Topic 8: Fukaya–Ono–Parker perturbations

1. Overview

Outline of the lecture.

(1) Applications that require counts beyond rational numbers: Arnold conjecture (discuss the consequence of the h-cobordism theorem), results on the Hofer–Zehnder conjecture, structural aspects of quantum connections.
(2) Examples of obtaining integrality: Donaldson–Uhlenbeck type compactification in the monotone setting via index counting, Hofer–Salamon's integral Floer theory in the Calabi–Yau setting via transversality.

Then introduce the FOP perturbation scheme as a black box following the exposition in [BPX].

To carry out the Fukaya–Ono–Parker (FOP) perturbation scheme, one needs a version of complex structures on orbifolds and the obstruction bundles. A **normal complex structure** on an effective orbifold $\mathcal{U}$ associates to each orbifold chart (U, Γ) and each subgroup $H \subset \Gamma$ an H-invariant complex structure on the normal bundle $NU^H \to U^H$ of the H-fixed point set $U^H \subset U$ such that orbifold coordinate changes respect these complex structures. If $\mathcal{E} \to \mathcal{U}$ is an orbifold vector bundle, a normal complex structure on $\mathcal{E}$ associates to each bundle chart (U, E, Γ) (where (U, Γ) is an orbifold chart and $E \to U$ is a Γ-equivariant vector bundle) and each subgroup $H \subset \Gamma$ an H-invariant complex structure on the subbundle $\check{E}^H \subset E|_{U^H}$ (which is the direct sum of nontrivial irreducible H-representations contained in $E|_{U^H}$) such that the bundle coordinate changes respect these complex structures. A normal complex structure on a D-chart $\mathcal{C} = (\mathcal{U}, \mathcal{E}, \mathcal{S})$ consists of a normal complex structure on $\mathcal{U}$ and a normal complex structure on $\mathcal{E}$. The notion of derived cobordism can be defined for normally complex D-charts if we require that stabilizations are via complex vector bundles.

Theorem 1.1. [BX22] *Given any normally complex and effective orbifold $\mathcal{U}$ and a normally complex orbifold vector bundle $\mathcal{E} \to \mathcal{U}$, there is a class of (single-valued) smooth sections, called* **FOP transverse sections** *which satisfy the following conditions.*

(1) *The FOP transversality condition is local; moreover, over the isotropy-free part of $\mathcal{U}$, $\mathcal{U}_{\mathrm{free}} \subset \mathcal{U}$ (which is a manifold), being FOP transverse is equivalent to being transverse in the classical sense.*
(2) *The condition that a smooth section $\mathcal{S}$ is FOP transverse only depends on the behavior of $\mathcal{S}$ near $\mathcal{S}^{-1}(0)$. In particular, any smooth section $\mathcal{S}$ is FOP transverse away from $\mathcal{S}^{-1}(0)$.*

(3) *Given a continuous norm on $\mathcal{E}$, for any continuous section $\mathcal{S}_0 : \mathcal{U} \to \mathcal{E}$ and any $\delta > 0$, there exists an FOP transverse section $\mathcal{S} : \mathcal{U} \to \mathcal{E}$ such that*

$$\|\mathcal{S}_0 - \mathcal{S}\|_{C^0} \leq \delta.$$

(4) **(CUDV property)** *Given any closed subset $C \subset \mathcal{U}$ and a smooth section $\mathcal{S}_0 : O \to \mathcal{E}$ defined over an open neighborhood O of C, if $\mathcal{S}_0$ is FOP transverse near C, then there exists an FOP transverse section $\mathcal{S} : \mathcal{U} \to \mathcal{E}$ which agrees with $\mathcal{S}_0$ near C.*

(5) **(Product Property.)** *Let $\mathcal{U}'$ be another normally complex orbifold and $\mathcal{E}' \to \mathcal{U}'$ be a normally complex vector bundle. Then the product map*

$$\Gamma(\mathcal{U}, \mathcal{E}) \times \Gamma(\mathcal{U}', \mathcal{E}') \to \Gamma(\mathcal{U} \times \mathcal{U}', \mathcal{E} \boxplus \mathcal{E}')$$

sends products of FOP transverse sections to FOP transverse sections.

