Hikita surjectivity for $\mathcal{N} /\!\!/\!\!/\!\!/ T$

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October 4, 2025

arXiv:2410.16217

Flag variety

$$\begin{aligned} \text{Variety } X &\rightsquigarrow \text{coordinate ring } \mathbb{C}[X] \\ &\rightsquigarrow \text{cohomology } H^*(X) \qquad (\text{or } H^*_T(X)) \end{aligned}$$

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Theorem (Borel, 1953)

Write $\mathcal{F}\ell_n :=$ (type A) flag variety in $\mathbb{C}^n$. Then

$$H^*(\mathcal{F}\ell_n) = \frac{\mathbb{C}[x_1, \dots, x_n]}{\langle p_1(\mathbf{x}), \dots, p_n(\mathbf{x})\rangle}, \qquad p_i(\mathbf{x}) := x_1^i + \cdots + x_n^i$$

$$H^*_{T_{\mathrm{SL_n}}}(\mathcal{F}\ell_n) = \frac{\mathbb{C}[x_1, \dots, x_n; y_1, \dots, y_n]}{\left\langle \begin{matrix} p_i(\mathbf{x}) - p_i(\mathbf{y}), i \in [n], \\ x_1 + \cdots + x_n \end{matrix} \right\rangle}$$

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where $\mathfrak{t} = \{x_1 + \cdots + x_n = 0\} \subseteq \mathbb{C}^n$.

Who is $H_T^*(\mathcal{F}\ell_n)$?

Claim

$\mathfrak{t} \times_{\mathfrak{t}/S_n} \mathfrak{t}$ *is your friend.*

$$\begin{aligned}\mathfrak{t} \times_{\mathfrak{t}/S_n} \mathfrak{t} &= \Big\{(x_1, \dots, x_n), (y_1, \dots, y_n) \colon \{\{x\}\} = \{\{y\}\}\Big\} \\ &= \bigcup_{w \in S_n} \underbrace{\{x_i = y_{w(i)} \text{ for all } i\}}_{:= V_w}\end{aligned}$$

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$$V_w = \mathrm{graph}\begin{pmatrix} \mathfrak{t} \to \mathfrak{t} \\ t_i \mapsto t_{w(i)} \end{pmatrix} \cong \mathfrak{t}$$

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Conclusion

$$\mathbb{C}[\mathfrak{t} \times_{\mathfrak{t}/S_n} \mathfrak{t}] = \{(f_w)_{w \in S_n} \in \mathbb{C}[\mathfrak{t}]^{\oplus n!} \colon f_w \text{ glue well}\} \qquad \text{(cf. [GKM])}$$

Who is $H^*(\mathcal{F}\ell_n)$?

$$\mathcal{N} = \{\text{nilpotent } n \times n \text{ matrices}\}$$

$$\mathbb{C}[\mathcal{N}] = \frac{\mathbb{C}[\mathrm{Mat}_{n,n}]}{\langle \mathrm{tr}(A), \mathrm{tr}(A^2), \dots \rangle}$$

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$$\mathbb{C}[\mathcal{N}^T] = \mathbb{C}[\mathcal{N} \cap \mathfrak{t}] = \frac{\mathbb{C}[x_1, \ldots, x_n]}{\langle p_1(\mathbf{x}), p_2(\mathbf{x}), \ldots \rangle} = H^*(\mathcal{F}\ell_n).$$

Hikita's observation

Observation/Conjecture (Hikita, 2017)

For many varieties $\widetilde{X}$ of representation theoretic interest,

$$H^*(\widetilde{X}) = \mathbb{C}[(X^!)^{T^!}]$$

for some other interesting variety $X^!$ and torus $T^!$.

Example (Hikita, 2017)

$$\widetilde{X} = \mathcal{F}\ell_n \rightsquigarrow X^! = \mathcal{N} \quad \text{[Borel, '53]}$$

$$\widetilde{X} = \text{Springer fiber} \rightsquigarrow X^! = \text{nilpotent orbit} \quad \text{[de Concini–Procesi, '81]}$$

$$\widetilde{X} = \mathrm{Hilb}^n(\mathbb{C}^2) \rightsquigarrow X^! = \mathrm{Sym}^n(\mathbb{C}^2)$$

$$\widetilde{X} = G/P \rightsquigarrow X^! = \text{Slodowy slice}$$

$$\widetilde{X} = \text{hypertoric variety} \rightsquigarrow X^! = \text{Gale dual}$$

Physics magic...

