A REMARK ON THE SYMPLECTIC COHOMOLOGY OF COTANGENT BUNDLES, AFTER KRAGH

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The cotangent bundle of any closed manifold N^n is a symplectic manifold T^*N, which is exact and of contact type at infinity. It comes with a trivialization of its bicanonical bundle $\mathcal{K}^2_{T^*N}$ (the complex line bundle representing -2 times the first Chern class), unique up to homotopy. Its symplectic cohomology $SH^*(T^*N)$ is therefore a $\mathbb{Z}$-graded module over an arbitrarily chosen coefficient ring $\mathbb{K}$. Here, we use the canonical orientations from [3] to determine the signs in the differential, and the trivialization of the bicanonical bundle to fix the grading for non-contractible loops [11]. The literature contains the claim (see for instance my own [12]) that

$$SH^*(T^*N) \cong H_{n-*}(\mathcal{L}N; o_N \otimes_{\mathbb{Z}} \mathbb{K}), \tag{1}$$

where o_N is the orientation local system, or rather its pullback by the base point map $\mathcal{L}N \to N$. If $2 = 0 \in \mathbb{K}$, so that $o_N \otimes_{\mathbb{Z}} \mathbb{K}$ is trivial, there are several different proofs of the isomorphism in the literature, see [13, 10, 1]. It may seem natural to assume that for general $\mathbb{K}$ the signs work out as indicated in (1) (twisting by o_N is required in order to have a natural candidate on the right hand side for the unit element $1 \in SH^0(T^*N)$, namely the class $[N]$ of constant loops).

However, recent work of Kragh [6] suggests a different formula:

$$SH^*(T^*N) \cong H_{n-*}(\mathcal{L}N; o_N \otimes_{\mathbb{Z}} \eta_N \otimes_{\mathbb{Z}} \mathbb{K}), \tag{2}$$

where $\eta_N \to \mathcal{L}N$ is another local system, which represents the image of the second Stiefel-Whitney class $w_2(TN)$ under the transgression map $H^2(N; \mathbb{Z}/2) \to H^1(\mathcal{L}N; \mathbb{Z}/2)$. Since η_N is trivial on constant loops, the right hand side of (2) still contains a natural class corresponding to the unit in symplectic cohomology. However, that class is no longer necessarily nontrivial.

Example 1. *Consider the case of an N such that $w_2(TN)|\pi_2(N) \neq 0$. Suppose that 2 is invertible in $\mathbb{K}$. Look at the based loop space homology with coefficients in η_N, more precisely*

$$H_*(\Omega N; \eta_N|\Omega N \otimes_{\mathbb{Z}} \mathbb{K}). \tag{3}$$

Since η_N is nontrivial on the component of contractible loops, the class of the base point in this group vanishes. But on the other hand, the whole group is an associative graded ring under a twisted version of the Pontryagin product (this works because η_N comes from an abelian gerbe on N), and the base point is the unit. Hence (3) *would then vanish altogether. In view of the base point spectral sequence, this implies the vanishing of the right hand side of* (2).

This note computes the following example:

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Theorem 2. *$SH^*(T^*\mathbb{C}P^2)$ vanishes if $\mathbb{K}$ is a field of characteristic $\neq 2$.*

This contradicts (1) but agrees with (2), as shown in Example 1 above. By design, our computation leading to Theorem 2 is entirely independent of the existing proofs of (1). Instead, it relies on Lefschetz fibrations of the type considered in [5, 4], and computational techniques from [8, 7, 2, **?**].

