Newton polytopes of Schubert and Grothendieck polynomials

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AlGeCom XXVI

Joint with Elena Hafner, Karola Mészáros, Avery St. Dizier
and with Jack Chen-An Chou

Schubert polynomials

Definition

For $w \in S_n$, the *Schubert polynomial* is:

$$\mathfrak{S}_w(\mathbf{x}) = \begin{cases} x_1^{n-1} x_2^{n-2} \dots x_{n-1} & \text{if } w = w_0 \\ \partial_i(\mathfrak{S}_{ws_i}(\mathbf{x})) & \text{if } \ell(w) < \ell(ws_i), \end{cases}$$

where $\partial_i(f) := \frac{f - s_i f}{x_i - x_{i+1}}$.

The $\mathfrak{S}_w$ lift *Schubert cycles* $[X_w] \in H^*(\mathcal{F}\ell(n))$.

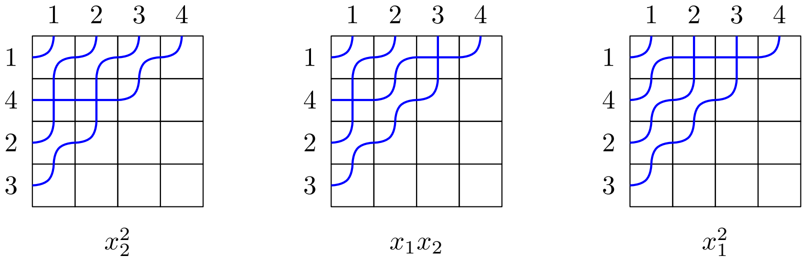
Pipe dreams

Theorem

Schubert polynomials are counted by reduced pipe dreams:

$$\mathfrak{S}_w = \sum_{\substack{P \in \mathrm{PD}(w) \\ P \text{ reduced}}} \mathbf{x}^{\mathrm{wt}(P)}.$$

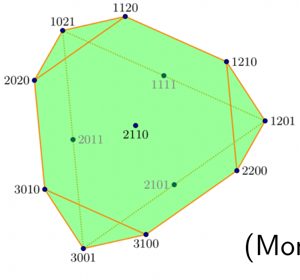
Reduced pipe dreams of $w = 1423$:



Saturation

Question

Assume that $\mathbf{x}^{\alpha-\beta}$ and $\mathbf{x}^{\alpha+\beta}$ appear in $\mathfrak{S}_w$. Does $\mathbf{x}^{\alpha}$ appear?



(Monomials in $\mathfrak{S}_{21543}$)

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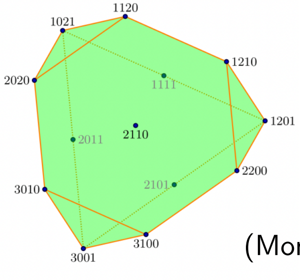

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Conjecture (Monical–Tokcan–Yong, '17)

$\mathrm{supp}(\mathfrak{S}_w) := \{\alpha\colon \mathbf{x}^{\alpha} \text{ appears in } \mathfrak{S}_w\}$ *is saturated.*

(*Saturated*: $S = \mathrm{conv}(S) \cap \mathbb{Z}^n$.)

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Definition

A set $S \subseteq \mathbb{Z}^n$ is M-convex if:

- S is saturated
- $\mathrm{conv}(S)$ is a generalized permutahedron

$$S,\ T \text{ are saturated } \not\Longrightarrow S + T \text{ is saturated.}$$
$$S,\ T \text{ are M-convex } \Longrightarrow S + T \text{ is M-convex.}$$

M-convexity!

Theorem (Fink–Mészáros–St. Dizier, '17)

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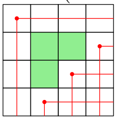
- Start with *Rothe diagram* $D(w)$ of w.

