The dual abelian variety

Nov 24, 2019

As usual I'm basically following Milne. But today I want to zoom toward a particular result and then reflect on what happened – so I'm going to be jumbling around the order. (I think Milne does it in his order because he has a responsibility to be rigorous. But not me :D)

Let us recall the following corollary of the Theorem of the Square:

Theorem 1. *For every invertible sheaf $\mathscr{L}$ on A, the map $\lambda_{\mathscr{L}}\colon A(k) \to \operatorname{Pic} A$ sending $a \mapsto t_a^*\mathscr{L} \otimes \mathscr{L}^{-1}$ is a homomorphism.*

Note that we also have multiplication and projection maps $m, p, q\colon A \times A \to A$ (here p is projection onto the first coordinate and q is projection onto the second coordinate). We can consider the sheaf $m^*\mathscr{L} \otimes p^*\mathscr{L}^{-1}$ on $A \times A$, which can be thought of as a family of invertible sheaves on $A = p(A \times A)$, parametrized by $A = q(A \times A)$. Thus, a choice of a point $a \in A(k)$ gives an element in this family, namely, $(m^*\mathscr{L} \otimes p^*\mathscr{L}^{-1})|_{A\times\{a\}}$.

What is this invertible sheaf? Well, on $A \times \{a\}$ the map m is actually t_a, and the map p is the identity, and so

$$(m^*\mathscr{L} \otimes p^*\mathscr{L}^{-1})|_{A\times\{a\}} = t_a^*\mathscr{L} \otimes \mathscr{L}^{-1} = \lambda_{\mathscr{L}}(a).$$

With this discussion in mind, let me define

Definition 2. Let $\mathscr{L}$ be an invertible sheaf on A. Define

$$K(\mathscr{L}) = \{a \in A\colon (m^*\mathscr{L} \otimes p^*\mathscr{L}^{-1})|_{A\times\{a\}} \text{ is trivial}\}.$$

This is a subset of A. The set of k-points of $K(\mathscr{L})$ is given by

$$K(\mathscr{L})(k) = \{a \in A(k)\colon \lambda_{\mathscr{L}}(a) = \mathscr{O}_A\}. \qquad \triangle$$

Apparently, $K(\mathscr{L})$ is a closed subset of A. It follows from the observation that $K(\mathscr{L}) = \operatorname{Supp}(q_*(\mathscr{L}_2)) \cap \operatorname{Supp}(q_*(\mathscr{L}_2^\vee))$, where $\mathscr{L}_2 = (m^*\mathscr{L} \otimes p^*\mathscr{L}^{-1})$. I don't think I understand this, but the observation that outside the support of D, the line bundle $\mathscr{L}(D)$ is trivial sounds relevant (see discussion after Definition 3.1 in these notes).

The homomorphism-ness of $\lambda_{\mathscr{L}}$ implies that $K(\mathscr{L})$ is actually a closed subgroup of A. Indeed, $K(\mathscr{L})$ commutes with extension of scalars, and for points $a, b \in \overline{k}$ we have $\lambda_{\mathscr{L}}(a+b) = \lambda_{\mathscr{L}}(a) \otimes \lambda_{\mathscr{L}}(b) = \mathscr{O}_A$, hence $K(\mathscr{L})(\overline{k})$ is a subgroup of $A(\overline{k})$. [I hope this is enough to guarantee that $K(\mathscr{L})$ is a subgroup scheme of A, but I'm not so sure...]

We hit our first important result.

Proposition 3 (Proposition 8.4 in Milne)**.** *Let $\mathscr{L}$ be an invertible sheaf on A. Then, the following conditions are equivalent:*

(a) $K(\mathscr{L}) = A$

(b) $t_a^*\mathscr{L} \cong \mathscr{L}$ *on* $A_{\overline{k}}$ *for all* $a \in A(\overline{k})$,

(c) $m^*\mathscr{L} \cong p^*\mathscr{L} \otimes q^*\mathscr{L}$.