Remark 3. *One can introduce twisted versions of symplectic cohomology with respect to any class $\alpha \in H^2(N;\mathbb{Z}/2)$, denoted by $SH^*(T^*N;\alpha)$, by taking twisted coefficients in the associated local system over $\mathcal{L}N$. Equivalently, as pointed out to me by Kragh, this amounts to choosing a different coherent orientation of the moduli spaces in Floer theory. Hence, even if* (2) *is the correct formula, it seems that ordinary loop space homology can be recovered by choosing $\alpha = w_2(TN)$. However, the difference between the two expressions* (2) *and* (1) *is still meaningful, because of Viterbo functoriality* [14]. *Namely, suppose that* (2) *is correct, and let N be as in Example 1. By combining the computation above with a standard Viterbo functoriality argument, it follows that there is no closed exact $L \subset T^*N$ such that $w_2(L)|\pi_2(L) = 0$. This resembles the use of Novikov-twisted symplectic cohomology in* [9].

Remark 4. *The formula* (2) *is also natural from the point of the open-closed string relationship, as pointed out to me by Abouzaid. In the case when N is* Pin*, one has a natural open-closed string map $SH^*(T^*N) \to HF^*(N,N) \cong H^*(N;\mathbb{K})$, which maps the unit element to the unit, hence shows that $SH^*(T^*N)$ must be nonzero. More generally, one expects to have maps $SH^*(T^*N;w_2(TN)) \to H^*(N;\mathbb{K})$, since the zero-section becomes as object of the Fukaya category formed with respect to the background class $w_2(TN) \in H^2(T^*N;\mathbb{Z}/2)$. This agrees with the idea that the twisted version of symplectic cohomology with $\alpha = w_2(TN)$, and not the untwisted version, should agree with ordinary loop space homology.*

I thank Thomas Kragh for explaining his work to me. Obviously, this note is essentially a minor remark on it. I also thank Mohammed Abouzaid for discussions regarding the implications of Kragh's work, and for correcting the initial version of this note, which was mostly wrong (he is not responsible for the possible persistence of this fact).

Clean intersection and signs

We consider a particularly simple instance of Lagrangian clean intersection. Namely, suppose that L_0, L_1 are Lagrangian submanifolds in a symplectic manifold M^{2n}, $n \geq 2$, intersecting cleanly along a circle $C = L_0 \cap L_1 \cong S^1$. In this situation, the symplectic form gives a nondegenerate pairing between the normal bundles ν_0, ν_1 of C inside the two submanifolds. By choosing a metric on the first normal bundle, we translate the pairing into an isomorphism

$$\nu_1 \longrightarrow \nu_0^* \cong \nu_0 \tag{4}$$

which is unique up to homotopy. Equivalently, suppose that we have a compatible almost complex structure J and associated metric g on M. The restriction of this metric allows us to see ν_0 as a subbundle of $TL_0|C$, and similarly for L_1. One can arrange J in such a way that $J(\nu_0) = \nu_1$, and this is the same isomorphism as defined before. Suppose now that L_0 and L_1 come with *Pin* structures. One can compare these locally near C by using (4) and tubular neighbourhoods. The difference between them is given by an element of $H^1(C;\mathbb{Z}/2)$, which we think of as classifying a local $\mathbb{Z}$-coefficient system $\rho_C \to C$.

An elementary example which is relevant for us is the affine quadric threefold

$$M = \{(x_1, \dots, x_4) \in \mathbb{C}^4 \;:\; x_1^2 + x_2^2 - x_3^2 - x_4^2 = 1\}. \tag{5}$$

Take $L_0 = M \cap (\mathbb{R}^2 \times i\mathbb{R}^2)$, which is a three-sphere, and $L_1 = M_{\mathbb{R}} = M \cap \mathbb{R}^4$, which is a plane bundle over the circle $C = L_0 \cap L_1 = S^1 \times \{0\}$. Clearly

$$\begin{aligned} (\nu_0)_{(x_1,x_2,0,0)} &= \{0\} \times i\mathbb{R}^2, \\ (\nu_1)_{(x_1,x_2,0,0)} &= \{0\} \times \mathbb{R}^2. \end{aligned} \tag{6}$$

As an abstract circle, C comes with a preferred *Pin* structure (the one that bounds a *Pin* structure on a disc, which means that it is not compatible with an actual trivialization of TC). Again abstractly, L_0 comes with its unique *Pin* structure, which is compatible with thinking of it as the boundary of the unit ball in $\mathbb{R}^2 \times i\mathbb{R}^2 \cong \mathbb{R}^4$. These two induce a *Pin* structure on ν_0, which then becomes a *Pin* structure on ν_1 by applying (4). All these *Pin* structures are compatible with the constant trivializations of (6). In particular, the *Pin* structure on ν_1 extends to one on the normal bundle of the disc $D^2 \times \{0\}$ inside $\mathbb{R}^4$. (One can substitute stable framings for *Pin* structures throughout this argument.)

