

Charting one's course through mirror symmetry

Paul Seidel

Geometry and physics have long gone hand in hand. All around us, physical processes play out in geometric terms, such as straight lines (rays of light), ellipses (planetary motion), or parallelograms (the combined effect of two forces). To earlier scientists, this meant that the universe was created to be comprehensible. Kepler went so far as to argue that God, in setting up the natural world, could use regular pentagons but never heptagons, since the heptagon can't be constructed with ruler and compass¹. Kepler's enthusiasm for geometry still resonates with modern mathematicians, even though we may not share his metaphysical certainties. Our views also differ in another important respect. For Kepler, the elements of geometry, as set out by Euclid, were immutable (after all, they constrained even God). Today it seems clear that, in order for geometric thinking to remain a source of new insights (in mathematics, physics, computer science...), geometry must continue to evolve.

One of the current challenges comes from quantum physics, which indicates that space should be an emergent concept, rather than a fundamental one. To a geometer, this is rather disconcerting, like demanding that a painter work without a canvas. As a more easily graspable compromise, we can take the familiar notion of space from Euclidean geometry as a starting point, and then put that through a process that moves pieces of it around, like a series of small earthquakes (called "quantum corrections"). If the earthquakes get too violent, the process will get out of control, and its outcome must lie beyond geometry in any ordinary sense. But if the modifications are small enough, we will be able to see a new space gradually emerging as the result. In this article, I will try to explain one such construction, which comes from "mirror symmetry". The specific structure governing the process is called the "tropical vertex group", and it is visualized through "scattering diagrams" (I have no intention of explaining any of those terms; but they do roll off the tongue beautifully). The construction was invented by Kontsevich and Soibelman². Unlike the heptagon, which Kepler was so concerned about, it is still a developing mathematical subject.

When going about the mind-bending business of revisiting our concept of space, how can the imagination keep a foothold? A long-standing tradition is to imagine oneself a traveller in a faraway place. At various times, this kind of fiction has enabled us to conceive of the

¹"[...] quo minus Heptagonus, et caeterae hujus figurae, a Deo fuerint adhibite ad ornatum Mundi". *Harmonices Mundi* (1619), Lib. I, Prop. XLV. Kepler's discussion of such "unknowable" (inscibile) figures is fascinating: he claims that even an Omniscient Mind cannot comprehend them "in a simple action".

²Kontsevich and Soibelman, *Affine structures and non-Archimedean analytic spaces*, in: *The Unity of Mathematics*, Birkhauser, 2006

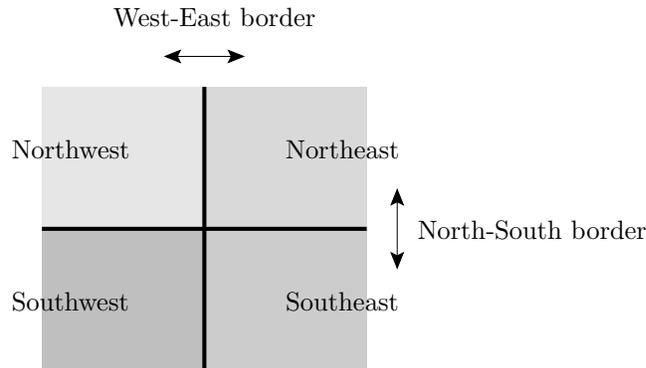


FIGURE 1.

moon and planets as earthlike bodies³; to have more or less than three dimensions of space⁴; or to run alongside a beam of light⁵. It will hopefully also help us here.

With that in mind... Once upon a time, there were two countries, Northlandia and Southlandia. Due to ideological disagreements, the countries' maps of the world use slightly different coordinates (x, y) . You may bring a South-made map to the North; when crossing the border, you will be handed a small paper slip with the formula for converting one kind of coordinates to the other, so that you can exchange geographical information with the locals. Here's what the slip says:

(x, y) in Southern coordinates translates into $(x, y + 0.01xy) = (x, y(1 + 0.01x))$ in Northern coordinates.

When looking at this slip, we see that North and South agree on what the x -coordinate should be (this has to do with the fact that the border between them is a horizontal West-East line), but disagree slightly on the y -coordinate. The strange-looking 0.01 is an arbitrarily chosen small number, measuring the ideological differences (if we replaced it with 0, the discrepancy between coordinate systems would disappear). Such coexisting coordinate systems, called “charts”, are generally unproblematic (just like using degrees Celsius and Fahrenheit doesn't mean that the notion of temperature itself is in doubt), as long as the conversion rules between them are consistent. To make things interesting, we have to look at a more complicated geography.

Take four countries, Northwestlandia, Southwestlandia, Southeastlandia, and Northeastlandia (Figure 1). Each has its own coordinate system, and here are the conversion rules handed out at border crossings:

³For an account of how literature and science in the 16th and 17th centuries worked together on that goal, see: F. Ait-Touati, *Contes de la lune: Essai sur la fiction et la science moderne*, 2011.

⁴Abbott, *Flatland: A Romance of Many Dimensions*, 1884.

⁵“Wenn man einer Lichtwelle mit Lichtgeschwindigkeit nachläuft”. Einstein, *Autobiographische Skizze*, in: Seelig (ed.), *Helle Zeit - Dunkle Zeit. In memoriam Albert Einstein, 1956*.