Note that condition (b) above is really $K(\mathscr{L})(\overline{k}) = A(\overline{k})$.

Proof. The equivalence of (a) and (b) is follows from the fact that $A \setminus K(\mathscr{L})$ is open, hence nonempty if and only if its base change to $\overline{k}$ has a closed point. That (c) implies (a) is also easy, since

$$(m^*\mathscr{L} \otimes p^*\mathscr{L}^{-1})|_{A\times\{a\}} \cong q^*\mathscr{L}|_{A\times\{a\}} = (x \mapsto a)^*\mathscr{L}|_{A\times\{a\}}$$

is trivial. That (b) implies (c) follows from the observations that for every $a \in \overline{k}$,

$$m^*\mathscr{L} \otimes p^*\mathscr{L}^{-1}|_{A\times\{a\}} = t_a^*\mathscr{L} \otimes \mathscr{L}^{-1}$$

is trivial, that the map q on $A \times \{a\}$ is a constant map and hence $q^*\mathscr{L}|_{A\times\{a\}}$ is trivial, that

$$m^*\mathscr{L} \otimes p^*\mathscr{L}^{-1}|_{\{0\}\times A} = q^*\mathscr{L}|_{\{0\}\times A} = \mathscr{L},$$

and that the Seesaw principle (Cor 5.18 in Milne) precisely says that (c) holds now. □

Definition 4. The set $\mathrm{Pic}^0(A) \subseteq \mathrm{Pic}(A)$ consists of line bundles satisfying the conditions of Proposition 3. Condition (b) of that result, along with theorem of the square, says that $\mathrm{Pic}^0(A)$ is a subgroup of $\mathrm{Pic}(A)$. △

Fact 5. *The k-points $A^\vee(k)$ of the dual of A shall be the group $\mathrm{Pic}^0(A)$.*

Lemma 6 (Lemma 8.8 in Milne)**.** *For an invertible sheaf $\mathscr{L}$ on A and any $a \in A(k)$, the invertible sheaf $\lambda_{\mathscr{L}}(a) = t_a^*\mathscr{L} \otimes \mathscr{L}^{-1}$ is in $\mathrm{Pic}^0(A) = A^\vee(k)$.*

Proof. Milne prefers to prove this in terms of divisors.

I think you can prove this by noting that

$$t_b^*(t_a^*\mathscr{L} \otimes \mathscr{L}^{-1}) \otimes (t_a^*\mathscr{L} \otimes \mathscr{L}^{-1})^{-1} \cong t_{a+b}^*\mathscr{L} \otimes t_b^*\mathscr{L}^{-1} \otimes t_a^*\mathscr{L}^{-1} \otimes \mathscr{L} \cong \mathscr{O}_A.$$ □

We are thus a stone's throw away from

Theorem 7 (Special case of Theorem 6.18 here)**.** *The maps $\lambda_{\mathscr{L}} \colon A(k) \to A^\vee(k)$ give a regular map $\varphi_{\mathscr{L}} \colon A \to A^\vee$ and its kernel is the subgroup scheme $K(\mathscr{L})$ of A.*

Proposition 8 (Proposition 8.1 in Milne)**.** *Let $\mathscr{L}$ be an invertible sheaf such that $\Gamma(A, \mathscr{L}) \neq 0$. Then $\mathscr{L}$ is ample if and only if $K(\mathscr{L})$ has dimension zero.*

Proof. Observe that $\Gamma(A, \mathscr{L}) \neq 0$-ness, ample-ness, and dimension 0-ness is preserved under base change to $\overline{k}$. The first two is 5.12 and 6.6 in Milne respectively, and the third fact is proven in much more generality here (it's also Hartshorne, (Ex II.3.20(f))).

