Class Field Theory

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We'll talk about adeles and ideles today, because homological algebra confuses me. References include Chapter 3 of Pete Clark's Algebraic Number Theory II Notes, Chapter 7 of Lang's Algebraic Number Theory, and Chapter 5 of Ramakrishnan and Valenza's Fourier Analysis on Number Fields. The first five lectures of Sam Mundy's Local Compactness and Number Theory also seem really good.

A lot of sources motivate these definitions by saying that adeles $\mathbb{A}_K$ and ideles $\mathbb{I}_K$ allow one to do harmonic/Fourier analysis on global fields (not that I will be doing any Fourier analysis today). Over local fields, this is doable: to do harmonic analysis, one only needs a locally compact abelian group, so it just kinda happens. Since I have the luxury of time I can say a few words about why we want locally compact groups:

Definition 1. A Radon measure μ on a topological space X is a measure defined on the Borel sets such that for any E,

$$\mu(E) = \sup_{K \subseteq E \text{ compact}} \mu(K) = \inf_{U \supseteq E \text{ open}} \mu(U).$$

A Radon measure μ on a locally compact abelian group G that is translation invariant in the sense that for all $g \in G$ and measurable E,

$$\mu(E) = \mu(gE)$$

is called a Haar measure.

It turns out that Haar measures always exist and are unique up to mulitplication by a positive real number.

Definition 2. Let G be a locally compact abelian group. Then the Pontryagin dual group $\hat{G}$ is the set of continuous group homomorphisms $\chi\colon G \to \mathbb{C}^\times$. The Fourier transform of a function $f \in L^1(G)$ is a function $\hat{f} \in L^1(\hat{G})$ given by

$$\hat{f}(\chi) = \int_G f(x)\overline{\chi(x)}\, d\mu(x)$$

where $\mu(x)$ is a Haar measure.

Back to not-analysis. We want to do harmonic analysis on global fields, so you'd want to complete the global field to even get started, and you want this $\mathbb{A}_K$ so that $K \hookrightarrow \mathbb{A}_K$. But if you're reasonable you want to not have to pick a completion (rather, you should consider them all at once), so the definition of the adeles is such that all of the (individually locally compact) completions are incorporated together to get a locally compact thingy. Unfortunately, for a global field K,

$$K \hookrightarrow \prod_{v \text{ a place}} K_v$$
$$x \mapsto (x, x, \dots)$$

is not necessarily locally compact:

Observation 3. Suppose $\{K_i\}_{i \in I}$ are locally compact. Then

$$\prod_{i \in I} K_i \text{ locally compact} \iff |\{i \in I \colon K_i \text{ not compact}\}| < \infty.$$

So how do we fix this? Well notice that for any fixed $x \in K$, the image of x in K_v actually lies in the the (compact!) valuation ring R_v for all but finitely many places v (only finitely many primes divide any fixed $y \in K$). We only need a locally compact group containing K, so instead of "reaching in" to the infinite direct product, we only ever need to reach into arbitrarily finitely many. With this in mind, we define:

Definition 4. For a finite set $S \subseteq \{v : v \text{ a finite place}\}$, define

$$\mathbb{A}_K(S) := \prod_{v \text{ infinite}} K_v \times \prod_{v \in S} K_v \times \prod_{v \notin S} \mathcal{O}_v.$$

These locally compact groups form a directed system: if $S_1 \subseteq S_2$ we get an embedding

$$\mathrm{id}\colon \mathbb{A}_K(S_1) \hookrightarrow \mathbb{A}_K(S_2),$$

so we can take a direct limit

$$\mathbb{A}_K := \lim_{S} \mathbb{A}_K(S)$$

and endow it with the final topology: the finest topology on $\mathbb{A}_K$ so that all of the $f_S\colon \mathbb{A}_K(S) \to \mathbb{A}_K$ are continuous (this is apparently a standard thing to do with direct limits of topological groups).

Observe that each $\mathbb{A}_K(S)$ is locally compact. It's a general fact that the directed limit of locally compact things is locally compact, and apparently this follows from the fact that $U \subseteq X := \lim_S X(S)$ is open in the final topology if and only if $f_S^{-1}(U)$ is open in $X(S)$.

Some sources seem to call this particular kind of directed limit, where you have groups $\{G_i\}_{i \in I}$ and $\{H_i\}_{i \in I}$ with $H_i \leq G_i$ and you look at "things which are in H_i for almost all i", as the *restricted directed product*.

Back to $\mathbb{A}_K$. Given an element $\alpha \in K$ we can define

$$||\alpha||_{\mathbb{A}_K} := \prod_{\tau \text{ complex}} |\alpha_\tau|_{\mathbb{C}}^2 \prod_{\sigma \text{ real}} |\alpha_\sigma|_{\mathbb{R}} \prod_{\mathfrak{p} \text{ finite}} |\alpha_{\mathfrak{p}}|_{\mathfrak{p}} = 1$$

by the product formula. This isn't so hard to prove; the version over $\mathbb{Q}$ is Ostrowski, and in general you can reduce it to $\mathbb{Q}$. In particular, this implies that K sits discretely inside $\mathbb{A}_K$.

