Quantum Steenrod operations \( \rightarrow \) part joint work with Wilkins

Recall

\[ H^*(\mathbb{R}P^n, \mathbb{F}_p) \]

is one-dimensional in each degree \( \geq 0 \).

The ordinary Steenrod operation

\[ S^k : H^k(M; \mathbb{F}_p) \rightarrow (H^*(M; \mathbb{F}_p) \otimes \mathbb{F}_p)^{k} \]

\[ S^k (x) = x^p + (\text{terms with } a + a^p \text{ or } \theta) \]

alternatively,

\[ S^k (x) = \pm \left( \frac{p-1}{2} \right) x + (\text{terms in } H^*(M), x > 1x1) \]
From now on, let $M$ be a closed monotone symplectic manifold. The quantum Steenrod operation is originally due to Fukaya.

\[ \text{QSt} : H^k(M; \mathbb{F}_p) \to (H^*(M; \mathbb{F}_p)[[t, G]])^p \]

\[ \text{QSt}(x) = x \underset{p}{\times} \ldots \underset{p}{\times} x + \text{(terms with } \theta \text{ a t)} \]

\[ \text{Example} \quad M = S^2, \quad p = 2. \]

\[ \text{QSt}(1) = 1, \quad \text{QSt}(\text{point}) = \text{point} + \text{point} + \text{point} \]

\[ H^2(M; \mathbb{F}_p) \]

graded mod 2, with curves in class $A$ contributing with grading $-2c_1(A)$

more computation: Wilkins' papers
Application to Hamiltonian dynamics

Suppose $H_x(M)$ is torsion-free. A Hamiltonian diffeomorphism $\phi: M \to M$ is called a "nondegenerate pseudorotation" if

- the only periodic points of $\phi$ are fixed points
- $\phi(x) = x = D\phi_x$ has no $\sqrt{1}$ as eigenvalues
- $|\text{Fix } \phi| = \text{rank } H_x(M)$

Thm (Salamon–Zehnder) $c_1(M) = 0$, then $M$ can't admit pseudo-rotations.

Thm (Ginzburg–Gürel) $c_1(M) = -[\omega_{\mathbb{R}}]$, then $M$ can't admit pseudo-rotations.

Ex. $\mathbb{R}^2$ irrational rotation
Theorem (Cintioli-Ginzburg-Gürel; Shelukhin) \( S_t \)

Suppose that for some \( p \), \( \ast^p(\{ \text{point} \}) = \{ \text{point} \} \).

Then \( M \) can't admit a pseudorotation.

Hence, we should focus on manifolds with lots of rational curves (rational curves through every point).

Example (S. Wilkins) The cubic surface \( \mathbb{CP}^2 \) blown up at 6 points, with its monotone symplectic form does not admit a pseudorotation. \( \ast_p \) computation

Example \( T^4 \) blown up at a point has no pseudorotations

Example \( \mathbb{CP}^2 \) blown up at \( \leq \) 3 points, with \( \omega \) taut, is tautic and hence has a pseudorotation.
Formal structure (this description is inaccurate in many ways)

Gromov-Witten theory

\[ \overline{M}_{0, p+1}(M; A) \]

Deligne-Mumford

evaluate at \( p \)

\[ M^p \]

evaluate locally

\[ M \times \overline{M}_{0, p+1} \]

Symp-equivariant theory

(permute p marked points)

\[ H^\ast_{\text{Symp}}(\overline{M}_{0, p+1}(M; A); \pm 1) \]

local \( \mathbb{F}_p \)-coefficients

pullback

\[ H^\ast_{\text{Symp}}(\overline{M}^p; (\pm 1)) \]

\[ H^\ast_{\text{Symp}}(M^p \times \overline{M}_{0, p+1}; \pm 1) \]

\[ H^\ast_{\text{Symp}}(M; \mathbb{F}_p) \otimes \]

equivariant \( p \)-th power

\[ \otimes H^\ast_{\text{Symp}}(\overline{M}_{0, p+1}; \pm 1) \]

quantum Steenrod
This means that operations are parametrized by $H^x_{\text{Symp}}(\overline{M}_g^{p+1}; \pm 1)$, which is unfortunately unknown. But we have the unique non-free orbit $O \subset \overline{M}_g^{p+1}$, with stabilizer $\mathbb{Z}_p$, and correspondingly can specialize the operations to

$$H^x(M_i; \mathbb{F}_p) \rightarrow H^x(M_i; \mathbb{F}_p) \otimes H^x_{\text{Symp}}(O_p; \pm 1)$$