(6) **(Stabilization property)** *Suppose $\pi_{\mathcal{F}} : \mathcal{F} \to \mathcal{U}$ is an orbifold complex vector bundle and $\tau_{\mathcal{F}} : \mathcal{F} \to \pi_{\mathcal{F}}^* \mathcal{F}$ is the tautological section. If $\mathcal{S} : \mathcal{U} \to \mathcal{E}$ is an FOP transverse section, then the section*

$$\pi_{\mathcal{F}}^* \mathcal{S} \oplus \tau_{\mathcal{F}} : \mathcal{F} \to \pi_{\mathcal{F}}^* \mathcal{E} \oplus \pi_{\mathcal{F}}^* \mathcal{F}$$

is a also an FOP transverse section.

(7) *If $\mathcal{Z} \subset \mathcal{U}$ is a closed and proper sub-orbifold whose normal bundle is an ordinary vector bundle (i.e., the fibers as representations of stabilizers are a direct sum of trivial representations, which implies $\mathcal{Z}$ is also normally complex), and if $\mathcal{S} : \mathcal{Z} \to \mathcal{E}$ is an FOP transverse section, then there exists an FOP transverse extension of $\mathcal{S}$ to $\mathcal{U}$.*

(8) *For any FOP transverse section $\mathcal{S} : \mathcal{U} \to \mathcal{E}$, the isotropy-free part of the zero locus*

$$\mathcal{S}^{-1}(0)_{\text{free}} := \mathcal{S}^{-1}(0) \cap \mathcal{U}_{\text{free}}$$

is a transverse intersection in the classical sense. Moreover, its boundary

$$\overline{\mathcal{S}^{-1}(0)_{\text{free}}} \setminus \mathcal{S}^{-1}(0)_{\text{free}} \subset \mathcal{U} \setminus \mathcal{U}_{\text{free}}$$

is the union of images of smooth maps from manifolds of dimension at most $\dim \mathcal{U} - \operatorname{rank} \mathcal{E} - 2$.

Now if we are given a normally complex effective derived orbifold chart $(\mathcal{U}, \mathcal{E}, \mathcal{S})$, we can choose an FOP transverse section $\mathcal{S}'$ which agrees with $\mathcal{S}$ outside a compact neighborhood of $\mathcal{S}^{-1}(0)$. Then $(\mathcal{S}')^{-1}(0)$ is compact and the isotropy-free part

$$(\mathcal{S}')^{-1}(0)_{\text{free}}$$

is a pseudocycle [1] of dimension $\dim \mathcal{U} - \operatorname{rank} \mathcal{E}$. Hence if $\mathcal{U}$ and $\mathcal{E}$ are oriented, this pseudocycle represents an integral homology class in $\mathcal{U}$. Moreover, given any two FOP transverse perturbations $\mathcal{S}_1', \mathcal{S}_2'$, the two pseudocycles $(S_1')^{-1}(0)_{\text{free}}$ and $(S_2')^{-1}(0)_{\text{free}}$ are cobordant. Therefore the integral homology class, which we call the FOP Euler class

$$\chi^{\text{FOP}}(\mathcal{C}) \in H_*(\mathcal{U}; \mathbb{Z}),$$

is well-defined.

[1] The notion of pseudocyle in orbifolds is defined in [BX22].

One typically needs to push forward the homology class into another space. To connect with classical cobordism theory, we restrict to the case when the derived orbifold charts are *stably complex*, meaning that the virtual vector bundle $T\mathcal{U} - \mathcal{E}$ has a stable complex structure (see [BX22, Definition 6.11, 6.14]). In particular, if $T\mathcal{U}$ and $\mathcal{E}$ are both complex vector bundles, then $\mathcal{C} = (\mathcal{U}, \mathcal{E}, \mathcal{S})$ is automatically stably complex and normally complex. In the special case when $\mathcal{U}$ is a manifold and $\mathcal{S} : \mathcal{U} \to \mathcal{E}$ is transverse, then a stable complex structure on $(\mathcal{U}, \mathcal{E}, \mathcal{S})$ induces a stable complex structure on the manifold $\mathcal{S}^{-1}(0)$. On the other hand, if a derived orbifold chart $\mathcal{C}$ is stably complex, then via a stabilization, it is equivalent to a normally complex one.