Let's prove $\mathscr{L}$ ample implies that $K(\mathscr{L})$ is dimension zero (since that's all Milne does). Let B be the connected component of $K(\mathscr{L})$ passing through 0. It is an abelian variety, hence $\mathscr{L}|_B$ is ample. For any $b \in B$, we have $t_b^*\mathscr{L}|_B \cong \mathscr{L}|_B$; Proposition 3 says that $m^*\mathscr{L}|_B \otimes p^*\mathscr{L}|_B^{-1} \otimes q^*\mathscr{L}|_B^{-1}$ on $B \times B$ is trivial. Take the inverse image of this sheaf by the regular map

$$\begin{aligned} B &\to B \times B \\ b &\mapsto (b, -b) \end{aligned}$$

to get that $\mathscr{L}|_B \otimes (-1_B)^*\mathscr{L}|_B$ is trivial. We have an ample sheaf $\mathscr{L}|_B$ so that $\mathscr{L}|_B \otimes (-1_B)^*\mathscr{L}|_B$ is trivial; we saw last time that this automatically implies $\dim B = 0$ and hence $B = 0$. (Last time, they key point was that on a connected variety V, the sheaf $\mathscr{O}_V$ can only be very ample if V consists of a single point.) □

Proposition 9 (Proposition 8.14 in Milne)**.** *If $\mathscr{L}$ is ample, then $\lambda_{\mathscr{L}} \colon A \to \mathrm{Pic}^0(A)$ is surjective.*

Proof. (He cites Mumford 1970, §8, p77 or Lang 1959, p99.) □

The previous two propositions, along with the fact that A always has an ample line bundle, say

Theorem 10. *Let A be an abelian variety and $A^\vee$ be its dual. Then A and $A^\vee$ are isogenous, and for every ample line bundle $\mathscr{L}$ with $\Gamma(A, \mathscr{L}) \neq 0$ the map $\varphi_{\mathscr{L}}$ is an isogeny. Furthermore, in characteristic zero, the geometric quotient $A/K(\mathscr{L})$ exists and $A^\vee \cong A/K(\mathscr{L})$.*

The maps $\lambda_{\mathscr{L}}\colon a \mapsto t_a^*\mathscr{L} \otimes \mathscr{L}^{-1}$ give isomorphisms $A^\vee(k') \cong A(k')/K(\mathscr{L})(k')$ for every $k' \supseteq k$, and I presume this means the regular $\varphi_{\mathscr{L}}$ gives an isomorphism $A^\vee \cong A/K(\mathscr{L})$.

(Of course, we don't even know that $A^\vee$ is a variety yet!)

Remark 11 (Remark 8.2 in Milne)**.** In Proposition 8 we said that if $\Gamma(A, \mathscr{L}) \neq 0$ then $\mathscr{L}$ is ample if and only if $K(\mathscr{L})$ has dimension zero. Well, an effective divisor D always has global sections [e.g., it always contains $k^\times$, right?] so Proposition 8 says that an effective divisor D is ample if and only if $\lambda_D\colon A(\overline{k}) \to \mathrm{Pic}(A_{\overline{k}})$ has finite kernel [is this because $K(\mathscr{L})$ has dimension zero if and only if it has finitely many closed points after base change to $\overline{k}$?]. △

Remark 12 (Remark 8.5 in Milne)**.** Let α, β be two regular maps $V \to A$. Their sum is the composition $m \circ (\alpha \times \beta)$. If $\mathscr{L} \in \mathrm{Pic}^0(A)$, then

$$m^*\mathscr{L} \cong p^*\mathscr{L} \otimes q^*\mathscr{L}.$$

Applying $(\alpha \times \beta)^*$ to both sides we obtain

$$(\alpha + \beta)^*\mathscr{L} \cong \alpha^*\mathscr{L} \otimes \beta^*\mathscr{L}.$$