$D = D(1432)$



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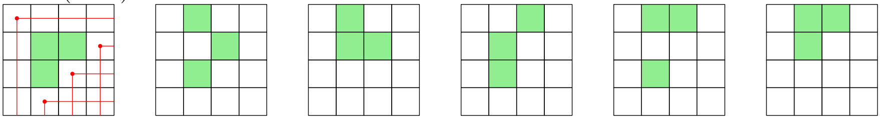

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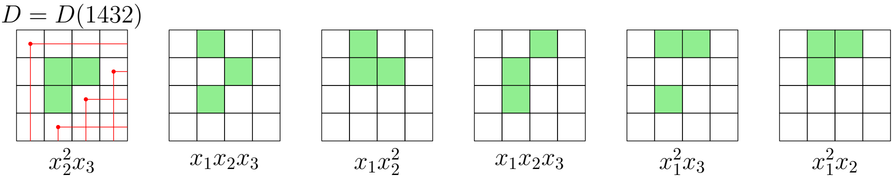

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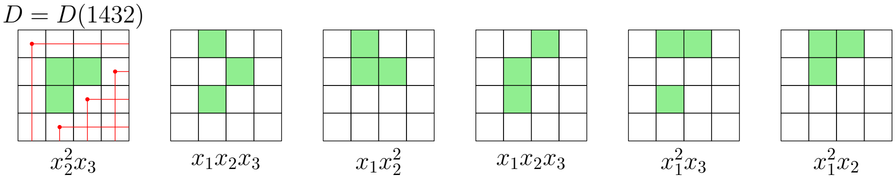

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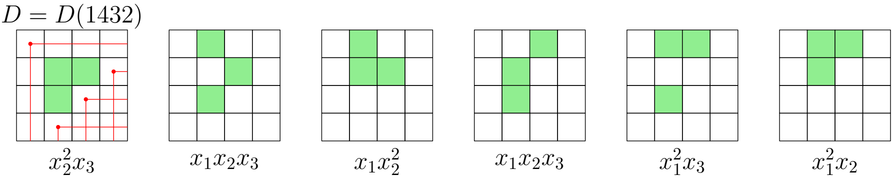

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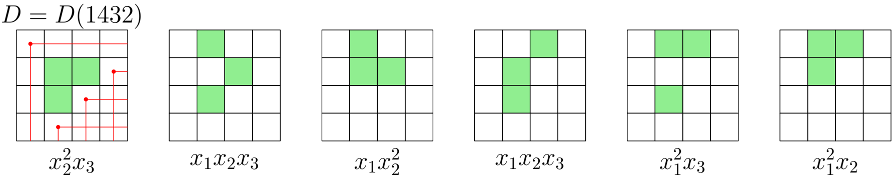
- "Easy": If C is a column, S_C is M-convex. □



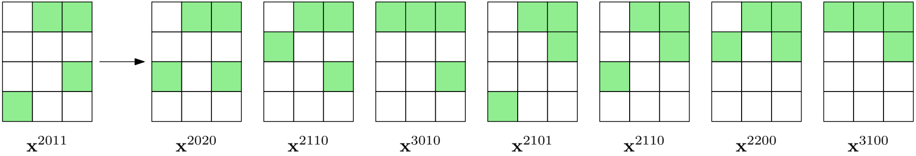
Schubitope

Definition

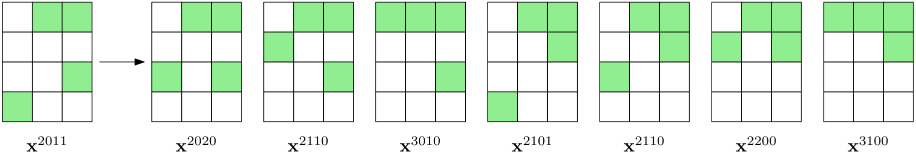
The *Schubitope* of a diagram D is

$$\mathcal{S}_D := \mathrm{conv}\{\mathrm{wt}(C) \colon C \in B_D\},$$

where $B_D = \{$diagrams obtained by bubbling boxes of D upwards$\}$.