For any topological space X, the stably complex derived orbifold bordism group

$$\Omega_k^{\mathrm{der},\mathbb{C}}(X)$$

is the abelian group generated by isomorphism classes of quadruples $(\mathcal{U}, \mathcal{E}, \mathcal{S}, f)$ where $(\mathcal{U}, \mathcal{E}, \mathcal{S})$ is an stably complex D-chart of virtual dimension $\dim\mathcal{U} - \mathrm{rank}\mathcal{E} = k$ and $f : \mathcal{U} \to X$ is a continuous map, modulo the equivalence relation induced from stabilization by complex vector bundles and cobordism respecting the stable complex structures. The assignment $\Omega_*^{\mathrm{der},\mathbb{C}}(-)$ actually defines a generalized homology theory. Then the pushforward of the FOP pseudocycle (and its homology class) induces a natural transformation of generalized homology theories

$$\Omega_*^{\mathrm{der},\mathbb{C}}(-) \to H_*(-; \mathbb{Z}).$$

In fact, when the target space is a manifold, this natural transformation factors through pseudocycles. As we do not know if pseudocycles up to cobordism is a homology theory or not, we only consider the naive properties. Namely, for any manifold X, one has a group homomorphism

$$\Omega_k^{\mathrm{deg},\mathbb{C}}(X) \to \mathcal{H}_k(X) \tag{1.1}$$

where $\mathcal{H}_k(X)$ is the abelian group of k-dimensional oriented pseudocycles up to cobordism.

2. Pseudocycles in characteristic p

Definition 2.1. *Let p be a prime number. An oriented k-dimensional p-pseudocycle in a smooth manifold X is a smooth map $f : W \to X$ from a k-dimensional oriented manifold with boundary W to X satisfying the following property:*

1. *$f(W)$ is precompact in X.*
2. *There is a p-fold oriented covering $\partial W \to V$ and a smooth map $g : V \to X$ such that $f|_{\partial W}$ is the pullback of g.*
3. *The frontier is small. More precisely, the Ω-set of f is*

$$\Omega_f := \bigcap_{K \subset W \text{ compact}} \overline{f(W \setminus K)}.$$

We require that it has dimension at most $k - 2$.

In particular, a pseudocycle is a p-pseudocycle with $\partial W = \emptyset$.

Let $\tilde{\mathcal{H}}_k^{(p)}(X)$ be the set of all oriented k-dimensional p-pseudocycles. It has the structure of an abelian group where the sum of two p-pseudocycles is their disjoint union and the inverse of a p-pseudocycle is the same map with domain orientation reversed.

Definition 2.2. *Two oriented k-dimensional p-pseudocycles $f_0 : W_0 \to X$, $f_1 : W_1 \to X$ are said to be p-cobordant if there exists another smooth map $g : V \to X$ from an oriented k-dimensional manifold V without boundary, a smooth map $\tilde{f} : \tilde{W} \to X$ from an oriented $k+1$-dimensional manifold $\tilde{W}$ with boundary with*

$$\tilde{f}(\tilde{W}) \text{ is precompact}, \ \dim\Omega_{\tilde{f}} \leq k-1,$$

an oriented diffeomorphism

$$\partial\tilde{W} \cong -\mathrm{Int}W_0 \sqcup \mathrm{Int}W_1 \sqcup W'$$

where W' is a k-dimensional manifold without boundary (having the induced orientation) such that the restriction of $\tilde{f}$ to $\mathrm{Int}W_0$ resp. $\mathrm{Int}W_1$ coincides with f_0 resp. f_1, and an oriented p-fold covering $W' \to V$ such that $\tilde{f}|_{W'}$ is the pullback of g.

One needs to do some simple modifications to make certain naive operations satisfy the above definition. For example, a p-pseudocycle $f : W \to X$ is p-cobordant to itself. However, the naive map

$$\tilde{f} : W \times [0,1] \to X, \ \tilde{f}(x,t) = f(x)$$

is a p-cobordism only after removing the corner $\partial W \times \{0,1\}$.

