Abelian Varieties (over $\mathbb{C}$)

Sept 15, 2019

I'm essentially just following Milne's notes, with some more details added here and there.
In case you have forgotten [since, low-key, I had forgotten...], let's just state

Definition 1. An *abelian variety* is a complete, connected group variety. △

Let me finish the last bit of Chapter 1. It's a nice structure theorem about morphisms to abelian varieties. (Last time, we proved the ridiculously nice structure theorem about the varieties themselves, namely, that the underlying group is abelian...)

Corollary 2. *Let V and W be complete varieties over k with k-rational points v_0 and w_0. Let π_V and π_W be the projection maps $V \times W \to V$ and $V \times W \to W$, respectively. Let A be an abelian variety. Then, a morphism $h\colon V \times W \to A$ such that $h(v_0, w_0) = 0$ can be written uniquely as $h = f \circ \pi_V + g \circ \pi_W$, with $f\colon V \to A$ and $g\colon W \to A$ morphisms with $f(v_0) = g(w_0) = 0$.*

Proof. Certainly, we would need $f = h|_{V\times\{w_0\}}$ and $g = h|_{\{v_0\}\times W}$. The difference map

$$\Delta \stackrel{\text{def}}{=} h - (f \circ \pi_V + g \circ \pi_W)$$

sends $V \times \{w_0\}$ and $W \times \{v_0\}$ to 0, so rigidity implies $\Delta = 0$. □

Let's talk about abelian varieties over $\mathbb{C}$. As Guanyu asserted last week, abelian varieties are projective (although this won't be proven until a long time from now.)

Since we haven't proven this yet, let us fix an abelian variety A over $\mathbb{C}$ and assume it is projective. [Keep in mind the case where A is an elliptic curve over $\mathbb{C}$.] Note that A is nonsingular, closed inside $\mathbb{P}^n(\mathbb{C})$, and connected for the Zariski topology (by definition). That is to say, A is a compact, connected complex manifold (i.e. with holomorphic transition maps). Furthermore, it has a (commutative) group structure, in particular $G \times G \to G$ given by $(x, y) \mapsto x - y$ is a regular map between complex algebraic varieties and hence[!] a holomorphic map.

Thus, $A(\mathbb{C})$ is a compact, connected, commutative, complex Lie group, which implies $A(\mathbb{C}) \cong \mathbb{C}^g/L$, where $g \stackrel{\text{def}}{=} \dim(A)$ and L is a full lattice in $\mathbb{C}^g$. Milne gives a more detailed proof; essentially everything follows from the existence of a homomorphism

$$\exp\colon T_0(A(\mathbb{C})) \to A(\mathbb{C})$$

so that the differential of $\exp$ at 0 is the identity map on $T_0(A(\mathbb{C}))$. If you have this fact, the inverse function theorem says that $\exp$ is a local isomorphism at 0, hence its image is clopen in the connected space $A(\mathbb{C})$. So it is surjective. But furthermore, $\exp$ is injective around 0, so its kernel is a lattice of $T_0(A(\mathbb{C})) \cong \mathbb{C}^g$, and since $A(\mathbb{C})$ is compact, its kernel must be a full lattice.

Remark 3. While we've shown that $A(\mathbb{C}) \cong \mathbb{C}^g/L$ for some full L, and for $g = 1$ every quotient is an elliptic curve, the converse does not hold in general. △

Remark 4. In fact, since abelian varieties were merely defined as complete, connected group varieties, we've shown that a compact, connected, complex Lie group is necessarily abelian. [... right?] △

Milne goes over some basic algebraic topology [which I think everybody in the room knows better than I do]. As manifolds, tori obey everything we might want it to, namely:

- It is a finite CW complex, so all cohomologies agree,

- It obeys the Künneth formula, in particular since $H^r((S^1)^x, \mathbb{Z})$ and $H^s((S^1)^y, \mathbb{Z})$ are free $\mathbb{Z}$-modules for all r, s, there is a canonical isomorphism

$$H^m((S^1)^x \times (S^1)^y, \mathbb{Z}) \cong \bigoplus_{r+s=m} H^r(X, \mathbb{Z}) \otimes H^s(Y, \mathbb{Z})$$

given by cup producting,

- It satisfies $H^1((S^1)^n, \mathbb{Z}) \cong \mathrm{Hom}(\pi_1((S^1)^n, x), \mathbb{Z})$, by the universal coefficient theorem and the fact that $H_1 = \pi_1^{\mathrm{ab}}$,

- There are canonical isomorphisms

$$H^r((S^1)^n, \mathbb{Z}) \cong H_{n-r}((S^1)^n, \mathbb{Z}) \cong H^{n-r}((S^1)^n, \mathbb{Z})^\vee,$$

the first coming from Poincaré and the second from universal coefficient theorem.