Schubitope fun facts



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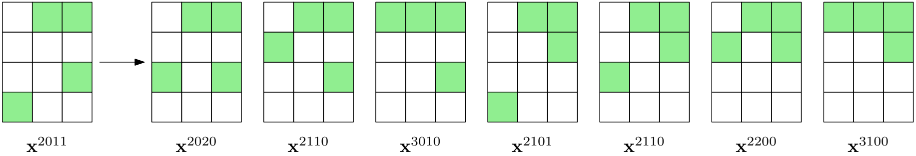

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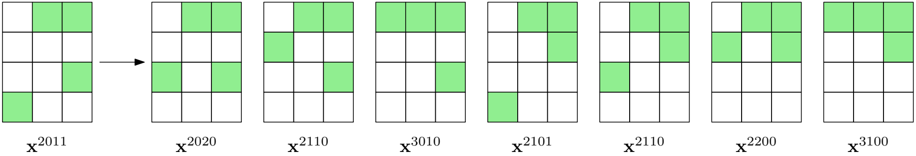

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- {Schubitopes} $\subset$ {generalized permutahedra} is "set of $\mathbb{Z}$-points of full dim subcone". [Hafner–Mészáros–S.–St. Dizier, '23]

Applications of Schubitopes

- M-convexity of $\mathrm{supp}(\mathfrak{S}_w)$
- Sufficient vanishing criteria for $c_{u,v}^w$ [St. Dizier–Yong, '20]

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  $\rightsquigarrow [x^\alpha]\mathfrak{S}_w = 1$ whenever α is a vertex of $\mathcal{S}_{D(w)}$.
- Lower bounds: $\mathfrak{S}_w(1, \ldots, 1) \geq |\mathcal{S}_{D(w)} \cap \mathbb{Z}^n|$.

Pipe dream maximizers

Motivation (Stanley, '17)

Write $u(n) := \max_{w \in S_n} \mathfrak{S}_w(1, \ldots, 1)$. Then

$$2^{n^2/4} \leq u(n) \leq 2^{n^2/2} \qquad \text{(up to "small" factors).}$$

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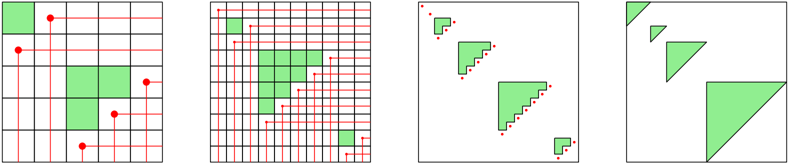
$\mathfrak{S}_w(1, \ldots, 1)$ *is maxed at a layered permutation.*



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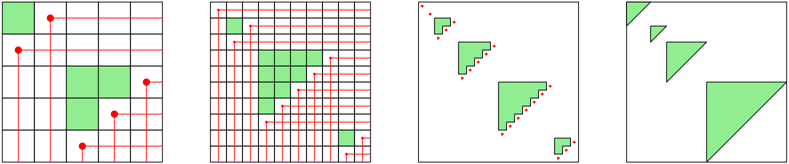

[Morales–Pak–Panova, '18]: Among layered $w \in S_n$, $\max \mathfrak{S}_w(\mathbf{1}) \approx 2^{0.293262762\ldots \cdot n^2}$

[Morales–Panova–Petrov–Yeliussizov, '24]: $\mathfrak{G}_w^{(\beta=1)}(\mathbf{1})$ is asymptotically maxed at layered.

Schubitope maximizers

Best lower bound for $\mathfrak{S}_w(1,\ldots,1)$:

Theorem (Guo–Lin, '24)

Let $p_u(w) :=$ number of u-patterns in w. *Then* $|\mathrm{supp}(\mathfrak{S}_w)|$ *is at least*

$$1+p_{132}(w)+p_{1432}(w)+p_{13524}(w)+3p_{14253}(w)+p_{14352}(w)+4p_{15243}(w)+\ldots$$

Key idea: produce many diagrams $C \in B_{D(w)}$.

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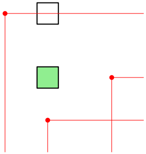

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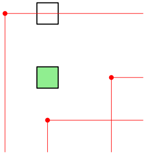
p_{132} $\rightsquigarrow$  (bubble this!)

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Question (Guo–Lin, '24)

What is the asymptotic behavior of $\beta(n) := \max_{w \in S_n} |\mathrm{supp}(\mathfrak{S}_w)|$*?*

Asymptotically maximal Schubitopes

Theorem (Chou–S., '25+)

$\beta(n) = \max_{w \in S_n} |\mathrm{supp}(\mathfrak{S}_w)|$ *satisfies*

$$\lim_{n \to \infty} \frac{\log(\beta(n))}{n \log(n)} = 1.$$

Also, $|\mathrm{supp}(\mathfrak{S}_w)|$ *is asymptotically maxed at a layered permutation.*

(Proof idea: produce a layered $w \in S_n$ with many diagrams $C \in B_{D(w)}$.)

