Growth and complexity of iterates

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who will hopefully forgive
any mistakes / misattributions
I may make

Growth of iterates

Let M^n be a closed manifold, $f: M \longrightarrow M$ a self-map. We consider $f^d = \underbrace{f \circ \cdots \circ f}_{d}$. Set

a_d = # of essential fixed points of f^d

= min { # Fix(g), for any g homotopic to f^d, and which has nondegenerate fixed points

Otherwise, one has to count fixed points with their multiplicities.

Example $f(x) = x + x^7$, $x = 0$ counts as 7 fixed points.

Fix additionally (sub)manifolds $L_0, L_1 \to M$, $\dim L_0 + \dim L_1 = n$. Set

b_d = # of essential intersection points of $f^d(L_0)$ and L_1

= min { intersection points after arbitrarily deforming f^d, L_0, L_1; again, assuming nondegeneracy }

Questions

- What is the growth behaviour of a_d, b_d as $d \to \infty$?
- How "complicated" can the sequences a_d, b_d be?
- Add geometric structure
- Analogies elsewhere in math?

Arnold, Bol. Soc. Brasil Mat. 1990

Elementary upper bounds

Theorem (Artin-Mazur, Arnol'd)
a_d, b_d grow at most exponentially: if $\|Df\| \leq C$, then

$$\|a_d\|, \|b_d\| \lesssim (C^n)^d$$

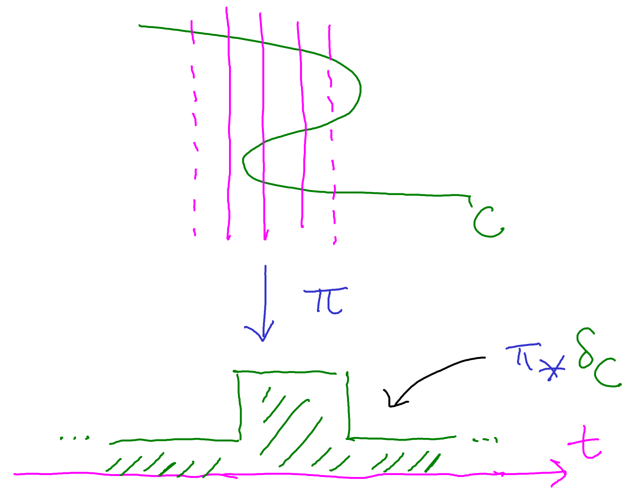
Arnol'd's argument (for a_d)

$$\mathrm{Graph}(f^d) = \{y = f^d(x)\} \subseteq M \times M$$

$$\mathrm{Fix}(f^d) = \mathrm{Graph}(f^d) \cap \mathrm{Diagonal}\,\{y = x\}$$

wiggle this randomly. The number of intersections is replaced by an averaged version, which is bounded by the n-dimensional volume of Graph (f^d) close to $\{y=x\}$.

Basic geometric picture:



Average intersection of C with one of $|$ is

$$\int_{\mathbb{R}} (\pi_* \delta_C)\,|dt| = \int_C \pi^* |dt| \leq \int_C d\mathrm{vol}_C$$

pushforward measure

Algebraic computation

Take $f_{*,i} : H_i(M;\mathbb{Q}) \longrightarrow H_i(M;\mathbb{Q})$.
The Lefschetz trace is

$$L(f) = \sum_{i=0}^{\dim M} (-1)^i \operatorname{Tr}(f_{*,i})$$

Then

$$|L(f^d)| \leq a_d \qquad (*)$$

If M is simply-connected ($\pi_1 M$ is trivial), $(*)$ is an equality.

Example $f = \text{identity}\ (f(x)=x)$, then $a_d = |\chi(M)|$ is $\pm$ the Euler characteristic.

Idea: $L(f)$ counts fixed points with ± 1 signs. Arrange cancellation between $\oplus$ and $\ominus$.

e.g. Babenko–Bogatyi, Izvestiya 1991

Corollary If $\pi_1 M$ is trivial, then a_d either grows exponentially as $d \longrightarrow \infty$, or else remains bounded.

