Abelian Varieties (are Projective)

Oct 5, 2019

A long time ago we saw that projective varieties are complete and that the converse is not true in general; a few weeks ago Guanyu asserted that abelian varieties were examples of complete varieties which are projective.

Let's begin with the case $k = \bar{k}$, and remove this assumption later.

To make sure we are all on the same page, let us recall some definitions:

Definition 1. Let V be a complete nonsingular variety over k. A *divisor* (Weil and Cartier agree here) of V is a formal sum of subvarieties of codimension 1. A divisor is *effective* if the coefficients are nonnegative. It is *principal* if it arises from a function

$$\operatorname{div}(f) \stackrel{\text{def}}{=} \sum_Y \operatorname{ord}_Y(f) \cdot Y,$$

where $\operatorname{ord}_Y(f) \stackrel{\text{def}}{=} \operatorname{length}(\mathscr{O}_{X,\xi}/\langle g \rangle) - \operatorname{length}(\mathscr{O}_{X,\xi}/\langle h \rangle)$, with $\xi \in Y$ generic, and $f = g/h \in \operatorname{Frac}(\mathscr{O}_{X,\xi}) = k(X)$. Two divisors are *linearly equivalent* if their difference is principal. △

Definition 2. A *complete linear system* $\mathfrak{d}$ is a nonempty linear equivalence class of effective divisors. Thus, if $D_0 \in \mathfrak{d}$, then

$$\mathfrak{d} = \{D_0 + \operatorname{div}(f) \colon f \in L(D_0)\}.$$

If $W \subset L(D_0)$ is a subspace, then

$$\{D_0 + \operatorname{div}(f) \colon f \in W\}$$

will be called a *linear system*. △

Remark 3. For any normal projective variety X and any divisor D on X one has $\dim L(D) < \infty$. Since group varieties are smooth schemes over a field, they are normal, so we may freely take (finite) bases of the vector space $L(D)$. △

Proposition 4. *If V is a closed subvariety of $\mathbb{P}^n$, then*

$$\{V \cap H \colon H \text{ hyperplane in } \mathbb{P}^n\}$$

is a linear system.

Proof. This follows from the fact that any two hyperplanes in $\mathbb{P}^n$ are linearly equivalent: if H_1, H_2 are hyperplanes then they are defined by equations $\ell_1, \ell_2 = 0$; now $f = \ell_1/\ell_2$ gives rise to the divisor $H_1 - H_2$. Let H_V be a hyperplane intersecting V, and note that

$$\{V \cap H \colon H \text{ hyperplane in } \mathbb{P}^n\}$$

is the image of the subspace $W \subseteq L(V \cap H_V)$ defined by

$$W = \{f \in L(V \cap H_V) \colon \text{zeros and poles of } f \text{ intersect } V\}$$

under the map $W \twoheadrightarrow \mathfrak{d} \subseteq \operatorname{Div}(V)$.

[... I think.] In any case, observe that defining the subspace W algebraically (rather than geometrically) would require using the closed immersion into $\mathbb{P}^n$ defining V. □

Now a complete linear system on V gives rise to a rational map $V \dashrightarrow \mathbb{P}^n$ in the following way: if

$$\mathfrak{d} = \{D_0 + \operatorname{div}(f) \colon f \in L(D_0)\}$$

is a complete linear system and $f_0, \dots, f_n$ is a basis for $L(D_0)$, then we get a rational map

$$P \mapsto (f_0(P) : \cdots : f_n(P))$$

on the open set of V where no f_i has a pole at P and at least one f_i is nonzero. A change of basis corresponds to a projective linear transformation, and if we replace D_0 with $D' = D_0 + \operatorname{div}(f)$ then we can replace the basis $f_0, \dots, f_n$ to $f_0/f, \dots, f_n/f$ gives the same rational map.

When does this rational map define an isomorphism from V onto a closed subvariety of $\mathbb{P}^n$?

Well, if there exists an effective divisor E so that $D \geq E$ for every $D \in \mathfrak{d}$, then this means for any $D_0 \in \mathfrak{d}$ we have $L(D_0 - E) = L(D_0)$ and

$$\mathfrak{d} - E \stackrel{\text{def}}{=} \{D_0 - E + \operatorname{div}(f) \colon f \in L(D_0 - E) = L(D_0)\}$$

is also a complete linear system, and moreover defines the same rational map as $\mathfrak{d}$. Such an effective divisor E is called a *fixed divisor*, and from the point of view of rational maps we may assume that $\mathfrak{d}$ does not contain any fixed divisors.

