Corollary 7.2.6. *There is an isomorphism of graded* $\mathbb{K}[u, q^{\pm 1}]$*-modules,*

$$H^*(M;\mathbb{K})[q^{\pm 1}, u] \cong \mathbb{K}[q^{\pm 1}] \otimes_{\mathbb{K}[q]} H_{q,u}. \tag{7.2.31}$$

The final result we need concerns connections. For simplicity, we'll state it only in q-inverted form (readers interested in what it looks like without inverting q are referred to [64, Section 9]).

Theorem 7.2.7. *The isomorphism* (7.2.31) *identifies the quantum connection* (1.2.1) *with the canonical connection on the* $H_{q,u}$.

7.3. A topological toy model

(7.3a) Differential forms basics. This section is informal motivation for Theorems 7.2.3 and 7.2.5. The aim is to show how, at least in principle, one can get from a familiar Morse-Bott picture for symplectic cohomology to the left hand sides of (7.2.28) and (7.2.30). We ignore what should be contributions from rational curves in our symplectic manifold. The outcome is a self-contained purely topological toy model, in terms of differential forms. The actual proof in [64] does not rely on this toy model, even though it uses an action-filtration whose associated graded is related to that model.

We start with a compact manifold with boundary N, and a free circle action on ∂N, with quotient $\pi : N \to D = N/S^1$. Collapsing the circle orbits yields a closed manifold M containing D. This comes with a map $\tilde{\pi} : N \to M$ such that $\tilde{\pi}|\partial N = \pi$, and $\tilde{\pi}|(N \setminus \partial N)$ is a diffeomorphism from that to $M \setminus D$. All differential forms will be $\mathbb{C}$-valued, and we will consistently use the following notation for such forms on the various spaces involved:

$$\eta \in \Omega^*(N), \;\; \beta \in \Omega^*(\partial N), \;\; \theta \in \Omega^*(D), \;\; \mu \in \Omega^*(M). \tag{7.3.1}$$

The pullback and pushforward (integration along the fibres) maps are

$$\begin{aligned} &\pi^* : \Omega^*(D) \longrightarrow \Omega^*(\partial N), \quad \pi_* : \Omega^*(\partial N) \longrightarrow \Omega^{*-1}(D), \\ &\pi_*(\beta\, \pi^*\theta) = (\pi_*\beta)\theta \;\; \Rightarrow \;\; \pi_*\pi^* = 0. \end{aligned} \tag{7.3.2}$$

Fix a connection one-form $\alpha \in \Omega^1(\partial N)$, and its curvature $F \in \Omega^2(D)$. By definition,

$$d\alpha = \pi^* F, \quad \pi_*\alpha = 1 \;\Rightarrow\; \pi_*(\alpha\, \pi^*\theta) = \theta. \tag{7.3.3}$$

In these conventions, the first Chern class of the circle bundle $\partial N \to D$ is represented by $-F$. The idempotent maps $\beta \mapsto \alpha\pi^*\pi_*\beta$ and $\beta \mapsto \pi^*\pi_*(\alpha\beta)$ are projections to complementary summands of $\Omega^*(\partial N)$,

$$\alpha\, \pi^*\pi_*\beta + \pi^*\pi_*(\alpha\beta) = \beta. \tag{7.3.4}$$

One can extend α to $\tilde{\alpha} \in \Omega^1(N)$, such that $d\tilde{\alpha} = \tilde{\pi}^*\tilde{F}$ for some $\tilde{F} \in \Omega^2(M)$. Then, $\tilde{F}$ represents (the Poincaré dual to) $-[D]$, for the co-orientation of that submanifold inherited from the circle action. Let's suppose from now on that $\tilde{\alpha}$ is supported in a collar neighbourhood of ∂N, and correspondingly $\tilde{F}$ in a tubular neighbourhood of D. Then, the pushforward map $\iota_* : H^*(D;\mathbb{C}) \to$

$H^{*+2}(M;\mathbb{C})$ can be realized by taking some closed $\theta \in \Omega^*(D)$, picking an extension $\tilde{\theta} \in \Omega^*(M)$ which is closed in our tubular neighbourhood of D, and then setting $\iota_*[\theta] = -[\tilde{F}\tilde{\theta}]$.