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(Proof idea: produce a layered $w \in S_n$ with many diagrams $C \in B_{D(w)}$.)

In fact,

$$\frac{n!}{4^n} \le \beta(n) \le n! \qquad \text{(up to "small" factors).}$$

Open: Does $\lim_{n\to\infty} \frac{\log(\beta(n)) - n\log(n)}{n}$ exist? If so, what is its value?
(Answer is yes for $\mathfrak{G}_w$.)

Part II: Newton polytopes of Grothendieck polynomials

Grothendieck polynomials

Definition

For $w \in S_n$, the *Grothendieck polynomial* is:

$$\mathfrak{G}_w(\mathbf{x}) = \begin{cases} x_1^{n-1} x_2^{n-2} \dots x_{n-1} & \text{if } w = w_0 \\ \overline{\partial}_i(\mathfrak{G}_{ws_i}(\mathbf{x})) & \text{if } \ell(w) < \ell(ws_i), \end{cases}$$

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Grothendieck polynomials are counted by pipe dreams:

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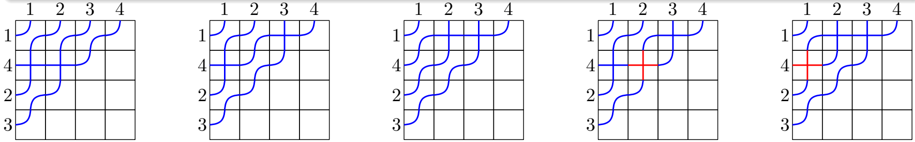

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Support of Grothendieck polynomials

Open: Is $\mathrm{supp}(\mathfrak{G}_w)$ M-convex?

or, at least, is it saturated?

or, is $\mathrm{conv}(\mathrm{supp}(\mathfrak{G}_w))$ a generalized permutahedron?

*Technically you should homogenize $\mathfrak{G}_w$...
*Known for some pattern avoidance classes: $\mathfrak{G}_w$ is symmetric, $\mathfrak{S}_w$ is zero-one...

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Equivalently:

$$\operatorname{supp}(\mathfrak{G}_w) = \bigcup_{\substack{\alpha \in \operatorname{supp}(\mathfrak{S}_w) \\ \gamma \in \operatorname{supp}(\mathfrak{G}_w^{\mathrm{top}})}} [\alpha, \gamma].$$

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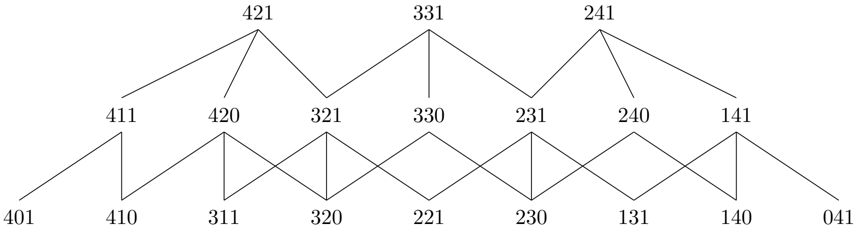

Figure: Hasse diagram for $\mathrm{supp}(\mathfrak{G}_{163245})$.

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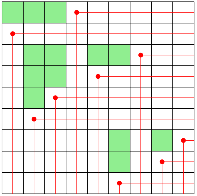
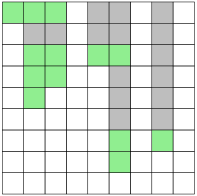
Proposition

For fireworks w, we have $\mathfrak{G}_w^{\mathrm{top}} = c_w \cdot \mathbf{x}^{\mathrm{wt}(\overline{D(w)})}$. $\overline{D(w)}$:= up-closure of $D(w)$.





$$\rightsquigarrow \mathfrak{G}_{417532986}^{\mathrm{top}} = c_w \mathbf{x}^{655432210}$$

Master formula for fireworks Grothendieck polynomials

Theorem (Chou–S., '25)

For fireworks w, the master formula holds:

$$\mathrm{supp}(\mathfrak{G}_w) = \bigcup_{\alpha \in \mathrm{supp}(\mathfrak{S}_w)} [\alpha, \mathrm{wt}(\overline{D(w)})].$$

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Goal: Polytope for $\bigcup_{\alpha \in \mathcal{S}_{D(w)}} [\alpha, \mathrm{wt}(\overline{D(w)})]$?