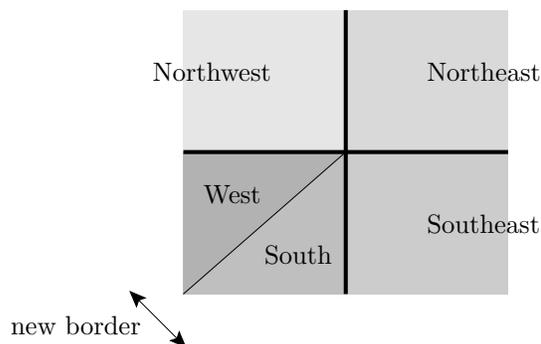


FIGURE 2.

When crossing a border from South to North, change (x, y) to $(x, y(1 + 0.01x))$.
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When crossing a border from West to East, change (x, y) to $(x(1 + 0.01y), y)$.
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The traveller trying to use these rules will be startled to find that the conversion from, say, Southwestern to Northeastern coordinates depends on whether you go through the Southeast or the Northwest. The difference is very small, on the order of 0.0001, but even the smallest discrepancy leads to logical inconsistencies. After all, the Cathedral of Northeastlandia should always be in the same place, no matter which route you choose on your visit from the Southwest. And if you take a trip all around the continent, the discrepancy means that landmarks in your home country would have shifted slightly when you return. Clearly, our cartography is insufficiently precise.

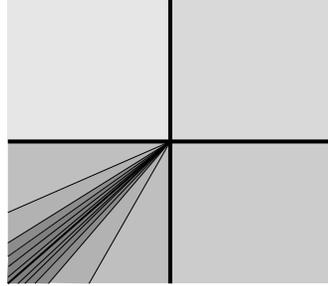
The local cartographers resolved this difficulty in the following way. Southwestlandia should be considered as two provinces, Southlandia and Westlandia, whose maps differ slightly. When crossing the border from one province to the other (the thinner line in Figure 2), another transition will be necessary, which had not been previously noticed since it's much smaller than the others:

New rule: change from (x, y) to $(x(1 + 0.0001xy), y/(1 + 0.0001xy))$.

With that taken into account, consistency is restored, and all landmarks stay in place when travelling, to any order of precision (the mathematical computation which shows that, by following the traveller all around, is simple but still entirely surprising). The appearance of the new border, which gives us five “charts” instead of the original four, is called “scattering” in this context. In (vaguely) physics-inspired terminology, the initial border crossing rules are the “first order quantum corrections”, given to us as part of setting up the problem; consistency then forced us to introduce a “second order correction”.

Now let's go through the same argument, but starting out with slightly different rules:

FIGURE 3.



When crossing from South to North, change (x, y) to $(x, y(1 + 0.01x)^2)$.

When crossing from West to East, change (x, y) to $(x(1 + 0.01y)^2, y)$.

This is like having each of the previous border crossings repeated twice (say, passport control and then customs), and it gives rise to the same inconsistency. We may think we already know how to address this— let’s repeat the previous trick, splitting one country in two, but now with terms raised to the appropriate power:

New rule: change from (x, y) to $(x/(1 + 0.0001xy)^4, y(1 + 0.0001xy)^4)$.

Unfortunately, this does not resolve all the inconsistencies! It leaves errors of order 0.000001. The actual solution involves an infinite number of new borders subdividing the former South-west territory, which are accompanied by smaller and smaller changes of coordinates (Figure 3). This kind of “infinite factorization” process, involving “quantum corrections” of higher and higher order, is what I’ve been aiming to show here. As before, the process starts with only the original “first order” North-South and West-East rules. From then on, everything is governed by the need to avoid paradoxes, which will determine the new borders and all the formulae associated with them⁶.

One could argue that what I’ve described is not a complicated geometry, but merely one in which ideological differences (and overly mathematically trained cartographers) have created artificial complications. After all, the countries in the story didn’t undergo any actual earthquakes, only the way in which their maps were related kept changing. This is partly the effect of an unfortunate mix-up of metaphors, and partly a consequence of sticking with a really simple example. One can still agree that describing the relative positions of places in different countries in a consistent way turned out to be unexpectedly difficult, far harder than in the usual Euclidean (x, y) plane; and that is certainly an interesting geometric phenomenon.

Since we’re already casting a critical eye back on our story, how about keeping track of the sizes of the various discrepancies? Undoubtedly 0.0001 is much smaller than 0.01, but

⁶For further developments, see: Gross, Pandharipade, and Siebert, *The tropical vertex*, Duke Math. J. 153 (2010).

small errors can easily accumulate. One would like to say that this not a problem, even for infinitely many border crossings, in the same sense as Achilles overtaking the tortoise wasn't impossible after all. But are we sure about that? This is a question of “convergence of a perturbative expansion”. One can circumvent it by replacing 0.01 with a fictitious infinitely small number; but then, all answers will be in terms of that fictitious number. For the purpose of doing geometry on the resulting spaces, an answer involving just the usual numbers would clearly be more satisfying⁷. The best I can say is that for many problems of this kind, “mirror symmetry” shows that the infinite process makes sense, by an indirect argument. This is one reason why the notion of “geometry emerging by quantum corrections” may still be regarded as one requiring deeper study.

⁷For a related construction where this is clearly an important problem, see: Gaiotto, Moore and Neitzke, *Four-dimensional wall-crossing via three-dimensional field theory*, Commun. Math. Phys. 299, 2010.