Definition 5. A point P of V is said to be a *base point* of $\mathfrak{d}$ if $P \in \operatorname{Supp}(D)$ for all $D \in \mathfrak{d}$. Even when there is no fixed divisor, there may be base points. △

Proposition 6. *The rational map $\varphi \colon V \to \mathbb{P}^n$ defined by $\mathfrak{d}$ is defined at P if and only if P is not a base point of $\mathfrak{d}$.*

Proof. Suppose P is not a base point of $\mathfrak{d}$, and let D_0 be an element of $\mathfrak{d}$ such that $P \notin \operatorname{Supp}(D_0)$. Pick a basis $f_0, \dots, f_n$ for $L(D_0)$, and note that the effectiveness of $\operatorname{div}(f_i) + D_0 \geq 0$ guarantees that f_i does not have a pole at P (since P isn't in the support of D_0). If $f_i(P) = 0$ for all i, then every $f \in L(D_0)$ has a root at P, but the constant functions should also be in $L(D_0)$ because D_0 is effective. Thus $P \mapsto (f_0(P) : \cdots : f_n(P))$ is defined.

Conversely, if $D_0 \in \mathfrak{d}$ and $f_0, \dots, f_n \in L(D_0)$ is a basis, and the map $P \mapsto (f_0(P) : \cdots : f_n(P))$ is not defined at P, then they're not all zero for the same reason as above so some f_i must have a pole at P; the fact that $\operatorname{div}(f_i) + D_0 \geq 0$ implies that $P \in \operatorname{Supp}(D_0)$.

[... I think.] [I don't know where I used that $\mathfrak{d}$ has no fixed divisor, but I assume it is at the last step.] □

Definition 7. Let $\mathfrak{d}$ be a linear system.

- We say $\mathfrak{d}$ *separate points* if for any $P, Q \in V$ there exists $D \in \mathfrak{d}$ so that $P \in \operatorname{Supp}(D)$ and $Q \notin \operatorname{Supp}(D)$.
- We say $\mathfrak{d}$ *separates tangent directions* if for every $P \in V$ and t tangent to V at P, there is $D \in \mathfrak{d}$ so that $P \in D$ but $t \notin T_P(D)$, where $T_P(D) \subseteq T_P(V)$ is the subspace of the tangent space defined by $(df)_P = 0$; in other words, only one prime divisor Z in D can contain P, and Z only occurs with multiplicity one, and that $t \notin T_P(Z)$. △

Proposition 8. *The map $\varphi \colon V \to \mathbb{P}^n$ defined by a complete linear system $\mathfrak{d}$ is a closed immersion if and only if $\mathfrak{d}$ separates points and separates tangent directions.*

According to Hartshorne, this proposition holds for projective schemes over algebraically closed fields.

Proof. Milne references Hartshorne, who simply asserts that this *is* Prop II.7.3, which states the following:

Proposition 9. *Let k be an algebraically closed field, let X be a projective scheme over k, and let $\varphi \colon X \to \mathbb{P}^n_k$ be a morphism (over k) corresponding to $\mathscr{L}$ [an invertible sheaf] and $s_0, \dots, s_n \in \Gamma(X, \mathscr{L})$ [That is, there is a unique morphism $\varphi \colon X \to \mathbb{P}^n_k$ such that $\mathscr{L} = \varphi^*(\mathscr{O}(1))$ and $s_i = \varphi^*(x_i)$]. Let $V \subseteq \Gamma(X, \mathscr{L})$ be the subspace spanned by the s_i. Then φ is a closed immersion if and only if elements of V separate points, i.e. for any two distinct closed coints $P, Q \in X$ there is $s \in V$ such that $s \in \mathfrak{m}_P \mathscr{L}_P$ but $s \notin \mathfrak{m}_Q \mathscr{L}_Q$, and elements of V separate tangent vectors, i.e. for each closed point $P \in X$, the set $\{s \in V \colon s_P \in \mathfrak{m}_P \mathscr{L}_P\}$ spans the k-vector space $\mathfrak{m}_P \mathscr{L}_P / \mathfrak{m}_P^2 \mathscr{L}_P$.*

The proof of this is kind of painful, with the forward direction being "not so bad".

Let me try to convince you that the definitions of "separating points" and "separating tangent directions" are the same.

Note that a complete linear system is a linear equivalence class of divisors, which are in bijection with isomorphism classes of invertible sheaves (that this is a bijection is a Hartshorne II.6 theorem: the map sends $D = \{(U_i, f_i)\} \mapsto \mathscr{L}(D)$, where $\mathscr{L}(D)$ is the subsheaf of $\mathscr{K}$ generated by f_i^{-1} on U_i; recall that a Cartier divisor is obtained by picking, for an open cover $\{U_i\}$ of V, sections $f_i \in \Gamma(U_i, \mathscr{K}^*)$ of the sheaf of "multiplicative groups of invertible elements in the total quotient ring of $\mathscr{O}$".) which intersect nicely (so f_i/f_j on $U_i \cap U_j$ is a section in $\Gamma(U_i \cap U_j, \mathscr{O}^*)$, where $\mathscr{O}^*$ is the sheaf of invertible elements in $\mathscr{O}$.