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Prediction (Dancer–Hanany–Kirwan, 2021)

Theory of *symplectic duality* predicts...

- The variety $\mathcal{N} /\!\!/\!\!/ T = \{x \in \mathcal{N} \colon \mathrm{diag}(x) = 0\} /\!\!/ T$ has a *symplectic resolution* $\widetilde{\mathcal{N} /\!\!/\!\!/ T}$,
- The Hikita conjecture holds for $\widetilde{X} = \widetilde{\mathcal{N} /\!\!/\!\!/ T}$ and $X^! = \overline{T^*(\mathrm{SL}_n/U)}$, where $U = \left\{ \begin{bmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{bmatrix} \right\}$.

Hikita surjectivity

Theorem (S.)

$\mathcal{N} /\!\!/\!\!/\!\!/ T$ has a symplectic resolution $\widetilde{\mathcal{N} /\!\!/\!\!/\!\!/ T}$, and there is a surjection

$$\mathbb{C}\left[\overline{T^*(\mathrm{SL}_n/U)}^{T \times B/U}\right] \xrightarrow{\varphi^*} H^*\left(\widetilde{\mathcal{N} /\!\!/\!\!/\!\!/ T}\right).$$

(The T and B/U actions on $\overline{T^*(\mathrm{SL}_n/U)}$ are induced by left, right mult. on SL_n/U.)

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Remark

- Hikita conj. is equivalent to $\dim \mathbb{C}[\cdot] = \dim H^*(\cdot)$. idk either dimension count
- $H^{\mathrm{odd}}(\widetilde{\mathcal{N} /\!\!/\!\!/\!\!/ T}) = 0$, so $\dim H^*(\cdot) = \chi(\cdot)$. $\mathbb{C}^\times$ acts on $\widetilde{\mathcal{N} /\!\!/\!\!/\!\!/ T}$ but idk $(\widetilde{\mathcal{N} /\!\!/\!\!/\!\!/ T})^{\mathbb{C}^\times}$

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- $\mathcal{N} /\!\!/\!\!/\!\!/ T$ is a Nakajima quiver variety; Kirwan map descends to φ^*.

Geometry of $\overline{T^*(\mathrm{SL}_n/U)}$

Theorem (S.)

For $A, B \subseteq [n]$, set $f_{A,B} := \prod_{\substack{a\in A\\ b\in B}}(x_a - y_b) \in \mathbb{C}[\mathfrak{t} \times_{\mathfrak{t}/S_n} \mathfrak{t}]$. Then

$$\mathbb{C}\left[\overline{T^*(\mathrm{SL}_n/U)}^{T\times B/U}\right] \cong \frac{\mathbb{C}[\mathfrak{t} \times_{\mathfrak{t}/S_n} \mathfrak{t}]}{\langle f_{A,B} \colon |A| + |B| = n\rangle}.$$

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- As a representation of B/U,

$$\mathbb{C}[\overline{T^*(\mathrm{SL}_n/U)}] \cong \bigoplus_{\alpha \in \Lambda_{B/U}} \Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}(\alpha)).$$

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- $T^*(\mathrm{SL}_n/U)$ is not S_n-invariant!

Thank you!

Theorem (S.)

Surjectivity part of Hikita conjecture for $X = \mathcal{N} /\!\!/\!\!/\!\!/ T$:

$$\mathbb{C}\left[\overline{T^*(\mathrm{SL}_n/U)}^{T\times B/U}\right] \xrightarrow{\varphi^*}\!\!\!\!\!\to H^*\left(\widetilde{\mathcal{N} /\!\!/\!\!/\!\!/ T}\right).$$

Theorem (S.)

Coordinate ring has an explicit description:

$$\mathbb{C}\left[\overline{T^*(\mathrm{SL}_n/U)}^{T\times B/U}\right] \cong \frac{\mathbb{C}[\mathfrak{t}\times_{\mathfrak{t}/S_n}\mathfrak{t}]}{\langle f_{A,B} \colon |A|+|B|=n\rangle}, \quad f_{A,B} := \prod_{\substack{a\in A\\ b\in B}}(x_a - y_b)$$

Hope and dream

Show $\dim \mathbb{C}[\cdot] = \dim H^*(\cdot)$, or compute anything...