Definition 5. Sitting inside the adeles $\mathbb{A}_K$ are the multiplicative units $\mathbb{A}_K^\times =: \mathbb{I}_K$, which we call the ideles. It is the restricted direct product of $\{K_v^\times\}_{v \text{ a place}}$ with $\{\mathcal{O}_v^\times\}_{v \text{ a place}}$, and it is given the topology that way. (Importantly, it does not agree with the subspace topology of $\mathbb{A}_K$)

The nonzero elements of K sit inside $\mathbb{I}_K$; these are called *principal ideles*. Thus the idele class group is defined to be

$$C_K := \mathbb{I}_K / K^\times$$

and it inherits a norm from $\mathbb{I}_K$ because it is trivial on $K^\times$. We let C_K^1 be the kernel of this norm map; this is sometimes called the norm 1 idele class group.

Here's something Kedlaya calls a proposition, but I promise it is a theorem:

Theorem 6. *The group C_K^1 is compact.*

Proof. We need to show that the inverse image of 1 under $|\cdot|\colon C_K \to \mathbb{R}_{>0}$. If $\rho > 0$ is any other number, then the inverse images are homeomorphic (just compose with multiplication by an element of norm ρ). Now Kedlaya asserts that there is c with the property that if $\rho > c$ then every idele of norm ρ is congruent mod $K^\times$ to an idele, all of whose components have norms in $[1, \rho]$.

We pick such a $\rho > c$. The inverse image of ρ is a closed subset of the set $\mathbb{I}_{K;\rho}$ of ideles whose components have norm in $[1, \rho]$. But

$$\mathbb{I}_{K;\rho} = \left\{ \alpha\colon \begin{cases} 1 \leq |\alpha_\tau|_\mathbb{C} \leq \sqrt{\rho}\colon \tau \text{ complex} \\ 1 \leq |\alpha_\sigma|_\mathbb{R} \leq \rho\colon \sigma \text{ real} \\ 1 \leq |\alpha_\mathfrak{p}|_\mathfrak{p} \leq \rho\colon \mathfrak{p}\colon N(\mathfrak{p}) \leq \rho \text{ finite} \\ |\alpha_\mathfrak{p}|_\mathfrak{p} = 1\colon N(\mathfrak{p}) > \rho \end{cases} \right\}$$

is a finite product of compact sets so we're compact! (Sorry for the sacrilegious tex) ☐

If you care, Ramakrishnan and Valenza's proof uses measures. Let's use this theorem to prove some cool results:

Corollary 7. *The class number is finite.*

Proof. We have a surjection $\mathbb{I}_K \twoheadrightarrow \mathcal{J}_K$ to the group of fractional ideals of K, given by

$$(\alpha_v)_{v \text{ a place}} \mapsto \prod_{\mathfrak{p}} \mathfrak{p}^{v_\mathfrak{p}(\alpha_\mathfrak{p})}$$

which is continuous when $\mathcal{J}_K$ is given the discrete topology. The principal ideles $\alpha \in K^\times$ map to principal ideals $\langle \alpha \rangle \in \mathcal{P}_K$, so we get a surjection

$$C_K = \mathbb{I}_K / K^\times \twoheadrightarrow \mathcal{J}_K / \mathcal{P}_K = \mathrm{Cl}_K.$$

Since the surjection does not depend on the infinite places, we actually get a surjection $C_K^1 \twoheadrightarrow \mathrm{Cl}_K$ (take the preimage of any ideal class and adjust the infinite places). Hence Cl_K is both discrete and compact. ☐

Corollary 8. *The group $\mathcal{O}_K^\times$ has rank $r + s - 1$, and more generally the group of S-units $K_S^\times$, ie. the elements with valuation zero at at each finite place not contained in S, has rank $\#S - 1$.*

As an example, note that $\mathbb{Q}_{\{3,5\}}^\times = \mathbb{Z}^\times[\frac{1}{3}, \frac{1}{5}] \cong \mathbb{Z}^2$.

Proof. Consider the map $\mathbb{I}_K(S) \to \mathbb{R}^{\#S}$ by taking log of the absolute value of the norm of each component in S. The image of $\mathbb{I}_K^1(S)$ is necessarily the hyperplane

$$H = \left\{ (x_1, \ldots, x_{\#S}) \in \mathbb{R}^{\#S}\colon \sum x_i = 0 \right\}.$$

Now $K_S^\times$ is discrete in this image, since given a bounded subset H_b of H, the preimage of H_b consists of things in $K_S^\times$ with bounded $|\cdot|_v$ for each $v \in S$, so the preimage of H_b is finite. Let W be the H-span of the image of $K_S^\times$. We have a continuous homomorphism

$$\mathbb{I}_K^1(S) / K_S^\times \to H/W,$$

and the real vector space H/W has a compact discrete subgroup, namely, the image of $\mathbb{I}_K^1(S)/K_S^\times$, so necessarily $\dim_\mathbb{R} H/W = 0$. ☐