It is clear that p-cobordant is an equivalence relation and respect the additive structure. Let $\mathcal{H}_k^{(p)}(X)$ be the set of p-cobordant classes of p-pseudocycles, which is an abelian group. As p times of any p-pseudocycle is p-cobordant to the empty set via the empty p-cobordism, $\mathcal{H}_k^{(p)}(X)$ is indeed an $\mathbb{F}_p$-vector space.

Before we discuss the relation between pseudocycles and homology classes, we define the intersection pairing between pseudocycles in finite characteristic.

Let

$$f_1 : W_1 \to X, \ f_2 : W_2 \to X$$

be two oriented p-pseudocycles in X of complimentary dimensions. When they intersect transversely, meaning that $\overline{f_1(W_1)} \cap \overline{f_2(W_2)} = f_1(\mathrm{Int}W_1) \cap f_2(\mathrm{Int}W_2)$ and the intersection is transverse, the signed count of intersection points, modulo p, is defined to be the intersection number. One can see easily that this intersection number is invariant under p-cobordism, provided that the p-cobordism is in general position.

In addition, if $f_1 : W_1 \to X$ is p-cobordant to the empty set via a p-cobordism $\tilde{f}_1 : \tilde{W}_1 \to X$, and if $\tilde{f}_1$ and f_2 are transverse, then $\tilde{f}_1(\tilde{W}_1) \cap f_2(W_2)$ is a compact oriented 1-dimensional manifold with boundary being

$$\Big(\tilde{f}_1(\partial\tilde{W}_1) \cap f_2(\mathrm{Int}W_2)\Big) \sqcup \Big(\tilde{f}_1(\mathrm{Int}\tilde{W}_1) \cap f_2(\partial W_2)\Big).$$

Besides the intersection $f_1(\mathrm{Int}W_1) \cap f_2(\mathrm{Int}W_2)$, other boundary intersections contribute to a multiple of p.

In general, if relevant intersections are not transverse, then one can perturb via ambient diffeomorphisms to achieve transversality. This allows us to define intersection numbers between any pair of p-pseudocycles of complementary dimensions and prove the independence of the choice of perturbations, as all nearby diffeomorphisms are homotopic. Therefore, we have defined a bilinear pairing

$$\mathcal{H}^{(p)}(X) \otimes \mathcal{H}^{(p)}(X) \to \mathbb{F}_p$$

Next we will define the map from homology to pseudocycles. Recall that one also has an abelian group $\mathcal{H}_k(X)$ of genuine k-dimensinoal pseudocycles up to genuine cobordism. There is an obvious group homomorphism

$$\mathcal{H}_*(X) \to \mathcal{H}_*^{(p)}(X).$$

Zinger constructed a natural isomorphism

$$\Phi_* : H_*(X;\mathbb{Z}) \cong \mathcal{H}_*(X).$$

Theorem 2.3. *There is a natural map*

$$\Phi_*^{(p)} : H_*(X;\mathbb{F}_p) \to \mathcal{H}_*^{(p)}(X)$$

satisfying the following conditions.

(1) *The following diagram commutes.*

$$\begin{array}{ccc} H_*(X;\mathbb{Z}) & \xrightarrow{\Phi_*} & \mathcal{H}_*(X) \\ \downarrow & & \downarrow \\ H_*(X;\mathbb{F}_p) & \xrightarrow[\Phi_*^{(p)}]{} & \mathcal{H}_*^{(p)}(X) \end{array} \tag{2.1}$$

(2) *Suppose a homology class $a \in H_*(X;\mathbb{F}_p)$ is represented by a smooth cycle $f : W \to X$ where W is a compact oriented manifold with boundary such that $f|_{\partial W}$ is a p-fold oriented covering of a map $g : V \to X$ from a compact oriented manifold V without boundary; in particular, f is a p-pseudocycle. Then $\Phi_*^{(p)}(a)$ is represented by f.*
(3) *The map $\Phi_*^{(p)}$ intertwines the Poincaré pairing on $H_*(X;\mathbb{F}_p)$ with the intersection pairing on $\mathcal{H}_*^{(p)}(X)$.*

Proof. We just need to consider the case that $p > 2$ as it is known that for $p = 2$ any homology class can be represented by a closed submanifold.

