Double orthodontia formulas and Lascoux positivity

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Outline

- Context: Flagged Schur modules $\mathcal{M}_D$, orthodontia, saturation, filtrations
- Orthodontia formula for double Grothendieck polynomials $\mathfrak{G}_w$
- Application: Lascoux positivity

Goal: Analogue of $\mathcal{M}_D$ for $\mathfrak{G}_w$.

Schubert polynomials

Definition

For $w \in S_n$, the *Schubert polynomial* is:

$$\mathfrak{S}_w(\mathbf{x}) = \begin{cases} x_1^{n-1} x_2^{n-2} \dots x_{n-1} & \text{if } w = w_0 \\ \partial_i(\mathfrak{S}_{ws_i}(\mathbf{x})) & \text{if } \ell(w) < \ell(ws_i), \end{cases}$$

where $\partial_i(f) := \frac{f - s_i f}{x_i - x_{i+1}}$.

The $\mathfrak{S}_w$ lift *Schubert cycles* $[X_w] \in H^*(\mathcal{F}\ell(n))$.

Schur polynomials

Example

Schur polynomial $s_\lambda := \mathrm{ch}(V_\lambda)$ is $\mathfrak{S}_{w_\lambda}$ for appropriate $w = w_\lambda$.

$\mathrm{ch}(W)$ is the polynomial $\sum_\alpha \dim(W_\alpha)\mathbf{x}^\alpha$, i.e. it records dims of weight spaces as coeffs.

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$\mathrm{ch}(W)$ is the polynomial $\sum_\alpha \dim(W_\alpha)\mathbf{x}^\alpha$, i.e. it records dims of weight spaces as coeffs.

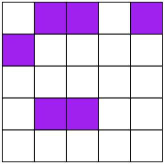
$$[X_u] \cdot [X_v] = \sum_w c_{uv}^w [X_w] \quad \leftrightsquigarrow \quad V_\lambda \otimes V_\mu = \bigoplus_\nu V_\nu^{\oplus c_{\lambda\mu}^\nu}$$

intersection nos. $\leftrightsquigarrow$ multiplicities of irreps

c_{uv}^w: “Littlewood–Richardson coefficients”

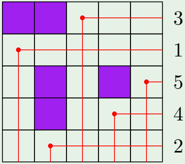
Central problem: Combinatorial formula for c_{uv}^w?

Diagrams

A diagram D is a collection of boxes, e.g. 

Example

$$w \rightsquigarrow D(w) \qquad \text{“\textit{Rothe diagram}”}$$

D(31542): 

Flagged Schur modules

$$D \rightsquigarrow \mathcal{M}_D \qquad \text{“\textit{flagged Schur module}”}$$

(representation of $B := \{\text{upper triangular matrices}\} \subseteq \mathrm{GL}_n$)

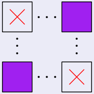
Theorem (Kraśkiewicz–Pragacz '87)

The character $\mathrm{ch}(\mathcal{M}_{D(w)})$ is the Schubert polynomial $\mathfrak{S}_w$.

(This extends $[X_w] \leftrightsquigarrow V_\lambda$; cf. [Watanabe '15].)

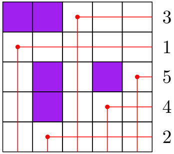
%-avoiding diagrams

Definition (Reiner–Shimozono '98)

A diagram is *%-avoiding* if there is no instance of  .

Example

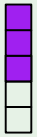
The Rothe diagram $D(w)$ is %-avoiding for all $w \in S_n$.



Orthodontic sequence

Game

Start with %-avoiding diagram D.

- If there is an "up-justified" column  of D, remove it;
- Otherwise, swap rows i and $i+1$, where $i = i_1(D)$:

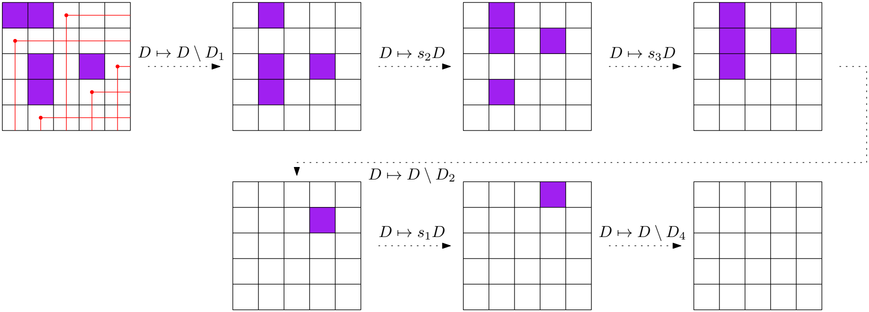
Orthodontic sequence

Game

Start with %-avoiding diagram D.