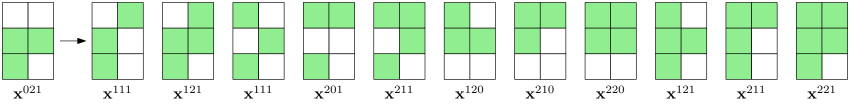
K-bubbling

Definition

Let D be a diagram. Define $B_{\mathrm{sp},D}$:= diagrams that can be obtained by:

- Any square can bubble normally
- Any square can bubble normally and leave behind a copy of itself.

Let $S_{\mathrm{sp},D} := \{\mathrm{wt}(C) \colon C \in B_{\mathrm{sp},D}\}$.



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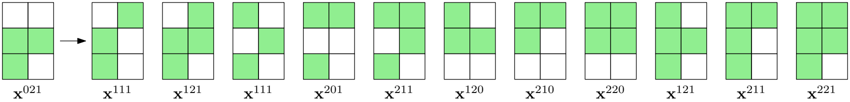

$\mathrm{conv}(S_{\mathrm{sp},D})$ will play the role of the Schubitope.

Proposition

- *Have a column-wise decomposition* $S_{\mathrm{sp},D} = S_{\mathrm{sp},D_1} + \cdots + S_{\mathrm{sp},D_n}$. *In particular, after homogenizing,* $\widetilde{S}_{\mathrm{sp},D}$ *is M-convex.*

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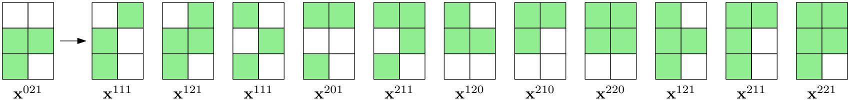

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- $S_{\mathrm{sp},D} = \bigcup_{\alpha \in S_D} [\alpha, \mathrm{wt}(\overline{D})]$

M-convexity of fireworks $\mathfrak{G}_w$

Corollary (Chou–S., '25)

For fireworks w,

- *There is a Minkowski sum decomposition*

$$\mathrm{supp}(\mathfrak{G}_w) = S_{\mathrm{sp},D(w)_1} + \cdots + S_{\mathrm{sp},D(w)_n}$$

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- $\mathrm{supp}(\widetilde{\mathfrak{G}}_w)$ *is M-convex.*
- *In particular, each* $\mathrm{supp}(\mathfrak{G}_w^{(c)})$ *is M-convex and* $|\mathrm{supp}(\mathfrak{G}_w^{(c)})|$ *is a log-concave sequence.*

(There are superexponentially many fireworks permutations.)

Application: Layered permutations

Theorem (Pechenik–Speyer–Weigandt, '21)

For every $w \in S_n$, there exists a unique inverse fireworks $\pi(w) \in S_n$ so that $\mathfrak{G}_w^{\mathrm{top}} = c_w \cdot \mathfrak{G}_{\pi(w)}^{\mathrm{top}}$ for some $c_w \in \mathbb{Z}$.

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Corollary (Chou–S., '25)

If w is fireworks, the layered permutation $\pi(w)$ satisfies

$$\mathrm{supp}(\mathfrak{G}_{\pi(w)}) \supseteq \mathrm{supp}(\mathfrak{G}_w).$$

As long as master formula holds for $\mathfrak{G}_w$ and $\mathfrak{G}_{\pi(w)}$, we get this inclusion.

Grassmannian permutations

Definition

$w \in S_n$ is *Grassmannian* if it has one descent.

Equivalently, w is 321, 2143, and 3142-avoiding.

Equivalently, $\mathfrak{S}_w$ is a Schur polynomial s_λ.

