The quantum connection and its mod p reduction

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FIRST PART : Pomerleano - S.
SECOND PART : S.
} writing up...

CLOSELY RELATED : Zihong Chen, arXiv 2409.03922

Motivation (with HMS in mind)

Hi Maxim  !

$f \in \mathbb{C}[x_1, \ldots, x_n]$ polynomial with an isolated critical point at $x = 0$.

Algebraic invariant The Milnor ring

$$\mathbb{C}[[x]]/(Df) = \mathbb{C}[[x_1, \ldots, x_n]]/(\partial_1 f, \ldots, \partial_n f)$$

$$\dim_{\mathbb{C}} \mathbb{C}[[x]]/(Df) = \mu < \infty$$

Thm (Briançon-Skoda) $f^n \in (Df)$

Thm (Le) $f \in (Df)$ iff the critical point is equivalent to a weighted homogeneous one.

Topological invariant H = cohomology of the Milnor fibre of f, carries the monodromy $\phi : H \xrightarrow{\cong} H$. In fact

$$H \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathbb{C}[[x]]/(Df)$$

and one can obtain ϕ from the algebraically defined GM connection.

Thm ϕ is quasi-unipotent (eigenvalues are roots of unity). The Jordan blocks are of size $\leq n$.

Thm (Varchenko, Scherk) If $f^k \in (Df)$, the Jordan blocks are of size $\leq k$

Example (Milnor) If f is weighted homogeneous, the monodromy is of finite order.

Quantum cohomology M projective smooth Fano variety of dimension n.

q a formal variable of degree 2,

$$\left(H^*(M;\mathbb{C}[q]),\ *_q = \cup + q\cdots \right)$$

$\swarrow$ $q=0$

classical cohomology ring

$\searrow$ invert q

(or reduce the grading mod 2 and set $q=1$), generally get a simpler ring

$$q^{-1} c_1(M) *_q : H^*(M;\mathbb{C}[q^{\pm 1}]) \longrightarrow H^*(M;\mathbb{C}[q^{\pm 1}])$$

has eigenvalues $\lambda_1, \dots, \lambda_m$, ring decomposes as

$$\bigoplus_{i=1}^{m} H^*(M;\mathbb{C}[q^{\pm 1}])_{\lambda_i}$$

Quantum connection t another deg 2 variable,

$$\left(H^*(M;\mathbb{C}[q,t]),\ \nabla_{tq\partial_q} = tq\partial_q + c_1(M)*_q \right)$$

$\swarrow$ $t=0$

quantum cohomology

$\searrow$ invert q and t-complete

$$H^*(M;\mathbb{C}[q^{\pm 1}][[t]])$$

An elementary formal power series argument shows:

Lemma There is a unique splitting of $H^*(M;\mathbb{C}[q^{\pm 1}][[t]])$ into m pieces, compatible with $\nabla_{tq\partial_q}$ and which for $t=0$ reduces to

It is more intuitive to consider

$$\nabla_{t\partial_q} = q^{-1}\nabla_{tq\partial_q}$$

which has degree zero, and to introduce $\tau = t/q$, so our space is $H^*(M;\mathbb{C}[q^{\pm 1}][[\tau]])$. In degree d we can write

grading in cohomology

$$\nabla_{t\partial_q} = \nabla_{-\tau^2\partial_\tau} = -\tau^2\partial_\tau + (q^{-1}c_1(M)*\cdot) - \tau\frac{Gr - d}{2}$$

$\nabla_{\partial_\tau} = -\tau^{-2}\nabla_{-\tau^2\partial_\tau}$ has a quadratic pole at $\tau = 0$ (corresponds to $q/t = \infty$)

<u>Thm</u> (Unramified exponential type; Chen 2024, or with added assumptions, Pomerleano-Seidel 2023) The λ_i summand of the quantum connection ∇_{∂_τ} (in degree d) is isomorphic to

$$-\lambda_i\tau^{-2} + \nabla^{\lambda_i}_{\partial_\tau}$$

quadratic pole is scalar

regular singularity (simple pole)

<u>Remark</u> This is elementary if $q^{-1}c_1(M)*_q$ is diagonalizable (see previous splitting)

We also know $\nabla^{\lambda_i}_{\partial_\tau}$ has quasi-unipotent monodromy. What is the Jordan normal form of the monodromy?