- If there is an "up-justified" column of D, remove it;
- Otherwise, swap rows i and $i+1$, where $i = i_1(D)$:

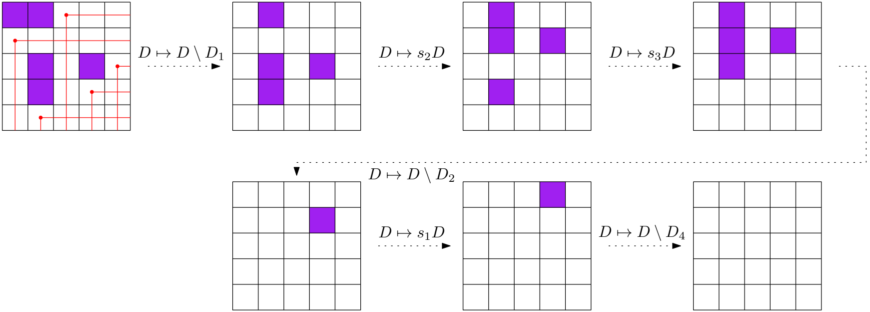
Let D_j = leftmost nonempty column of D. The *first missing tooth* $i_1(D)$ is $\min\{i\colon i \notin D_j, i+1 \in D_j\}$.



Orthodontia formula for flagged Schur modules

Theorem (Magyar '98, "orthodontia formula")

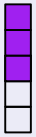
- *Every diagram which appears is again %-avoiding.*
- *The empty diagram eventually appears.*



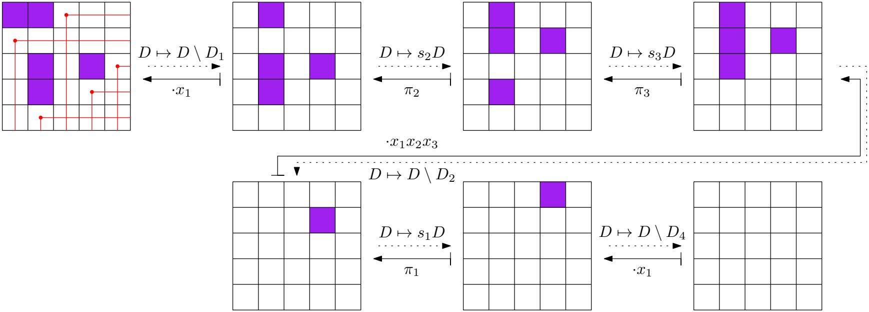
Orthodontia formula for flagged Schur modules

Theorem (Magyar '98, "orthodontia formula")

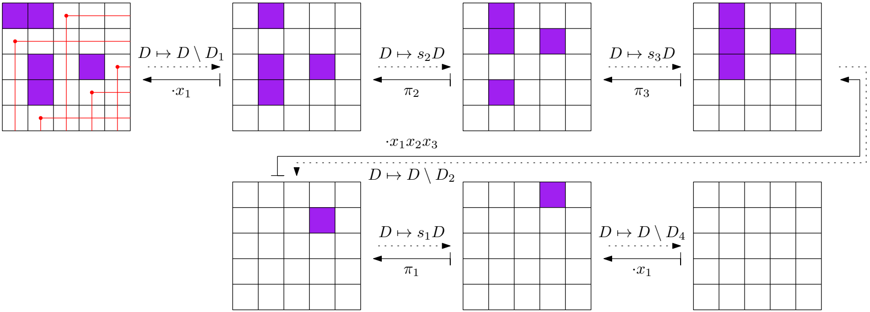
- *Every diagram which appears is again %-avoiding.*
- *The empty diagram eventually appears.*
- $$\mathrm{ch}(\mathcal{M}_D) = \begin{cases} x_1 \ldots x_i \cdot \mathrm{ch}(\mathcal{M}_{D \setminus D_j}) & \text{if } \exists\, D_j = \square \text{ with } i \text{ squares} \\ \pi_i(\mathrm{ch}(\mathcal{M}_{s_i D})) & \text{otherwise,} \end{cases}$$

  where $i = i_1(D)$ and $\pi_i(f) := \partial_i(x_i f)$.