Now note that

$$\Gamma(X, \mathscr{L}(D)) = L(D)/k^\times = \{f \in k(X)^\times : \operatorname{div}(f) + D \geq 0\}/k^\times.$$

Picking a divisor $D \in \mathfrak{d} = \{D_0 + \operatorname{div}(f) : f \in L(D_0)\}$ corresponds to picking $f \in L(D_0)$ up to multiplication by a constant (this requires a theorem, namely II.7.7(c) in H). This is an element $s \in \Gamma(X, \mathscr{L}(D))$, so we need to show that $P \in \operatorname{Supp}(D)$ if and only if $s \in \mathfrak{m}_P \mathscr{L}(D_0)_P$. This is true because if we picked D_0 to be such that $P \notin \operatorname{Supp}(D_0)$ and $P \in \operatorname{Supp}(D)$ then $D \geq 0$ means that s must vanish at P. But a section s vanishes at P if and only if $s \in \mathfrak{m}_P \mathscr{L}(D_0)_P$. [Thanks, Kabir.]

I think you can say something similar for separating tangent spaces: an element in $\mathfrak{m}_P \mathscr{L}(D_0)_P / \mathfrak{m}_P^2 \mathscr{L}(D_0)_P$ is represented by a function vanishing with order exactly one at P, which comes from a divisor D so that P only appears in one prime divisor Z, with Z appearing only once. But there are details here that I won't check. Sorry :(□

Theorem 10. *Every abelian variety is projective.*

Proof. Suppose we had a finite set of prime divisors, say n many $Z_1, \ldots, Z_n$, so that the divisor $D = \sum Z_i$ separates 0 from every other point, and separates tangent directions at 0. Precisely, we want

$$\bigcap Z_i = \{0\} \subseteq A \quad \text{and} \quad \bigcap \mathrm{T}_0(Z_i) = \{0\} \subseteq \mathrm{T}_0(A).$$

We'll prove the existence of this in a bit. Our claim is that $3D$ defines a complete linear system $\mathfrak{d} = \{3D + \operatorname{div}(f) : f \in L(3D)\}$ that separates all points and all tangent directions. This would make the map φ a closed immersion.

To show that $3D$ separates all points and all tangent directions, we first recall the Theorem of the Square, which was proven in a previous section of the book but we haven't proven yet:

Theorem 11. *Let $\mathscr{L}$ be a line bundle on an abelian variety A and let $a, b \in A(k)$ be two k-rational points. Then*

$$t_{a+b}^* \mathscr{L} \otimes \mathscr{L} \cong t_a^* \mathscr{L} \otimes t_b^* \mathscr{L},$$

where t_x is translation by x.

Thus,

$$t_{a+b}^* \mathscr{L} \otimes \mathscr{L}^{-1} \cong (t_a^* \mathscr{L} \otimes \mathscr{L}^{-1}) \otimes (t_b^* \mathscr{L} \otimes \mathscr{L}^{-1})$$

implies that $a \mapsto t_a^* \mathscr{L}^{-1}$ is a homomorphism $A \mapsto \operatorname{Pic} A$. In particular, we have for any $a, b \in A$ the isomorphism

$$t_a^* \mathscr{L} \otimes t_b^* \mathscr{L} \otimes t_{-a-b}^* \mathscr{L} \cong \mathscr{L}^{\otimes 3}$$

In light of the fact that the bijection $D \mapsto \mathscr{L}(D)$ from divisors mod linear equivalence to line bundles mod isomorphisms, we get a linear equivalence

$$Z_a + Z_b + Z_{-a-b} \sim 3Z$$

for any prime divisor Z. Let us pick $2n$ points $\{a_1, \ldots, a_n; b_1, \ldots, b_n\}$ of A, and note that

$$3D = \sum 3Z_i \sim \sum (Z_{i,a_i} + Z_{i,b_i} + Z_{i,-a_i-b_i}).$$

Let us also pick two distinct points $a, b \in A$, and observe that $b - a \neq 0$ so some Z_i (WLOG Z_1) doesn't contain $b - a$. Let $a_1 = 1$, and observe that Z_{1,a_1} passes through a but not b. Since