Our desired application is to symplectic Lefschetz fibrations with real involutions. Namely, let $\pi : E \to \mathbb{C}$ be a symplectic Lefschetz fibration. A real involution is an involution ι of E which reverses the sign of the symplectic form, and also satisfies $\pi \circ \iota = \bar{\pi}$. In particular, the restriction of π to the fixed locus $E_{\mathbb{R}}$ of the involution is then automatically a Morse function $\pi_{\mathbb{R}}$. The specific situation we're interested in is the following one. First of all, E is of real dimension 8. Next, $E_{\mathbb{R}}$ is compact, and $\pi_{\mathbb{R}}$ has a unique local maximum p_1, as well as another critical point p_0 which has Morse index 2. Finally, there are no critical points of $\pi_{\mathbb{R}}$ in the level sets between $\pi_{\mathbb{R}}(p_0)$ and $\pi_{\mathbb{R}}(p_1)$.

Choose a point $z \in \mathbb{R}$ which is slightly larger than $\pi(p_0)$, and let $M = \pi^{-1}(z)$ be the fibre at that point. By going left and right along the real line from z to p_0 and p_1, respectively, we get two vanishing cycles $L_0, L_1 \subset M$. It is not hard to see that $L_1 = M_{\mathbb{R}}$ is just the fixed point set of $\iota|M$. On the other hand, after possibly adjusting the symplectic form slightly, we can find local coordinates around p_0 which reduce us to the situation considered above. In particular, $C = L_0 \cap L_1$ is a circle in M, which bounds a small disc $D_0 \subset E_{\mathbb{R}}$ having p_0 at the center. Take the canonical *Pin* structures on C (as an abstract circle) and L_0, use them to construct a *Pin* structure on ν_0, and transfer that to ν_1 using (4). Note that ν_1 is the normal bundle of $C \subset M_{\mathbb{R}}$. Then, that *Pin* structure extends one on the normal bundle of $D_0 \subset E_{\mathbb{R}}$, by the previous purely local discussion. On the other hand, note that C bounds another disc $D_1 \subset E_{\mathbb{R}}$ with center p_1, obtained by flowing upwards along the gradient flow of $\pi_{\mathbb{R}}$. Using the canonical *Pin* structures on C and L_1, we can get another *Pin* structure on ν_1, which has the property that it extends to the normal bundle of $D_1 \subset E_{\mathbb{R}}$. If these two *Pin* structures agree, there is an induced *Pin* structure on the normal bundle of the sphere

$$S = D_0 \cup_C D_1 \subset E_{\mathbb{R}}. \tag{7}$$

In particular, that sphere must have even selfintersection. Conversely, if $S \subset E_{\mathbb{R}}$ has odd selfintersection, the two *Pin* structures must be different, hence the local coefficient system $\rho_C \to C$ will be nontrivial.

To conclude, we return to the general framework of two Lagrangian submanifolds L_0, L_1 intersecting cleanly along a circle. In addition to the *Pin* structures, assume that L_0 and L_1 are oriented, and that suitable exactness conditions hold, so that $HF^*(L_0, L_1)$ is defined as a $\mathbb{Z}/2$-graded $\mathbb{K}$-vector space, and can be computed from contributions which are local to C. Then

$$HF^*(L_0,L_1) \cong H^{*+\mu}(C;\rho_C \otimes_{\mathbb{Z}} \mathbb{K}), \tag{8}$$

where $\mathbb{K}$ is the coefficient ring, and $\mu \in \mathbb{Z}/2$ depends on the orientations. So, if $\mathbb{K}$ is a field of characteristic $\neq 2$, one finds that $HF^*(L_0,L_1)$ vanishes if ρ_C is nontrivial. Finally, note that in the situation where the Lagrangian submanifolds are graded, (8) lifts to an isomorphism of $\mathbb{Z}$-graded groups.