Consider the complex of singular chains with $\mathbb{F}_p$-coefficients. A homology class in $\mathbb{F}_p$ coefficients is represented by a singular cycle

$$C = \sum_{i=1}^{N} a_i h_i$$

where $a_i \in \mathbb{F}_p$ and $h_i : \Delta_k \to X$ is a continuous map from the k-simplex Δ_k. We can always choose the representative such that each h_i is smooth. Let $\delta^j(h_i)$ be the j-th face of h_i, which is a $k-1$-simplex. Moreover, let $\tilde{a}_i \in \{1, \ldots, p-1\}$ be a lift of a_i in $\mathbb{Z}$. For any smooth map $g : \Delta_{k-1} \to X$ which may appear as a face of h_i, consider

$$\Delta^{\pm}_{i,g}(C) = \Big\{ j \ | \delta^j(h_i) = \pm g \Big\}$$

and

$$\Delta^{\pm}_g(C) = \bigcup_{i=1}^{N} \left(\underbrace{\Delta^{\pm}_{i,g}(C) \sqcup \cdots \sqcup \Delta^{\pm}_{i,g}(C)}_{\tilde{a}_i} \right)$$

Then since C is a cycle, one has

$$\#\Delta^+_g(C) - \#\Delta^-_g(C) \in p\mathbb{Z}.$$

Without loss of generality, assume $\#\Delta^+_g(C) \geq \#\Delta^-_g(C)$. Then choose an injection $\Delta^-_g(C) \hookrightarrow \Delta^+_g(C)$ and glue the corresponding (interior of) faces, and removing all codimension two or higher facets from the simplexes, one obtains a topological k-manifold with boundary $N(C)$ together with a map $f : N(C) \to X$. Notice that the boundary of $N(C)$ can be identified as p copies of a manifold. If we fix a certain standard way of gluing standard k-simplexes along a face, $N(C)$ is then equipped with a canonical smooth structure. One can perturb $f : N(C) \to X$ to a smooth map, and hence a smooth p-pseudocycle. The cobordism class of the pseudocycle f is independent of the choice of the perturbation.

On the other hand, by using the same method as Zinger, one can show that two homologous cycles induce cobordant p-pseudocycles. The details are left to the reader. Hence we have constructed the map

$$\Phi^{(p)}_* : H_*(X; \mathbb{F}_p) \to \mathcal{H}^{(p)}_*(X).$$

It is easy to see that this is an $\mathbb{F}_p$-linear map making the diagram (2.1) commutative and satisfying (2) of Theorem 2.3. Moreover, by comparing the definition of the Poincaré pairing (which is essentially counting transverse intersections of cycles) and the intersection pairing, (3) of Theorem 2.3 is also true. □

3. Some flavors of FOP perturbations

Let G be a finite group and $\boldsymbol{V}, \boldsymbol{W}$ be two finite-dimensional complex representations of G. We require that the G-action on $\boldsymbol{V}$ is effective. Define

$$Z^G(\boldsymbol{V}, \boldsymbol{W}) = \{(v, P) \in \mathrm{Poly}^G(\boldsymbol{V}, \boldsymbol{W}) \mid P(v) = 0 \in \boldsymbol{W}\}$$

the zero locus of the evaluation map

$$\mathrm{ev} : \boldsymbol{V} \times \mathrm{Poly}^G(\boldsymbol{V}, \boldsymbol{W}) \to \boldsymbol{W}.$$

Its cut-off at any degree d is

$$Z^G_d(\boldsymbol{V}, \boldsymbol{W}) := Z^G(\boldsymbol{V}, \boldsymbol{W}) \cap (\boldsymbol{V} \times \mathrm{Poly}^G_d(\boldsymbol{V}, \boldsymbol{W})).$$

Similarly, one can define the family of the Z-variety for the parametrized case. Given a smooth manifold M and $V, W \to M$ smooth complex vector bundles with fiberwise complex linear G-actions, the zero locus of

$$\mathrm{ev} : V \oplus \mathrm{Poly}^G_d(V, W) \to W.$$

is denoted by

$$Z^G(V,W) := \{(v,P) \in V \oplus \mathrm{Poly}^G(V,W) \mid P(v) = 0\}$$

and $Z_d^G(V,W)$ is defined similarly by considering fiberwise polynomial maps of degree at most d.