Thm (Pomerleano - S. 2023) If M admits a smooth anticanonical divisor, the monodromy of $\nabla^{\lambda_i}_{\partial_\tau}$ has Jordan blocks of size

$$\le \begin{cases} n+1 & \text{unipotent blocks} \\ n & \text{other roots of unity} \end{cases}$$

Petrov - Vaintrob - Vologodski

The proof uses a noncommutative version of the monodromy theorem.

Remark If M had no rational curves, (impossible in algebraic geometry, but maybe possible symplectically) $*_q = \cup$, $\nabla_{t\partial_q}$ has regular singularities at $q/t = 0, \infty$ and the monodromy has a unipotent block of size $n+1$.

Thm (Pomerleano - S.) Suppose that the λ_i-Jordan blocks of $q^{-1}c_1(M) *_q \cdot$ are of size $\le k_i$. Then the same applies to the monodromy of $\nabla^{\lambda_i}_{\partial_\tau}$

Corollary If $q^{-1}c_1(M) *_q \cdot$ is diagonalizable, the monodromy of $\nabla^{\lambda_i}_{\partial_\tau}$ is of finite order.

Remark Dubrovin's work shows that, if the quantum cohomology is semisimple, the monodromy is ± 1 (depending on degree d)

Following Zihong Chen's work, the $\boxed{\text{theorem}}$ is proved by reduction to characteristic p (meaning, using cohomology with coefficients in a characteristic p field F). The behaviour of connections in >0 characteristic is largely governed by their p-curvature (see Katz' work), the q-linear map

$$\psi_p = \nabla_{tq\partial_q}^p - t^{p-1}\nabla_{tq\partial_q} = t^p \nabla_{t\partial_q}^p$$

Lemma (Pomerleano-S., "Jae's conjecture" in the Fano case):

$$\psi_p = Q\Sigma_{c_1(M)}$$

with $c_1(M)$ is the quantum Steenrod operation.

Quantum Steenrod operations are an equivariant form of enumerative invariants based on rational curves,

$$Q\Sigma_b : H^*(M; F[q^{\pm 1}, t, \theta]) \circlearrowleft \quad \text{degree } p|b|$$

$b \in H^*(M; F[q^{\pm 1}])$

θ is a variable of degree 1, which forms $H^*(B\mathbb{Z}/p; F) = F[t, \theta]$. Facts:

- $Q\Sigma_b$ commutes with $\nabla_{tq\partial_q}$ (covariant constancy, S.-Wilkins)
- $Q\Sigma_{b_1} \circ Q\Sigma_{b_2} = Q\Sigma_{b_1 *_q b_2}$

Cor If $f \in \mathbb{Z}[x]$ is the minimal polynomial of $q^{-1} c_1(M) *_q$, then

$$f(\nabla_{t\partial_q}^p) = 0$$

Question What can one say about the Jordan block size of $q^{-1} c_1(M) *_q \cdot$ (bounded by $\dim_{\mathbb{C}}(M)$) ?

motivation very shaky (mirror not obliged to have isolated critical points)

(deep breath)

Splittings More generally, suppose in $H^*(M;\mathbb{C}[q^{\pm 1}])$ we have

$$1 = e_1 + \dots + e_m \qquad e_i *_q e_j = \begin{cases} e_i & i=j \\ 0 & i \neq j \end{cases}$$

Idea:

splitting of quantum cohomology by the e_i

closed-open map

splitting of the Fukaya category

need open-closed to be an isomorphism

splitting of other HMS structures, such as the quantum connection

Conjecture There is a canonical splitting of $H^*(M;\mathbb{C}[q^{\pm 1}][[t]])$ which is compatible with $\nabla_{tq\partial_q}$ and which, when reduced to $t=0$, recovers the splitting of $H^*(M;\mathbb{C}[q^{\pm 1}])$ given by the e_i.