$$\{b_1 \colon Z_{1,b_1} \text{ passes through } b\} \cup \{b_1 \colon Z_{1,-a_1-b_1} \text{ passes through } b\}$$

is a union of proper closed subsets, there exists b_1 in neither. We can also choose a_i, b_i so that none of $Z_{i,a_i}, Z_{i,b_i}, Z_{i,-a_i-b_i}$ pass through b, and a is in the support of $\sum(Z_{i,a_i} + Z_{i,b_i} + Z_{i,-a_i-b_i})$, but b is not. Hence $3D$ separates points. The proof that it separates tangents is similar; for any $a \in A$ and $t \neq 0 \in \mathrm{T}_a(A)$ we have

$$\bigcap \mathrm{T}_0(Z_i) = \{0\} \subseteq \mathrm{T}_0(A) \implies \bigcap \mathrm{T}_0(Z_{i,a}) = \{0\}$$

so there exists j so that $t \notin Z_{j,a}$. Then we pick $a_j = a$ and all other a_i, b_i to be so that $Z_{i,a_i}, Z_{i,b_i}, Z_{i,-a_i-b_i}$ are out of the way. I think this proof is really cute.

As promised, I have to tell you why there exist Z_i so that

$$\bigcap Z_i = \{0\} \subseteq A \quad \text{and} \quad \bigcap \mathrm{T}_0(Z_i) = \{0\} \subseteq \mathrm{T}_0(A).$$

We first show that for any point $P \neq 0 \in A$, there is an open affine containing both 0 and P. Indeed, if U is an open affine containing 0, take a point $u \in U \cap (U + P)$ and observe that $u \in U + P$ means that $0 \in U + P - u$, and $u \in U$ means that $P \in U + P - u$. Hence $U' = U + P - u$ is an open affine neighborhood. Thus we can identify U' with a closed subset of $\mathbb{A}^n$, take a hyperplane H passing through 0 but not P, and take Z_1 to be the closure of $H \cap U'$ in A. If there is P' on Z_1 other than 0, choose Z_2 to pass through 0 but not P'. Continue in this fashion; by the descending chain condition we win after finitely many steps. Then $\cap Z_i = \{0\}$. Now choose any open affine U around 0, and suppose $t \in \mathrm{T}_0(Z_i)$ for all i. Then embed $U \hookrightarrow \mathbb{A}^n$ and choose a hyperplane through 0 not containing t. Add the closure Z of $H \cap U$ in A to the set $\{Z_i\}$ and continue in this way until $\cap \mathrm{T}_0(Z_i) = \{0\}$. [I don't understand what Milne does, I hope it is a typo] □

Definition 12. A divisor D on V is *very ample* if the complete linear system it defines gives a closed immersion of V into $\mathbb{P}^n$. A divisor D on V is *ample* if nD is very ample for some $n \geq 0$. △

[In the case that V is a smooth projective variety over algebraically closed k, a divisor is (very) ample if and only if the isomorphism class of invertible sheaves associated to its complete linear system is (very) ample. I think this is essentially the content of the Hartshorne proposition above.]

Milne asserts that if D is ample on an abelian variety A, then $3D$ is always very ample. He also asserts that it's difficult to prove (and doesn't prove it), but we saw an example of this a bit earlier.

Proposition 13. *We have:*

(a) If D and D' are ample, then so is $D + D'$.

(b) If D is an ample divisor on V, then $D|_W$ is ample for any closed subvariety W of V, assuming $D|_W$ is defined.

(c) A divisor D on V is ample if and only if its extension of scalars to $\bar{k}$ is ample

(d) A variety V has an ample divisor if $V_{\bar{k}}$ has an ample divisor.

Proof. For part (a), take n so that nD and nD' are very ample; observe that nD' is linearly equivalent to an effective divisor D_2, and $L(nD + D_2) \supseteq L(nD)$. Hence $nD + D_2$ is very ample, and since $nD + D_2$ defines the same complete linear system as $nD + nD'$, it follows that $nD + nD'$ is very ample, we win.

For part (b), we observe that "the restriction of the map defined by D to W is the map defined by the restriction of D to W".

For part (c), the map obtained by the extension of scalars from the map $V \to \mathbb{P}^n$ defined by D is that defined by $D_{\bar{k}}$.

For part (d), let D be an ample divisor on $V_{\bar{k}}$. Since D is defined over some finite extension k' of k, the set $\{\sigma D \colon \sigma \in \mathrm{Aut}(\bar{k}/k)\}$ is finite. The sum of the σD, which is again a divisor we will denote by D_0, is ample by part (a). Note that D_0 is defined over a purely inseparable extension of k. Possibly having to multiply by some power of char k, we get that D_0 is defined over k. Part (c) finishes the proof. □