The cotangent bundle of $\mathbb{C}P^2$

$T^*\mathbb{C}P^2$ appears as the total space of a Lefschetz fibration with real structure, which can be thought of as a complexification of the standard Morse function on $\mathbb{C}P^2$. To obtain a concrete description, let $E \subset \mathfrak{gl}_3(\mathbb{C})$ be the subset of matrices of rank 1 and trace 1, which means that they are of the form $a = v \otimes w$ for two vectors with complex scalar product $wv = 1$. This is a smooth subvariety, and if we equip it with the restriction of a suitable Kähler form, it becomes symplectically isomorphic to $T^*\mathbb{C}P^2$. The real involution is $\iota(a) = a^*$. One identifies $\mathbb{C}P^2 \cong E_{\mathbb{R}}$ by taking a unit vector $v \in S^5$ to the matrix $a = v \otimes v^*$. The Lefschetz fibration can be taken to be

$$\pi(a) = a_{11} - a_{33}. \tag{9}$$

This example was considered in detail in [5] from a related perspective, so we only summarize the results. There is a basis of vanishing cycles (V_0,V_1,V_2), which is such that each intersection $V_i \cap V_j$ $(i \neq j)$ is clean and a circle. The argument from the previous section applies to both $V_1 \cap V_2$ and (after reversing the sign of π) $V_0 \cap V_1$. In either case, the sphere (7) is a generator of $H_2(\mathbb{C}P^2;\mathbb{Z})$, hence has odd selfintersection. It follows that, if we choose coefficients in a field $\mathbb{K}$ of characteristic $\neq 2$, then

$$HF^*(V_0,V_1) = 0, \quad HF^*(V_1,V_2) = 0. \tag{10}$$

Note that we have not completely determined $HF^*(V_0,V_2)$, which could be either zero or the cohomology of a circle.

Consider the Lefschetz thimbles $\Delta_0,\Delta_1,\Delta_2$ associated to our vanishing cycles, which are noncompact Lagrangian submanifolds in E. We have the following general criterion, taken from [7].

Lemma 5. *Suppose that the wrapped Floer cohomology $HW^*(\Delta_2,\Delta_2)$ is nonzero. Let $\mathcal{B}$ be the Fukaya category of the fibre M. Then, in $D(\mathcal{B})$, the object V_2 must be a direct summand of a twisted complex*

$$Z = \{hom(V_1,V_2) \otimes hom(V_0,V_1) \otimes V_0 \xrightarrow{\delta_C} hom(V_0,V_2) \otimes V_0 \oplus hom(V_1,V_2) \otimes V_1\}. \tag{11}$$

Suppose that in our situation, the assumption of this Lemma were satisfied. Since V_2 is a nonzero object, we then know that $HF^*(V_0,V_2)$ must be nonzero, so it is isomorphic to $H^*(S^1;\mathbb{K})$, and possibly after a shift, we have

$$Z \cong V_0 \oplus V_0[-1]. \tag{12}$$

But this splits as a direct sum only in the obvious way, so V_2 would have to be isomorphic to V_0 up to a shift. That can't be the case, since the generators of $HF^*(V_0,V_2)$ lie in adjacent degrees, so we arrive at a contradiction, which shows that $HW^*(\Delta_2,\Delta_2) = 0$.

It is easy to see that any of the Δ_k is Lagrangian isotopic to a cotangent fibre. Hence, the vanishing of wrapped Floer cohomology implies that the symplectic cohomology must be zero as well, by a very general result of [2].

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