For each subgroup $H \subset G$, denote

$$\boldsymbol{V}_H := \mathring{\boldsymbol{V}}^H := \{v \in \boldsymbol{V} \mid H \subset G_v\}, \qquad \boldsymbol{V}_H^* := \{v \in \boldsymbol{V} \mid G_v = H\}.$$

The top stratum is also called the isotropy-free part of $\boldsymbol{V}$, denoted by

$$\boldsymbol{V}^{\mathrm{free}} := \{v \in \boldsymbol{V} \mid G_v = \{e\}\}.$$

Then we have the decomposition

$$\boldsymbol{V} = \bigsqcup_{H \subseteq G} \boldsymbol{V}_H^*.$$

Proposition 3.1. *For any finite group G, there exists a positive integer d_0 satisfying the following conditions. Let $\boldsymbol{V}, \boldsymbol{W}$ be finite-dimensional complex representations. Suppose G acts on $\boldsymbol{V}$ faithfully. Then for any $d \geq d_0$ and for each subgroup $H \subset G$, the set*

$$Z_{d,H}^* := Z_{d,H}^*(\boldsymbol{V},\boldsymbol{W}) := Z_d^G(\boldsymbol{V},\boldsymbol{W}) \cap (\boldsymbol{V}_H^* \times \mathrm{Poly}_d^G(\boldsymbol{V},\boldsymbol{W}))$$

is a nonsingular complex algebraic set of complex dimension

$$\dim_{\mathbb{C}} Z_{d,H}^* = \dim_{\mathbb{C}} \mathrm{Poly}_d^G(\boldsymbol{V},\boldsymbol{W}) + \dim_{\mathbb{C}} \mathring{\boldsymbol{V}}^H - \dim_{\mathbb{C}} \mathring{\boldsymbol{W}}^H.$$

We first prove a lemma.

Lemma 3.2. *There exists $d_0 > 0$ such that for all $\mathring{v}_H \in \mathring{\boldsymbol{V}}_H^*$ and $\mathring{w}_H \in \mathring{\boldsymbol{W}}_H$, there exists $P \in \mathrm{Poly}_{d_0}^G(\boldsymbol{V},\boldsymbol{W})$ such that $P(\mathring{v}_H) = \mathring{w}_H$.*

Proof. By decomposing $\boldsymbol{W}$ into irreducible components, we may assume that $\boldsymbol{W}$ is an irreducible representation of G. Define the G-vector space

$$\boldsymbol{U} := \bigoplus_{\gamma \in G} \mathbb{C}\{\langle\gamma\rangle\}$$

with G-action defined as

$$g\left(\sum_\gamma c_\gamma \cdot \langle\gamma\rangle\right) = \sum_\gamma c_\gamma \cdot \langle\gamma g^{-1}\rangle = \sum_\gamma c_{\gamma g} \cdot \langle\gamma\rangle.$$

Since $\boldsymbol{U}$ is a regular representation, there is a G-equivariant homomorphism $\Psi : \boldsymbol{U} \to \boldsymbol{W}$ and an element $\boldsymbol{u} \in \boldsymbol{U}$ such that

$$\Psi(\boldsymbol{u}) = \Psi\left(\sum_\gamma w_\gamma \cdot \langle\gamma\rangle\right) = \mathring{w}_H.$$