Orthodontia formula for flagged Schur modules, II



Orthodontia computes $\mathfrak{S}_w = \mathrm{ch}(\mathcal{M}_{D(w)})$, but using non-Rothe diagrams!

Double Grothendieck polynomials

Definition

For $w \in S_n$, the *double Grothendieck polynomial* is

$$\mathfrak{G}_w(\mathbf{x};\mathbf{y}) = \begin{cases} \prod_{i+j\leq n}(x_i + y_j - x_i y_j) & \text{if } w = w_0 \\ \overline{\partial}_i(\mathfrak{G}_{ws_i}(\mathbf{x};\mathbf{y})) & \text{if } \ell(w) < \ell(ws_i), \end{cases}$$

where $\overline{\partial}_i(f) := \partial_i((1 - x_{i+1})f)$.

The $\mathfrak{G}_w(\mathbf{x};\mathbf{y})$ lift structure sheaves $[\mathcal{O}_{X_w}] \in K_T^*(\mathcal{F}\ell_n)$.

Orthodontia for double Grothendieck polynomials

Goal

For %-avoiding D, define $\mathscr{G}_D$ *so that* $\mathscr{G}_{D(w)} = \mathfrak{G}_w(\mathbf{x};\mathbf{y})$.

(No known analogue of $\mathcal{M}_D$ for $\mathfrak{G}_w$!)

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Theorem (S.–St. Dizier)

For %-avoiding D, define

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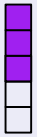

where $i = i_1(D)$ and $j := k - \#\{\ell \leq i_1(D) : \ell \notin D_k\}$. Then $\mathscr{G}_{D(w)} = \mathfrak{G}_w(\mathbf{x};\mathbf{y})$.

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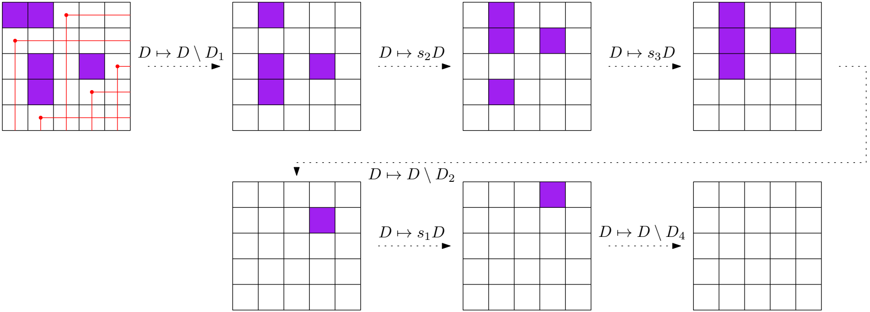

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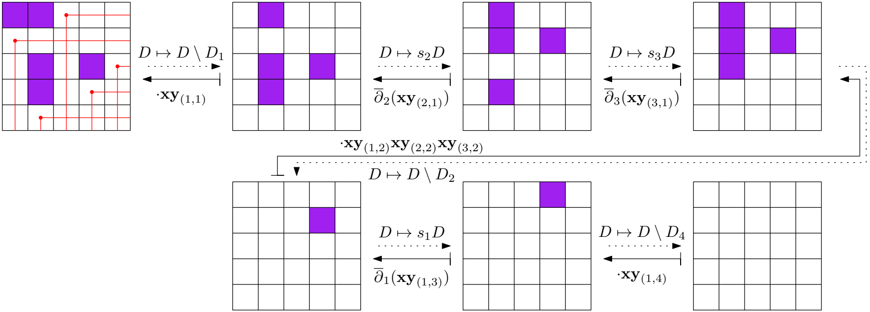