Lemma 7.3.1. *The following complexes (obtained by collapsing double complexes, and where each piece carries the de Rham differential) are acyclic:*

$$\Omega^*(M) \xrightarrow{\tilde{\pi}^*} \Omega^*(N) \xrightarrow{\pi_*(\cdot|\partial N)} \Omega^{*-1}(D) \tag{7.3.5}$$

$$\Omega^*(D) \xrightarrow{\pi^*} \Omega^*(\partial N) \xrightarrow{\pi^*\pi_*} \Omega^{*-1}(\partial N) \xrightarrow{\pi^*\pi_*} \cdots \tag{7.3.6}$$

$$\Omega^*(M) \xrightarrow{\tilde{\pi}^*} \Omega^*(N) \xrightarrow{\pi^*\pi_*(\cdot|\partial N)} \Omega^{*-1}(\partial N) \xrightarrow{\pi^*\pi_*} \Omega^{*-2}(\partial N) \xrightarrow{\pi^*\pi_*} \cdots \tag{7.3.7}$$

Proof. The acyclicity of (7.3.5) follows from that of the top and bottom row in

$$\begin{array}{ccccc}
\Omega^*(M,D) & \xrightarrow{\tilde{\pi}^*} & \Omega^*(N,\partial N) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^*(M) & \xrightarrow{\tilde{\pi}^*} & \Omega^*(N) & \xrightarrow{\pi_*(\cdot|\partial N)} & \Omega^{*-1}(D) \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^*(D) & \xrightarrow{\pi^*} & \Omega^*(\partial N) & \xrightarrow{\pi_*} & \Omega^{*-1}(D).
\end{array} \tag{7.3.8}$$

Note that the bottom row is acyclic even without the de Rham differential; one can get a splitting from (7.3.4). The same applies to (7.3.6), which is in fact constructed from that bottom row. Finally, one gets (7.3.7) by combining (7.3.5) and (7.3.6). □

(7.3b) The chain complex. As usual, we have formal variables $|q| = 2$, $|u| = 2$. We also use a variable z of degree 0, which geometrically plays the same role as in (7.2.24) (algebraically it's just a bookkeeping device; our structures will not be z-linear). Define a complex of $\mathbb{C}[q,u]$-modules

$$\begin{aligned}
&C_{q,u} = \big(\Omega^*(N) \oplus \Omega^*(\partial N)z \oplus \Omega^*(\partial N)z^2 \oplus \cdots\big)[q,u], \\
&d_{q,u}(\eta) = d\eta + qz\,\pi^*\pi_*(\eta|\partial N), \quad d_{q,u}(z^i\beta) = z^i(d\beta + iu\pi^*\pi_*\beta) + qz^{i+1}\pi^*\pi_*\beta.
\end{aligned} \tag{7.3.9}$$

It carries a $\mathbb{C}[u]$-linear connection

$$\begin{aligned}
&\nabla_{u\partial_q} : C^*_{q,u} \longrightarrow C^*_{q,u}, \quad [\nabla_{u\partial_q}, q] = u, \\
&\nabla_{u\partial_q}(\eta) = z(\eta|\partial N), \quad \nabla_{u\partial_q}(z^i\beta) = z^{i+1}\beta.
\end{aligned} \tag{7.3.10}$$

Remark 7.3.2. *Let's explain the symplectic cohomology motivation for* (7.3.9)*. As in* (7.2.24)*, the $\Omega^*(N)$-component corresponds to the contribution of ordinary cohomology to symplectic cohomology (for a suitable choice of Hamiltonian, these would be constant periodic orbits), and the $\Omega^*(\partial N)z^i$-components to orbits winding i times around D. The i-fold fibrewise rotation gives an endomorphism $H^*(\partial N) \to H^{*-1}(\partial N)$, which one can see in the $iu\pi^*\pi_\beta$ term in* (7.3.10)*. The most interesting part is the $qz^{i+1}\pi^*\pi_*\beta$ term. In the geometric setup from [64] where Floer trajectories are allowed to go through D, this is expected to represent the contribution of low-energy Floer cylinders that intersect D once, and correspondingly connect winding number i and $(i+1)$ orbits (one can visualize those cylinders easily in the simplest case of $M = \mathbb{C}P^1$). Finally, the* *definition of connection on deformed symplectic cohomology in [64, Sections 8–9] involves cylinders with an extra marked point going through D; thinking of the same low-energy contribution as before motivates the part of* (7.3.10) *taking* z^i *to* z^{i+1}.