Similarly,

$$|f_*^d(L_0) \cdot L_1| \leq b_d$$

Corollary Under suitable topological assumptions, b_d is either bounded, grows polynomially, or else exponentially.

degree of polynomial growth $< \dim H_*(M;\mathbb{Q})$

Example $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^d = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$

Skolem-Mahler-Lech theorem

Let $A \in GL_N(\mathbb{Z})$ be an invertible integer matrix. Fix a point $p \in \mathbb{Z}^N$ and an affine (linear) subspace $K \subseteq \mathbb{Z}^N$. Consider

$$(*) \quad \{ d \in \mathbb{Z} : A^d(p) \in K \} \subseteq \mathbb{Z}$$

Theorem The subset $(*)$ is quasi-arithmetic (= union of a finite set and a periodic set = union of a finite set and finitely many arithmetic progressions)

If f is invertible (a diffeomorphism), this applies to

$$\{ d \in \mathbb{Z} : a_d = \text{fixed number} \}$$
$$\{ d \in \mathbb{Z} : b_d = \text{fixed number} \}$$

The proof of Skolem-Mahler-Lech uses p-adic analysis: fix a prime p. For suitable $d \gg 0$,

$$A^d \equiv \mathbb{1} \mod p, \text{ or } p^2, \ldots$$

Hence, A^d is p-adically close to the identity, and admits a p-adic logarithm. This transforms the problem from a "discrete" $(\mathbb{Z})$ to a "continuous" $(\mathbb{Z}_p)$ one.

Corollary If f is invertible and $\{b_d\}$ is bounded, it is eventually periodic. The same holds for $\{a_d\}$.

The Thurston classification

This concerns invertible maps $f: M \longrightarrow M$ for a surface M.

Theorem The fixed point numbers a_d are either periodic in d, or grow exponentially.

This is not obvious from the general discussion, since M is not simply-connected.

Theorem The numbers b_d (for L_0, L_1 curves on a surface) are either periodic, or grow linearly, or exponentially

No higher degree polynomial growth!

In fact, surface diffeomorphisms up to deformation (homotopy) fall into 3 classes:

- periodic
- completely reducible (anything grows at most linearly, but a_d remains bounded)
- Pseudo-Anosov, where everything grows exponentially (curves get exponentially stretched, etc.)

Growth in symplectic topology

Let M be a symplectic manifold and f a symplectic automorphism. Fix two Lagrangian submanifolds L_0, L_1. We define

Phase space of a classical mechanical system

fix position or momenta

a_d = essential fixed points of f^d

b_d = essential intersection points of $f^d(L_0)$ and L_1

but where only deformations that are symplectic (given by Hamiltonian flows) are allowed.

Have examples with exponential growth (easy, since the topology gives lower bounds); and of polynomial growth (automorphisms of integrable systems; geodesic flow on nilmanifolds)

$\boxed{\text{Question: What can one say about the growth of } a_d \text{ or } b_d\text{?}}$

What about high degree polynomials? Bounded sequences? Intermediate growth?

Bell, J. LMS 2006

Periodicity phenomena?

Fix a symplectic manifold M, an invertible self-map f, and Lagrangian submanifolds L_0, L_1 (as before, but now with some simplifying technical assumptions, e.g. exactness). Consider

(*) $\{ d \in \mathbb{Z} :$ $f^d(L_0)$ is Floer-theoretically isomorphic to L_1 with some choice of flat line bundle (which can depend on d) $\} \subseteq \mathbb{Z}$

$\boxed{\text{Conjecture: The set (*) is always quasi-arithmetic.}}$

For examples, apply mirror symmetry to algebraic automorphisms of cluster varieties.

Compare Nonlinear Skolem-Mahler-Lech:

$\boxed{\begin{array}{l}\textbf{Theorem (Bell)}\ M \text{ affine algebraic variety, } f \text{ an automorphism, } p \in M \text{ a point,}\\ K \subseteq M \text{ a subvariety. Then}\\ \qquad \{ d \in \mathbb{Z} : f^d(p) \in K \}\\ \text{is quasi-arithmetic.}\end{array}}$

The analogue for birational maps is a major open conjecture (the "dynamical Mordell-Weil conjecture" of Zhang)

For this, we would like to apply Skolem-Mahler-Lech to "moduli spaces of Lagrangians".

Cutkowski-Srinivas, Annals, 1993

The growth of linear systems

Let X^n be a smooth projective variety. Choose a line bundle L and a vector bundle E.

Consider

$$c_d = h^0(E \otimes \overbrace{L \otimes \cdots \otimes L}^{d})$$

Lemma c_d grows at most polynomially (and its degree is bounded by $n = \dim X$)

This is easy: choose an ample line bundle N such that $L \otimes N$ is ample. Then

$$c_d \leq h^0(E \otimes L^{\otimes d} \otimes N^{\otimes d})$$

and the dimension of the larger space grows polynomially $\sim d^n$ (Riemann-Roch + vanishing).