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$$\{\text{Grassmannian permutations}\} \longleftrightarrow \{\text{Partitions}\}$$

Theorem (Lenart, '00)

For Grassmannian $w = w_\lambda$,

$$\mathfrak{G}_{w_\lambda}(\mathbf{x}) = \sum_{\mu} (-1)^{|\mu|-|\lambda|} a_{\lambda,\mu} s_\mu(\mathbf{x})$$

where $a_{\lambda,\mu} = \#\{\text{strictly increasing row-flagged tableaux of shape } \mu/\lambda\}$.

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Support of degree component $\mathrm{supp}(\mathfrak{G}_w^{(c)})$ is equal to $\mathrm{supp}(s_\mu(\mathbf{x}))$.

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- $\rightsquigarrow \mathrm{supp}(\mathfrak{G}_w^{(c)})$ is a Schubitope for all c.
- $\rightsquigarrow \mathrm{supp}(\widetilde{\mathfrak{G}}_w)$ is M-convex.
- $\rightsquigarrow \mathrm{supp}(\mathfrak{G}_w)$ satisfies master formula.

Vexillary permutations

Definition

$w \in S_n$ is *vexillary* if it is 2143-avoiding.

Equivalently, $\mathfrak{S}_w$ is a flagged Schur polynomial, or is a key polynomial.

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For vexillary w, master formula holds:

$$\mathrm{supp}(\mathfrak{G}_w) = \bigcup_{\substack{\alpha \in \mathrm{supp}(\mathfrak{S}_w) \\ \gamma \in \mathrm{supp}(\mathfrak{G}_w^{\mathrm{top}})}} [\alpha, \gamma].$$

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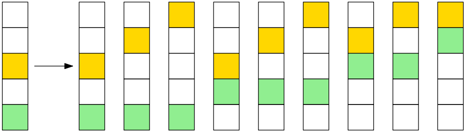
Goal: Polytope for $\bigcup_{\substack{\alpha \in \mathrm{supp}(\mathfrak{S}_w) \\ \gamma \in \mathrm{supp}(\mathfrak{G}_w^{\mathrm{top}})}} [\alpha, \gamma]$?

($B_{\mathrm{sp},D}$ construction only produces $|\mathrm{supp}(\mathfrak{G}_w^{\mathrm{top}})| = 1$...)

Bubbling diagrams



Definition

Write $\mathcal{D} = (D, A)$ where D is a diagram and $A \subseteq D$.

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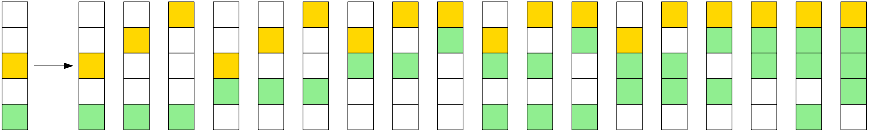
Bubbling diagrams



Definition

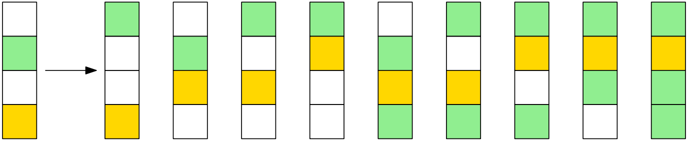
Write $\mathcal{D} = (D, A)$ where D is a diagram and $A \subseteq D$.

- Squares in A are called *distinguished*,
- Any square can bubble normally,
- Distinguished squares can bubble and leave behind a normal square.

$$\mathcal{B}(\mathcal{D}) := \{\text{diagrams obtained from } \mathcal{D} \text{ using these moves}\}.$$
$$S_{\mathcal{D}} := \{\mathrm{wt}(C) \colon C \in \mathcal{B}(\mathcal{D})\}.$$

Example: B_{sp}

Take $A := \{\text{lowest square in each column}\} \subseteq D$.



In this case: $\mathcal{B}(\mathcal{D}) = B_{\mathrm{sp},D}$.

Corollary

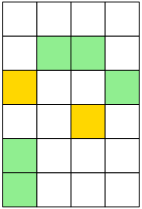
For fireworks w, $\mathrm{supp}(\mathfrak{S}_w)$ is computed by $\mathcal{B}(\mathcal{D}(w))$.