Proposition 7.3.3. *(i) The* $\mathbb{C}[u]$*-linear chain map*

$$\begin{aligned} &\Omega^*(M)[q,u] \oplus (u/q)\Omega^*(D)[u/q,u] \xrightarrow{\cong} C_{q,u}, \\ &q^j\mu \mapsto q^j\tilde{\pi}^*\mu, \quad (u/q)^i\theta \mapsto z^i\pi^*\theta \end{aligned} \tag{7.3.11}$$

where the domain just carries the standard de Rham differential, is a quasi-isomorphism.

(ii) Set $H_{q,u} = H^*(M)[q,u] \oplus (u/q)H^*(D)[u/q,u] \cong H(C_{q,u})$*. The* $\mathbb{C}[q]$*-module structure on* $H_{q,u}$ *carried over through* (7.3.11) *is:*

$$\begin{aligned} &q \cdot q^j[\mu] = q^{j+1}[\mu], \\ &q \cdot (u/q)^i[\theta] = (u/q)^{i-1}\big(\iota^*\iota_*[\theta] - (i-1)u[\theta]\big) \;\; \textit{for } i > 1, \\ &q \cdot (u/q)[\theta] = \iota_*[\theta]. \end{aligned} \tag{7.3.12}$$

(iii) The connection on $H_{q,u}$ *induced by* (7.3.10) *is*

$$\begin{aligned} &(\nabla_{u\partial_q})([\mu]) = (u/q)\iota^*[\mu], \\ &(\nabla_{u\partial_q})(q^j[\mu]) = ujq^{j-1}[\mu] + q^{j-1}\iota_*\iota^*[\mu] \;\; \textit{for } j > 0, \\ &(\nabla_{u\partial_q})((u/q)^i[\theta]) = (u/q)^{i+1}[\theta]. \end{aligned} \tag{7.3.13}$$

Proof. (i) The complex (7.3.9) splits as the direct sum of $\mathbb{C}[u]$-module pieces, compatibly with the map (7.3.11). The pieces are

$$\begin{aligned} &q^k\Omega^*(M)[u] \longrightarrow \big\{q^k\Omega^*(N)[u] \to q^{k+1}z\Omega^{*-1}(\partial N)[u] \to q^{k+2}z^2\Omega^{*-2}(\partial N)[u] \to \cdots\big\}, \\ &(u/q)^j\Omega^*(D)[u] \longrightarrow \big\{z^j\Omega^*(\partial N)[u] \to qz^{j+1}\Omega^*(\partial N)[u] \to \cdots\big\}, \end{aligned} \tag{7.3.14}$$

After setting $u = 0$, one recovers the situation from Lemma 7.3.1, showing that both maps become quasi-isomorphisms. After that, one uses the u-filtration (convergence is not an issue, because the filtration is bounded in each degree) to derive the same result for the original maps.

(ii) The first line of (7.3.12) is obvious. As for the second line, we have (assuming $d\theta = 0$ and $i > 1$) $qz^i\pi^*\theta = d_{q,u}(z^{i-1}\alpha\pi^*\theta) - z^{i-1}(F + (i-1)u)\pi^*\theta$. For the final case, we extend θ to $\tilde{\theta} \in \Omega^*(M)$, so that $\tilde{\theta}$ is closed near D, as in our previous discussion of ι_*. Then, $qz\pi^*\theta = d_{q,u}(\tilde{\alpha}\,\tilde{\pi}^*\tilde{\theta}) - \tilde{\pi}^*(\tilde{F}\tilde{\alpha})$.

(iii) The first and last lines of (7.3.13) are obvious. The second one then follows from the first, since $(\nabla_{u\partial_q})(q^j[\mu]) = ujq^{j-1}[\mu] + q^j \cdot (\nabla_{u\partial_q})[\mu] = ujq^{j-1}[\mu] + q^{j-1}\,\iota_*\iota^*[\mu]$. ☐

Remark 7.3.4. *The description given above is clearly compatible with Theorem 7.2.5, even though the two statements are formulated slightly differently. The reason is that in our topological toy model case, the q-action on the whole of* $H_{q,u}$ *can be easily computed, whereas Theorem 7.2.5 only partially determined that action (a complete description is expected to involve relative Gromov-Witten theory). The same applies to the connection: Theorem 7.2.7 obviously involves Gromov-Witten invariants, but a complete description of its action on* $H_{q,u}$ *should again involve more enumerative geometry.*