$\text{QST}$

$$\cong H^x(M_i; \mathbb{F}_p) \otimes H^x_{\mathbb{Z}_p}(\text{part}) = H^x(M_i; \mathbb{F}_p) \prod \delta \eta.$$

By localization, $H^x_{\text{Symp}}(\emptyset)$ is "most of" $H^x_{\text{Symp}}(\overline{M}_g^{p+1})$. 

$\{ \cdots \}$
looking at the geometry, we see that there is a further natural operation,

$$\Phi \Sigma : H^*(M; \mathbb{F}_p) \otimes H^*(M; \mathbb{F}_p) \rightarrow H^*(M; \mathbb{F}_p)[t, \theta]$$

$$(\alpha, \chi) \mapsto \Phi \Sigma_{\alpha}(\chi)$$

Let's extend $\Phi \Sigma$ to an endomorphism of $H^*(M; \mathbb{F}_p)[t, \theta]$.

\[
\begin{align*}
\Phi \Sigma_{\alpha}(1) &= \Phi \mathcal{S}t(\alpha) \\
\Phi \Sigma_{\alpha} \circ \Phi \Sigma_{\beta} &= \Phi \Sigma_{\alpha \ast \beta} \\
\Rightarrow \Phi \Sigma_{\alpha}(\Phi \mathcal{S}t(\beta)) &= \Phi \mathcal{S}t(\alpha \ast \beta)
\end{align*}
\]
Theorem (S-Wilkins) For any $\alpha$, $Q\Sigma_\alpha$ is covariantly constant with respect to the quantum connection.

For that to make sense, we define our cohomology over a ring $\mathbb{F}_p$:

$$\sum_{A \in H^2(M; \mathbb{Z})} C_A q^A, \quad C_A \in \mathbb{F}_p$$

$A = 0$ or $\text{wt}(A) > 0$

Any $\beta \in H^2(M; \mathbb{Z})$ gives an operation $\triangledown_{\beta} (q^A) = (\beta \cdot A) q^A$

The quantum connection is:

$$\triangledown_{\beta} = t\partial_{\beta} + \partial_{\beta} \star$$

For Gromov-Witten theory, determines $Q\Sigma$ up to the ideal formed by $q^A$, $A \not= 0$, $A \cdot \beta = 0$ (p) for all $\beta$. $\triangledown_{\beta} q^A = 0$ for all $\beta$.

The first case that eludes computation: double covers of $(-1)$-curves in $4d$ (p=2).

- classical sheafroid
Homological algebra

Recall that quantum Steenrod operation on $H^k(M;\mathbb{F}_p)$ are parametrized by $H^1_\text{Sym} (\overline{M}_{0,p+1}; (-1)^k)$. There is a classical analogue, Cohen's computation of the equivariant homology of configuration space. Essentially, the only interesting class is

$$H^*_{\text{Sym}}(\text{Conf}_p(C); -1) \cong \mathbb{F}_p.$$
The consequence is that

\[ G = t^{p-1} \text{-coefficient of } \Delta S, \]

acting on \( H^{\text{odd}}(k, \mathbb{T}_p) \)

(if \( p > 2 \))

has an elementary meaning in homological algebra, as part of the operations carried by any \((\mathbb{F}_p)\)-algebra \( \mathbb{T}_p \), such as the Hochschild cohomology of an algebra over \( \mathbb{T}_p \).