This means that

$$\mathrm{Hom}(V, A) \to \mathrm{Hom}(\mathrm{Pic}^0(A), \mathrm{Pic}(V))$$
$$\alpha \mapsto (\mathscr{L} \mapsto \alpha^*\mathscr{L})$$

is a homomorphism of groups. I want to claim to you that for $V = A$ that this becomes a map $\mathrm{End}(A) \to \mathrm{End}(\mathrm{Pic}^0(A)) \subseteq \mathrm{Hom}(\mathrm{Pic}^0(A), \mathrm{Pic}(A))$. If you believe that $\mathrm{Pic}^0(A) = A^\vee$ is equal to the connected component of the identity in $\mathrm{Pic}(A)$ then we win. Alternatively, we have a homomorphism

$$\mathrm{Hom}(V, A) \to \mathrm{Hom}(\mathrm{Pic}^0(A_{\overline{k}}), \mathrm{Pic}^0(V_{\overline{k}}))$$
$$\alpha \mapsto (\mathscr{L} \mapsto \alpha_{\overline{k}}^*\mathscr{L}),$$

because if $\mathscr{L}_2 = m_{\overline{k}}^*\mathscr{L} \otimes p_{\overline{k}}^*\mathscr{L}^{-1}$ is trivial on $A_{\overline{k}} \times \{a\}$ then $(\alpha_{\overline{k}} \times 1_{A_{\overline{k}}})^*(\mathscr{L}_2)$ is also trivial on $A_{\overline{k}} \times \{a\}$ (see Example 4.13 in here). [I'm not sure why this is true but it seems base change to $\overline{k}$ is needed here.]

In particular, $n_A \in \mathrm{End}(A)$ gets mapped to $(\mathscr{L} \mapsto \mathscr{L}^n) \in \mathrm{End}(\mathrm{Pic}^0(A))$. That is to say, $(n_A)^*\mathscr{L} \cong \mathscr{L}^n$ for every $\mathscr{L} \in \mathrm{Pic}^0(A)$. Milne specifically notes that when $\mathscr{L}$ is symmetric then we've seen before $(n_A)^*\mathscr{L} \cong \mathscr{L}^{n^2}$, and that $(n_A)^*\mathscr{L} \cong \mathscr{L}^n$ is not a contradiction to this because if $\mathscr{L} \in \mathrm{Pic}^0(A)$ then $(-1)_A^*\mathscr{L} \cong \mathscr{L}^{-1}$, so $\mathscr{L}$ is antisymmetric. △

Remark 13 (Remark 8.6 in Milne)**.** Let $\alpha\colon A \to B$ be an isogeny, and suppose $\ker(\alpha) \subseteq A_n$ lives inside the n-torsion of A. Then α factors into

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C,$$

where $\beta \circ \alpha = n$ and $\deg \alpha \cdot \deg \beta = n^{2g}$. I suspect he means that $C = A/A_n$ exists and that $\beta \circ \alpha$ is the composite map

$$A \xrightarrow{n_A} A \xrightarrow{\pi} A/A_n.$$

△

I hope I have just enough time to state what the dual abelian variety really is.

Definition 14. Consider a pair $(A^\vee, \mathscr{P})$ where $A^\vee$ is an algebraic variety over k and $\mathscr{P}$ is an invertible sheaf on $A \times A^\vee$. Assume that $\mathscr{P}|_{A \times \{b\}} \in \mathrm{Pic}^0(A_b)$ for all $b \in A^\vee$, and $\mathscr{P}|_{\{0\} \times A^\vee}$ is trivial.

We say $A^\vee$ is the dual abelian variety of A and $\mathscr{P}$ the Poincaré sheaf if $(A^\vee, \mathscr{P})$ has the following universal property: for any pair $(T, \mathscr{L})$ consisting of a variety T over k and an invertible sheaf $\mathscr{L}$ such that $\mathscr{L}|_{A \times \{t\}} \in \mathrm{Pic}^0(A_t)$ for all $t \in T$ and $\mathscr{L}|_{\{0\} \times T}$ is trivial, there is a unique regular map $\alpha\colon T \to A$ so that $(1 \times \alpha)^*\mathscr{P} = \mathscr{L}$.

△