(7.3c) Inverting u, inverting q. In parallel with previously encountered situations (Notation 7.1.13), $H^*(\mathbb{C}[u^{\pm 1}] \otimes_{\mathbb{C}[u]} C_{q,u})$ is a classical D-module in each degree, by the action of $\bar{q} = q/u$ and $\nabla_{\partial_{\bar{q}}} = \nabla_{u\partial_q}$. Proposition 7.3.3 translates into the following:

Corollary 7.3.5. *(i) In any given degree d, we have*

$$\begin{aligned} &H^d_{\bar{q}} \stackrel{\text{def}}{=} H^{d \bmod 2}(M;\mathbb{C})[\bar{q}] \oplus \bar{q}^{-1} H^{d \bmod 2}(D;\mathbb{C})[\bar{q}^{-1}] \xrightarrow{\cong} H^d(\mathbb{C}[u^{\pm 1}] \otimes_{\mathbb{C}[u]} C_{q,u}), \\ &q^j[\mu] \longmapsto u^{(d-|\mu|)/2-j} q^j \tilde{\pi}^*[\mu], \quad q^{-i}[\theta] \longmapsto u^{(d-|\mu|)/2}[z^i \pi^* \theta]. \end{aligned} \tag{7.3.15}$$

(ii) The $\bar{q}$-module structure on $H^d_{\bar{q}}$ is

$$\begin{aligned} &\bar{q} \cdot \bar{q}^j[\mu] = \bar{q}^{j+1}[\mu], \\ &\bar{q} \cdot \bar{q}^{-i}[\theta] = \bar{q}^{1-i}\big(\iota^*\iota_*[\theta] - (i-1)[\theta]\big) \text{ for } i > 1, \\ &\bar{q} \cdot \bar{q}^{-1}[\theta] = \iota_*[\theta]. \end{aligned} \tag{7.3.16}$$

(iii) The connection on $H^d_{\bar{q}}$ is

$$\begin{aligned} &\nabla_{\partial_{\bar{q}}}([\mu]) = \bar{q}^{-1}\iota^*[\mu], \\ &\nabla_{\partial_{\bar{q}}}(\bar{q}^j[\mu]) = j\bar{q}^{j-1}[\mu] + \bar{q}^{j-1}\iota_*\iota^*[\mu] \text{ for } j > 0, \\ &\nabla_{\partial_{\bar{q}}}(q^{-i}[\theta]) = q^{-i-1}[\theta]. \end{aligned} \tag{7.3.17}$$

Corollary 7.3.6. *The D-module $H^d_{\bar{q}}$ admits the following description: it has generators $m \in H^{d \bmod 2}(M;\mathbb{C})$ and $t \in H^{d \bmod 2}(D;\mathbb{C})$, with relations*

$$\nabla_{\partial_{\bar{q}}}(m) = \iota^* m, \quad \nabla_{\partial_{\bar{q}}}(\bar{q}t) = \iota^*\iota_* t. \tag{7.3.18}$$

With respect to our previous description, the generators are $m = [\mu]$ for $[\mu] \in H^*(M;\mathbb{C})$ and $t = \bar{q}^{-1}[\theta]$ for $[\theta] \in H^*(D;\mathbb{C})$, and they clearly satisfy the given relations. If we take the abstract D-module with those generators and relations, it has a $\mathbb{C}$-basis is given by $\bar{q}^j m$ and $\nabla^i_{\partial_{\bar{q}}} t$. Since the same is true for $H^d_{\bar{q}}$, our set of relations is complete.

Next, we look at what happens if we invert q rather than u. It follows from (7.3.12) that every element of $H_{q,u}$ is mapped into $H^*(M)[q,u]$ by a sufficiently high power of q. As a consequence, we have an induced isomorphism

$$H^*(M)[q^{\pm 1}, u] \cong H(\mathbb{C}[q^{\pm 1}] \otimes_{\mathbb{C}[q]} C_{q,u}). \tag{7.3.19}$$

Here, the $\mathbb{C}[q]$-module structure on the left hand side is the standard one, and from (7.3.13) one gets $\nabla_{u\partial_q} = u\partial_q + q^{-1}\iota_*\iota^*$; note that $\iota_*\iota^*$ is the cup product with $[D]$.

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