The Künneth formula gives

$$\dim H^r((S^1)^n, \mathbb{Z}) = \binom{n}{r},$$

and even more explicitly we have

Theorem 5. *Let X be the torus V/L. There are canonical isomorphisms*

$$\bigwedge^r H^1(X, \mathbb{Z}) \to H^r(X, \mathbb{Z}) \to \mathrm{Hom}\left(\bigwedge^r L, \mathbb{Z}\right).$$

Proof. The first isomorphism is given by taking the cup product; because X is a manifold it is an isomorphism. (Alternatively, because all cohomology theories agree, we can take $H^* = H^*_{dR}$.) In light of $H^1(X, \mathbb{Z}) \cong \mathrm{Hom}(\pi_1(X, x), \mathbb{Z})$ [cf. algtop final!], and the fact that V is a universal cover of $X = V/L$, we get $\pi_1(X, x) = L$ and

$$H^1(X, \mathbb{Z}) \cong \mathrm{Hom}(L, \mathbb{Z}).$$

Furthermore, we have a pairing

$$\bigwedge^r L^\vee \times \bigwedge^r L \to \mathbb{Z} \qquad (f_1 \wedge \cdots \wedge f_r, e_1 \wedge \cdots \wedge e_r) \mapsto \det(f_i(e_j))$$

realizing each group as the $\mathbb{Z}$-linear dual of the other. Since $L^\vee = H^1(X, \mathbb{Z})$, we get

$$\bigwedge^r H^1(X, \mathbb{Z}) \cong \mathrm{Hom}\left(\bigwedge^r L, \mathbb{Z}\right).$$

□

Let me recall some definitions/theory, which I promise will be useful for us.

Definition 6. A *Hermitian form* on a complex vector space V is a map $H\colon V \times V \to \mathbb{C}$ that is linear in the first coordinate and conjugate-symmetric, so:

1. $H(av_1 + bv_2, w) = aH(v_1, w) + bH(v_2, w)$, and
2. $H(v, w) = \overline{H(w, v)}$.

A *skew-symmetric form* on a complex vector space V is a map $E\colon V \times V \to \mathbb{C}$ that is, bilinear and well, skew-symmetric, so:

1. $E(av_1 + bv_2, w) = aE(v_1, w) + bE(v_2, w)$
2. $H(v, w) = -H(w, v)$

△

Lemma 7. *Let V be a complex vector space. There is a one-to-one correspondence*

$$\{\textit{Hermitian forms on } V\} \leftrightarrow \{\mathbb{R}\textit{-valued, } \mathbb{R}\textit{-bilinear, skew-symmetric forms on } V \textit{ with } E(iv, iw) = E(v, w)\}$$
$$H(v, w) \mapsto Im(H(v, w))$$
$$E(iv, w) + iE(v, w) \leftarrow E(v, w)$$

Proof. Check that the forms satisfy everything they should, and check that these maps are inverses. □

Definition 8. Let $X = V/L$ be a complex torus of dimension g. Let E be a skew-symmetric form $L \times L \to \mathbb{Z}$. Since $L \otimes \mathbb{R} = V$, we can extend E to a skew-symmetric $\mathbb{R}$-bilinear form $E_{\mathbb{R}} \colon V \times V \to \mathbb{R}$. We say E is a *Riemannian form* if:

1. $E_{\mathbb{R}}(iv, iw) = E_{\mathbb{R}}(v, w)$
2. The Hermitian form is positive-definite. △

Definition 9. We say X is *polarizable* if it admits a Riemannian form. △

Remark 10. Most other sources I've found define a Riemann form to be a Hermitian form $H \colon V \times V \to \mathbb{C}$ so that $\text{Im}(H(L \times L)) \subseteq \mathbb{Z}$. Then a Riemann form in the sense of Milne is a positive-definite Riemann form in the sense of other sources, and a polarization is a choice of positive-definite Riemann form. △

Remark 11. With the above definition it's really easy to check that when $X = V/L$ has complex dimension 1, then X is polarizable; let $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, and check that

$$H(z, w) = \frac{z\overline{w}}{\text{Im}(\omega_1\overline{\omega_2})}$$

works. △

Remark 12. "Most" complex tori are not polarizable. △

Theorem 13. *A complex torus X is of the form $A(\mathbb{C})$ if and only if it is polarizable.*

Proof. The data of a positive-definite Riemannian form is the same as the data of an ample line bundle (this is the *Appel-Humbert theorem*$^{++}$), which is more or less the same as giving an embedding of X into projective space. I'm stealing punchlines from page 3 of Brian Conrad's writeup. The proof sketch in Milne essentially says the same thing; my understanding is that this is quite a nontrivial theorem. □

Definition 14. A morphism $V/L \to V'/L'$ between complex tori is a $\mathbb{C}$ linear map $V \to V'$ mapping L into L'. (These are in fact all the holomorphic homomorphisms $X \to X'$.) △

Theorem 15. *The functor $A \mapsto A(\mathbb{C})$ is an equivalence from the category of abelian varieties over $\mathbb{C}$ to the category of polarizable tori.*

Definition 16. An *isogeny* of polarizable tori is a surjective homomorphism with finite kernel. The *degree* of the isogeny is the order of he kernel. Polarizable tori X and Y are said to be *isogenous* if there exists an isogeny $X \to Y$. △