Since $\mathring{w}_H \in \mathring{W}_H$, for all $h \in H$, one has

$$\Psi(h\boldsymbol{u}) = h\Psi(\boldsymbol{u}) = \mathring{w}_H.$$

Hence by taking average over H, one may assume that

$$\gamma' H = \gamma'' H \Longrightarrow w_{\gamma'} = w_{\gamma''}.$$

Now we claim that for some $d_0 > 0$ which only depends on G, one can choose a polynomial $f : \boldsymbol{V} \to \mathbb{C}$ (not necessarily G-invariant) of degree at most d_0 such that

$$\forall \gamma \in G,\ f(\gamma \mathring{v}_H) = w_\gamma.$$

Indeed, there are $n := |G/H|$ distinct elements in the G-orbit of $\mathring{v}_H$. One can choose a linear decomposition $\boldsymbol{V} = \boldsymbol{V}_1 \oplus \boldsymbol{V}_2$ such that $\boldsymbol{V}_1$ is one-dimensional and that the projection of these n distinct elements are still distinct in $\boldsymbol{V}_1$. Then by Lagrange's method of interpolation, one can find a complex polynomial $f : \boldsymbol{V}_1 \to \mathbb{C}$ of degree at most $|G|$ taking the prescribed values w_γ at the corresponding projection image of $\gamma \mathring{v}_H$ in $\boldsymbol{V}_1$. Extend f trivially to V one obtains a polynomial $f : \boldsymbol{V} \to \mathbb{C}$ satisfying the required conditions. Now define $P : \boldsymbol{V} \to \boldsymbol{W}$ by

$$P(v) = \Psi\left(\sum_{\gamma \in G} f(\gamma v) \cdot \langle \gamma \rangle\right).$$

Then this is a G-equivariant polynomial map sending $\mathring{v}_H$ to $\mathring{w}_H$. □

Proof of Proposition 3.1. For each subgroup $H \subset G$, we can write any polynomial map $P \in \mathrm{Poly}_d^G(\boldsymbol{V}, \boldsymbol{W})$ as

$$P = (\mathring{P}^H, \check{P}^H) : \boldsymbol{V} \to \mathring{\boldsymbol{W}}^H \oplus \check{\boldsymbol{W}}^H.$$

Then the equivariance implies that

$$\check{P}^H|_{\mathring{\boldsymbol{V}}_H} \equiv 0.$$

Therefore

$$Z^*_{d,H} := \{(v, P) \in \mathring{\boldsymbol{V}}^*_H \times \mathrm{Poly}_d^G(\boldsymbol{V}, \boldsymbol{W}) \mid \mathring{P}^H(v) = 0\}.$$

Then when $d \geq d_0$, the Lemma and the *faithfulness* of the G-action on $\boldsymbol{V}$ imply that $Z^*_{d,H}$ is a nonsingular complex algebraic set of dimension

$$\dim_{\mathbb{C}} \mathrm{Poly}_d^G(\boldsymbol{V}, \boldsymbol{W}) + \dim_{\mathbb{C}} \mathring{\boldsymbol{V}}^H - \dim_{\mathbb{C}} \mathring{\boldsymbol{W}}^H.$$

□

Remark 3.3. *If $\boldsymbol{W}$ is the trivial representation, then d_0 can be taken to be 0. In fact, all constant maps from $\boldsymbol{V}$ to $\boldsymbol{W}$ is G-equivariant and $Z_G^0(\boldsymbol{V}, \boldsymbol{W}) = \boldsymbol{V} \times \{0\} \subset \boldsymbol{V} \times \mathrm{Poly}_0^G(\boldsymbol{V}, \boldsymbol{W}) \cong \boldsymbol{V} \times \boldsymbol{W}$ so the*

$$Z^*_{0,H} = \boldsymbol{V}^*_H \times \{0\}.$$

Because $Z^*_{d,H}$ is a smooth complex algebraic variety, its Zariski closure has the property that the boundary has real codimension at least 2. For a generic polynomial P, this implies that $P^{-1}(0) \cap Z^*_{d,H}$ gives rise to a pseudo-cycle. This is the basis of the FOP perturbation scheme.

References

[BX22] Shaoyun Bai and Guangbo Xu, *An integral Euler cycle in normally complex orbifolds and $\mathbb{Z}$-valued Gromov–Witten type invariants*, https://arxiv.org/abs/2201.02688, 2022.