}$$

Here the "=" arrow is induced from the identity $\mathcal{F}_{\gamma\alpha} = \mathcal{F}_{\gamma\beta}|_{\mathcal{U}_\alpha} \oplus \mathcal{F}_{\beta\alpha}$.

Now consider a derived orbifold lift of a flow category.

Definition 4.5. *A* **scaffolding** *of a derived orbifold lift of a flow category $T^{\mathcal{P}}$ consists of a collection of scaffoldings for the induced system of derived orbifold chart presentations of T_{pq}*

$$\left(\left(\mathcal{F}_{\beta\alpha}, \boldsymbol{\theta}_{\beta\alpha}\right)_{\alpha\le\beta}\right)_{p<q}$$

satisfying

(1) *Suppose* $pr_1 \cdots r_l q = \alpha \leq \beta = ps_1 \cdots s_m q$. *Denote* $\alpha_0 = pr_1 \cdots s_1, \ldots, \alpha_m = s_m \cdots r_l q$. *Then as subbundles of* $\mathcal{E}_\beta|_{\mathcal{U}_\alpha}$, *in view of the identification* $\mathcal{U}_\alpha = \mathcal{U}_{\alpha_0} \times \cdots \times \mathcal{U}_{\alpha_m}$, *one has*

$$\mathcal{F}_{\beta\alpha} = \mathcal{F}_{ps_1,\alpha_0} \boxplus \cdots \boxplus \mathcal{F}_{s_m q,\alpha_m}.$$

(2) *With respect to the last identity, one has (as germs of maps)*

$$\boldsymbol{\theta}_{\beta\alpha} = \boldsymbol{\theta}_{ps_1,\alpha_0} \times \cdots \times \boldsymbol{\theta}_{s_m q,\alpha_m}.$$

Definition 4.6. *Given two flow categories* $T^{\mathcal{P}}$ *and* $T^{\mathcal{P}'}$ *endowed with derived orbifold lifts* $\mathfrak{D}^{\mathcal{P}}$, $\mathfrak{D}^{\mathcal{P}'}$ *respectively, let* M *be a flow bimodule from* $T^{\mathcal{P}}$ *to* $T^{\mathcal{P}'}$ *endowed with a compatible derived orbifold lift* $\mathfrak{D}^M$. *Suppose* $\mathfrak{D}^{\mathcal{P}}$ *and* $\mathfrak{D}^{\mathcal{P}'}$ *come with scaffoldings*

$$\left(\{\mathcal{F}^{\mathcal{P}}_{\beta\alpha}, \boldsymbol{\theta}^{\mathcal{P}}_{\beta\alpha}\}_{\alpha\leq\beta}\right)_{p<q}, \ \left(\{\mathcal{F}^{\mathcal{P}'}_{\beta'\alpha'}, \boldsymbol{\theta}^{\mathcal{P}'}_{\beta'\alpha'}\}_{\alpha'\leq\beta'}\right)_{p'<q'}.$$

A ***scaffolding*** *of such a derived orbifold lift* ***compatible with*** *the given scaffoldings is given by a collection of scaffoldings for the induced system of derived orbifold chart presentations*

$$\left(\{\mathcal{F}_{\tilde\beta\tilde\alpha}, \boldsymbol{\theta}_{\tilde\beta\tilde\alpha}\}_{\tilde\alpha\leq\tilde\beta}\right)_{\tilde\alpha,\tilde\beta\in\mathbb{A}^M_{pp'}}$$

satisfying similar conditions as in Definition 4.5 using the factorization of the boundary strata (see Definition 4.3 (C)).

5. Stable complex structures on flow categories

To make sense of complex orientations on a derived orbifold lift of $T^{\mathcal{P}}$, it requires the following addtional data. We omit the strict Π-equivariance for simplicity.

First, for each object $p \in \mathcal{P}$, we associate it with a virtual vector space $V_p = (V_p^+, V_p^-)$. We can additionally require that V_p^- is a complex vector space.

Second, we can discuss the meaning of a "relative" complex structure on a derived orbifold chart $C_{pq} = (\mathcal{U}_{pq}, \mathcal{E}_{pq}, \mathcal{S}_{pq}, \psi_{pq})$ of the morphism space $T^{\mathcal{P}}_{pq}$. It is the data of a complex orbifold vector bundle $I^{\mathbb{C}}_{pq} \to \mathcal{U}_{pq}$, an orbifold vector bundle $W_{pq} \to \mathcal{U}_{pq}$, together with an isomorphism of vector bundles

$$T\mathcal{U}_{pq} \oplus \underline{V}_q^+ \oplus \underline{V}_p^- \oplus \underline{\mathbb{R}} \oplus W_{pq} \cong \mathcal{E}_{pq} \oplus I^{\mathbb{C}}_{pq} \oplus \underline{V}_q^- \oplus \underline{V}_p^+ \oplus W_{pq},$$

which can be more compactly written as

$$T\mathcal{U}_{pq} \ominus \mathcal{E}_{pq} \oplus \underline{V}_q \oplus \underline{\mathbb{R}} \oplus W_{pq} \cong I^{\mathbb{C}}_{pq} \oplus \underline{V}_p \oplus W_{pq}.$$

To be consistent with our convention, we ask W_{pq} is a direct sum of a real vector bundle pulled back from the coarse space $|\mathcal{U}_{pq}|$ and a complex orbifold vector bundle over $\mathcal{U}_{pq}$.

Third, we need an associativity relation. Note that for a triple $p, r, q \in \mathcal{P}$, we know that ∂C_{pq} defines a stabilization of $C_{pr} \times C_{rq}$ via the difference bundle $\mathcal{F}_{pq,prq}$. We ask $\mathcal{F}_{pq,prq}$ to be a complex vector bundle. Then the associativity relation is expressed as the commutativity of equivalences

$$
\begin{array}{ccc}
\begin{array}{c} T\mathcal{U}_{pr} \ominus \mathcal{E}_{pr} \oplus \underline{V}_r \oplus \underline{\mathbb{R}} \oplus W_{pr} \\ T\mathcal{U}_{rq} \ominus \mathcal{E}_{rq} \oplus \underline{V}_q \oplus \underline{\mathbb{R}} \oplus W_{rq} \end{array} & \longrightarrow & \begin{array}{c} I^{\mathbb{C}}_{pr} \oplus \underline{V}_p \oplus W_{pr} \\ I^{\mathbb{C}}_{rq} \oplus \underline{V}_r \oplus W_{rq} \end{array} \\
\Big\downarrow & & \Big\downarrow \\
T\mathcal{U}_{pq} \ominus \mathcal{E}_{pq} \oplus \underline{V}_q \oplus \underline{\mathbb{R}} \oplus W_{pq} & \longrightarrow & I^{\mathbb{C}}_{pq} \oplus \underline{V}_p \oplus W_{pq}
\end{array}
$$

The $\mathbb{R}$ denotes the translation direction one needs to mod out.

Exercise 5.1. *Define the notation of normal complex structures on a derived orbifold lift and show that stable complex structures give rise to normal complex structures.*

References

[MP97] I. Moerdijk and D. A. Pronk, *Orbifolds, sheaves and groupoids*, K-Theory **12** (1997), no. 1, 3–21.