Lemma If L is topologically trivial ("numerically trivial"), c_d is bounded

This is also easy: one can get a uniform bound over $\forall L \in \mathrm{Pic}^0$.

> **Theorem** (Cutkowski-Srinivas) If L is topologically trivial, c_d is eventually periodic.

Uses Skolem-Mahler-Lech techniques

> **Note** There are examples (X a $\mathbb{P}^1$-bundle over an abelian surface) where
>
> $$c_d \sim \left(6 + 2\frac{\sqrt{3}}{9}\right) d^3$$
>
> irrational!

Algebras of intermediate growth

e.g. arXiv 1506.01241

Let $\mathfrak{g}$ be a Lie algebra.
The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is generated by elements of $\mathfrak{g}$, with the relations

$$x \cdot y - y \cdot x = [x, y]$$

Example If $\mathfrak{g}$ = vector fields, $\mathcal{U}(\mathfrak{g})$ = differential operators

Poincaré-Birkhoff-Witt describes $\mathcal{U}(\mathfrak{g})$ as a vector space;

$$\mathcal{U}(\mathfrak{g}) \cong \mathrm{Sym}^*(\mathfrak{g}).$$

Suppose $\mathfrak{g}$ carries an additional grading (not a super-Lie-algebra)
Let's count dimensions:

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \qquad \boxed{\gamma_d = \dim \mathfrak{g}_d}$$

$$\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{g})_0 \oplus \cdots \qquad \boxed{\mu_d = \dim \mathcal{U}(\mathfrak{g})_d}$$

PBW implies

$$\sum_{d=0}^{\infty} \mu_d q^d = \prod_{d=1}^{\infty} (1-q^d)^{-\gamma_d}$$

Corollary If γ_d are > 0 and bounded,

$$\mu_d \sim e^{\sqrt{d}} \quad \sim \#\text{ of partitions}$$

There are simple examples where γ_d has this property, and $\mathfrak{g}$ is finitely generated (hence, we can forget the grading).

Question What about the "higher cohomology" of $\mathcal{U}(\mathfrak{g})$?

Dynamics of the tensor product

Let A be a finite-dimensional dg algebra with good homological properties (say, a directed dga). We consider the derived category of bimodules over A, and the dynamics is iterated (derived) tensor product:

Start with a bimodule P, how does

$$P^{\otimes_A d} = \underbrace{P \otimes^L_A \cdots \otimes^L_A P}_{d}$$

behave? Here, we measure the size of P by the dimension of $H^*(P)$. At most exponential growth holds.

How about lower bounds? Suppose A is over a finite field. Then

$$\#\left(\begin{array}{l}\text{bimodules of size} \leq s \\ \text{up to isomorphism}\end{array}\right) \lesssim C^{s^2}$$

> **Corollary** If the size of $P^{\otimes_A d}$ grows too slowly,
>
> $$\sim \varepsilon \sqrt{\log d} \qquad (\varepsilon \text{ small})$$
>
> then $P^{\otimes_A d}$ is eventually periodic.

Proof Pigeon-hole principle: there are

$$C^{\varepsilon(\sqrt{\log d})^2} = d^{\varepsilon \log C}$$

possibilities for $P^{\otimes_A d}$, but if $\varepsilon \log C < 1$, they have to repeat.

Question Geometric interpretation?

Return to classical topology

Let M again be a manifold, but not assumed to be simply-connected, and $\tilde{M} \longrightarrow M$ its universal cover. Take $f : M \longrightarrow M$ and lift it to $\tilde{f} : \tilde{M} \longrightarrow \tilde{M}$. The Nielsen fixed point "number" is a kind of trace of

$$\tilde{f}_* : H_*(\tilde{M};\mathbb{Q}) \longrightarrow H_*(\tilde{M};\mathbb{Q}),$$

taking into account its nature as a module over $R = \mathbb{Q}[\pi_1 M]$. It is an element

$$L(\tilde{f}) \in \frac{R}{[R,R]} = HH_0(R).$$

More concretely, this "number" is a sum over conjugacy classes with integer coefficients,

$$L(\tilde{f}) = \sum_{C \subset \pi_1(M)} \gamma_C \langle C \rangle$$

Define a single number

$$|L(\tilde{f})| = \sum_C |\gamma_C| \geq 0.$$

Question (As far as I know, completely open) What is the behaviour of $|L(\tilde{f}^d)|$?

We know at most exponential growth. But, does boundedness imply periodicity ?

— Finis —