Application: positivity results

Definition

For $\alpha \in \mathbb{Z}^n_{\geq 0}$, the *key polynomial* and *Lascoux polynomial* are:

$$\kappa_\alpha(\mathbf{x}) = \begin{cases} x_1^{\alpha_1} \dots x_n^{\alpha_n} & \text{if } \alpha_1 \geq \cdots \geq \alpha_n \\ \pi_i(\kappa_{s_i\alpha}(\mathbf{x})) & \text{if } \alpha_i < \alpha_{i+1}, \end{cases}$$

$$\mathfrak{L}_\alpha(\mathbf{x}) = \begin{cases} x_1^{\alpha_1} \dots x_n^{\alpha_n} & \text{if } \alpha_1 \geq \cdots \geq \alpha_n \\ \pi_i((1 - x_{i+1}) \cdot \mathfrak{L}_{s_i\alpha}(\mathbf{x})) & \text{if } \alpha_i < \alpha_{i+1}. \end{cases}$$

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For %-avoiding D, $\mathrm{ch}(\mathcal{M}_D)$ is a $\mathbb{Z}_{\geq 0}$-sum of κ_α.

Idea.

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Orthodontia: $\mathrm{ch}(\mathcal{M}_D)$ can be obtained from $1 \in \mathbb{C}[\mathbf{x}]$ by repeated application of $\cdot x_1 \dots x_i$ and π_i.

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Theorem (Reiner–Shimozono '98)

For %-avoiding D, $\mathrm{ch}(\mathcal{M}_D)$ *is a* $\mathbb{Z}_{\geq 0}$*-sum of* κ_α*.*

Idea.

Orthodontia: $\mathrm{ch}(\mathcal{M}_D)$ can be obtained from $1 \in \mathbb{C}[\mathbf{x}]$ by repeated application of $\cdot x_1 \dots x_i$ and π_i. Easy: $\pi_i(\kappa_\alpha) = \kappa_{\alpha'}$.
[RS'98] proves $x_1 \dots x_i \kappa_\alpha$ is κ-positive. ("Pieri rule for κ_α") $\square$

A curious Lascoux positivity conjecture

Conjecture (S.–St. Dizier)

If D is %-avoiding, $x_1^n \dots x_n^n \mathscr{G}_D^{\mathrm{bot}}(x_n^{-1}, \dots, x_1^{-1}; -1, \dots, -1)$ is a graded nonnegative sum of Lascoux polynomials.

Proof??

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- $f \mapsto \pi_i((1 - x_{i+1})f)$,
- $f \mapsto x_1 \dots x_i(1 - x_{i+1}) \dots (1 - x_n)f$.

Easy: $\pi_i((1 - x_{i+1})\mathfrak{L}_\alpha) = \mathfrak{L}_{\alpha'}$.

Enough to show: $\mathfrak{L}_\alpha \cdot x_1 \dots x_i(1 - x_{i+1}) \dots (1 - x_n)$ is graded $\mathfrak{L}$-positive. $\diamondsuit$

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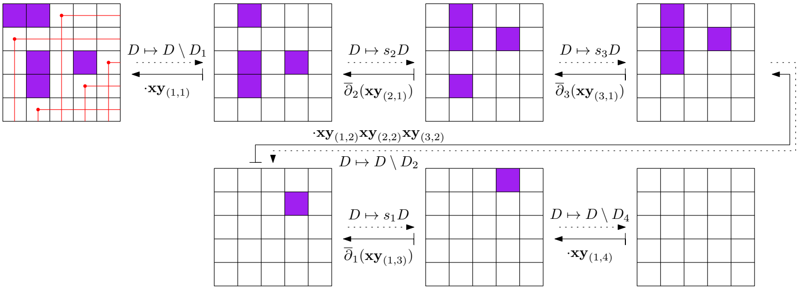
Theorem (S.–St. Dizier)

Conjecture holds when columns of D can be ordered by inclusion.

Thank you!

Theorem (S.–St. Dizier)

When $D = D(w)$ is a Rothe diagram, $\mathscr{G}_{D(w)} = \mathfrak{S}_w(\mathbf{x};\mathbf{y})$.



Theorem (S.–St. Dizier)

When the columns of D can be ordered by inclusion, the polynomial $x_1^n \dots x_n^n \mathscr{G}_D^{\mathrm{bot}}(x_n^{-1}, \dots, x_1^{-1}; -1, \dots, -1)$ is graded $\mathfrak{L}$-positive.