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**Example** A algebra over \( \mathbb{T}_p \);

if \( D : A \to A \) is a derivation,

then so \( \overset{\circ}{D} \) is \( D \in H^1(A) \)

\( D^p = D \circ \cdots \circ D \).

For \( p = 3 \), \( [D^p] \in H^3(A) \)

\[ D^3(ab) = D^2(\alpha D(b) + D(a)b) \]

\[ = D(D(a)D(b) + \alpha D^2(b)) \]

\[ + D(a)D(b) + D^2(a)b \]

\[ = \alpha D^3(b) + D^3(a)b \]

\[ + 3 D^2(a)D(b) + 3 D(a)D^2(b). \]
For symplectic geometry, this means that if $F(M)$ is the Fukaya category over $\mathbb{F}_p$, then

$$H^{\text{odd}}(M, [F_p]) \xrightarrow{\phi_{\mathcal{F}}} H^{\text{odd}}(M, [F_p])$$

The particular operation is easy to construct on (non-$S^1$-equivariant) symplectic cohomology or string topology.

"Exercise" look at superpotentials with $1d$ critical locus.

There is an explicit expression for the algebraic operations (Tourtchine).
Different homological algebra interpretation

\[ \text{\(\mathbb{Z}_p\)-equivariant version of Hochschild homology, easy to construct through cyclic tensor product of bimodules.} \]

\[ \text{\(\mathbb{Z}_p\)-equivariant cyclic homology} \]

\[ \text{\(A = A_{\infty}\)-algebra} / F_p \]

We take the \(p\)-fold module structure of \(\text{HH}^*(A)\) acting on \(\text{HH}^*(A)\), and that is naturally \(\mathbb{Z}_p\)-equivariant (think of it as endomorphisms of the bimodule \(A\) acting on the cyclic tensor product).
The outcome is expected to be $a = \text{Fuk}(\mathcal{M})$.

\[ H^*(\mathcal{M}) \otimes H^*(\mathcal{M})[[t, \theta]] \xrightarrow{\Psi} H^*(\mathcal{M})[[t, \theta]] \]

\[ \downarrow \quad \text{OC} \otimes \text{OC}_{\delta_1} \downarrow \quad \text{OC}_{\delta_1} \]

\[ HH^*(\mathcal{A}) \otimes HH^*(\mathcal{A})[[\theta]] \xrightarrow{\text{Z}^\text{equivariant module}} HH^*(\mathcal{A})[[\theta]] \]

The image of $\text{LEH}^*(\mathcal{M})$ under $\text{OC}_{\delta_1}$ describes the Calabi-Yau structure of the Fukaya category (Equation).

We can specialize to that class to get

\[ HH^*(\mathcal{A}) \xrightarrow{\text{spectral sequence}} HH^*(\mathcal{A})[[\theta]] \]

If we use the CY structure and a splitting of the Hodge-de Rham spectral sequence, we can get “algebraic Steenrod operations” on $HH^*(\mathcal{A}, \mathcal{A}) \simeq HH^*(\mathcal{A}, \mathcal{A})$. 


what might this look like concretely?

On category $\mathcal{D}^b \text{Clo}_e(X)$, $p > \dim(X)$,

$$H^0(X) \cong \mathcal{H}^0(t, \mathcal{O}_X) \exists f \text{ act on the de Rham complex by}$$

$$\eta \mapsto f^p \eta$$

(note $d(f^p \eta) = f^p d \eta + p f^{p-1} \frac{df}{dt} \eta$)

If we have $\xi \in H^0(t, \mathcal{F}) \subseteq \mathcal{H}^1(X)$ that should act on the de Rham complex (enhanced with $t \xi$ and $\theta$) by operations of degree $p$.

$p = 2 \xi^2$ = square of our vector field as a derivation

$$\eta \mapsto \left( t \xi \xi \eta + t \xi \eta \right) \theta.$$

After applying $d \circ - \circ d$. 

$\xi \eta \mapsto t \xi \eta$