Bubbling and M-convexity

Proposition (Hafner–Mészáros–S.–St. Dizier, '23)

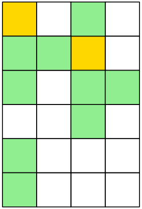
- *Have a column-wise decomposition* $S_{\mathcal{D}} = S_{\mathcal{D}_1} + \cdots + S_{\mathcal{D}_n}$.
- *After homogenizing,* $\widetilde{S}_{\mathcal{D}}$ *is M-convex.*
- $\mathrm{conv}(S_{\mathcal{D}}^{\mathrm{top}})$ *is a Schubitope. Not true for other degree components* $S_{\mathcal{D}}^{(c)}$*!*

$\mathcal{D} =$



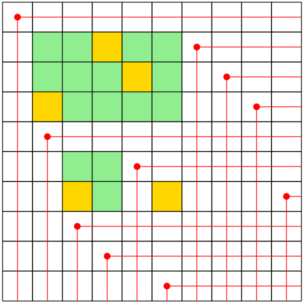
$\rightsquigarrow$

$D^{\mathrm{top}} =$



M-convexity of vexillary $\mathfrak{G}_w$

We give a recipe for $\mathcal{D}(w) = (D(w), A)$:



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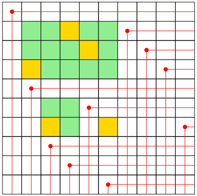


Theorem (Hafner–Mészáros–S.–St. Dizier, '23)

For vexillary $w \in S_n$, $\mathrm{supp}(\mathfrak{S}_w) = S_{\mathcal{D}(w)}$.

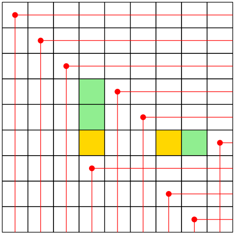
In particular, $\mathrm{supp}(\widetilde{\mathfrak{S}}_w)$ is M-convex and $\mathrm{supp}(\mathfrak{S}_w^{\mathrm{top}})$ is a Schubitope.

More generally, $\mathrm{supp}(\mathfrak{L}_\alpha^{\mathrm{top}})$ is a Schubitope; cf. Snow diagrams [Pan–Yu, '23], [Yu, '23]

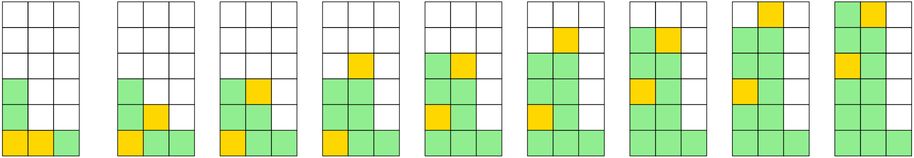
- For other c, $\mathrm{supp}(\mathfrak{S}_w^{(c)})$ is not always a Schubitope.
- For nonvexillary w, $\mathrm{supp}(\mathfrak{S}_w^{\mathrm{top}})$ is not always a Schubitope.

Back to Grassmannian permutations

Example: Grassmannian case.



Can show directly that $\mathrm{supp}(\mathfrak{G}_w^{(c)})$ is a Schubitope, with supports:



$S^{(5)}_{\mathcal{D}(w)}$ $S^{(6)}_{\mathcal{D}(w)}$ $S^{(7)}_{\mathcal{D}(w)}$ $S^{(8)}_{\mathcal{D}(w)}$ $S^{(9)}_{\mathcal{D}(w)}$ $S^{(10)}_{\mathcal{D}(w)}$ $S^{(11)}_{\mathcal{D}(w)}$ $S^{(12)}_{\mathcal{D}(w)}$ $S^{(13)}_{\mathcal{D}(w)}$

Thank you!

Theorem (Chou–S., '25+)

$\max_{w \in S_n} |\mathrm{supp}(\mathfrak{S}_w)| \approx n!$, *and is asymptotically maxed at layered.*

Theorem (Chou–S., '25)

For fireworks $w \in S_n$, *we have* $\mathrm{supp}(\mathfrak{S}_w) = \bigcup_{\alpha \in \mathrm{supp}(\mathfrak{S}_w)} [\alpha, \mathrm{wt}(\overline{D(w)})]$.
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Vexillary $w \in S_n$: $\mathrm{supp}(\mathfrak{S}_w)$ *is M-convex and computed by* $\mathcal{B}(\mathcal{D}(w))$.