weasel word

Example If e_i are the projectors to the generalized eigenspaces of $q^{-1}c_1(M)*_q$, elementary theory yields the splitting (as we mentioned before).

unique

Example If $H^*(M;\mathbb{C}[q^{\pm 1}]) = \bigoplus \mathbb{C}[q^{\pm 1}]\, e_i$ (semisimplicity), we have such a splitting from Dubrovin's work (uses higher WDVV).

preferred

We take inspiration from a different partial result: if we use coefficients in a field F of characteristic p, then

$$Q\Sigma_{e_i} : H^*(M;F[q^{\pm 1},t]) \to H^*(M;F[q^{\pm 1},t])$$

(no 0-component in this case)

is the desired splitting, and works without formal completion in t. Such a global splitting (over all of $\tau = t/q$) would be impossible in characteristic zero.

From now on: $R = \mathbb{Z}_p$ (or the ring of integers in a p-adic field). We assume for simplicity that the homology of M has no p-torsion.

Definition $R\langle\langle\tau\rangle\rangle \subseteq R[[\tau]]$ is the ring of those power series $x = \sum_{k=0}^{\infty} x_k \tau^k$ with this property:

> there is an $A>0$ such that for all $m \geq 1$, reduction mod p^m yields a polynomial in τ of degree $\leq A \cdot p^m$

(A slight strengthening of convergence on the closed unit disc). We define $H^*(M; R[q^{\pm 1}]\langle\langle t\rangle\rangle)$ in the same way. Because of the grading, this means:

$$x = \sum_{k=0}^{\infty} x_k t^k, \quad x_k \in H^*(M; R[q^{\pm 1}]) \text{ and}$$

after reduction mod p^m, x_k has only powers $q^{-A \cdot p^m}$ and higher.

Theorem (S.) Given $e_i \in H^*(M; R[q^{\pm 1}])$, there is a canonical splitting of the quantum connection over $R[q^{\pm 1}]\langle\langle t \rangle\rangle$

$$\begin{tikzcd} & \text{quantum connection} \arrow[dl, "t=0"'] \arrow[dr, "\text{reduce mod } p"] & \\ \text{splitting of } H^*(M; R[q^{\pm 1}]) \text{ given by } e_i & & \text{splitting of } H^*(M; R/p\,[q^{\pm 1}, t]) \text{ given by } Q\Sigma_{e_i} \end{tikzcd}$$

This implies (and is stronger than) our original conjecture, over R. The construction lifts quantum Steenrod operations $Q\Sigma_b$ to p-adic ones, but works only for an idempotent $b = e_i$ (or more precisely, b which has p^m-th roots for all m)

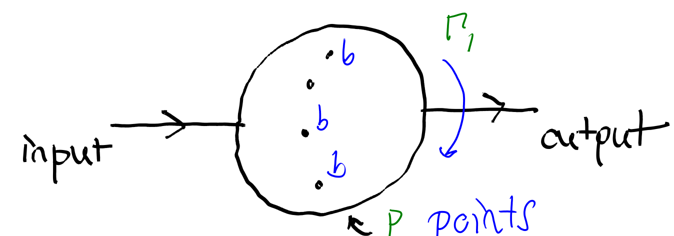
let $\Gamma_m = \mathbb{Z}/p^m\mathbb{Z}$ as a group, with $\Gamma_m \leq \Gamma_{m+1}$ and Γ_∞ their union.

$$H^*(B\Gamma_m; R) = \begin{cases} R & * = 0 \\ 0 & * \text{ odd} \\ R/p^m & * > 0 \text{ even} \end{cases}$$

and the limit is a discrete replacement for S^1:

$$H^*(B\Gamma_\infty; R) = \varprojlim H^*(B\Gamma_m; R) = R[t]$$

Reminder of quantum Steenrod $Q\Sigma_b$,



We can use R-coefficients, but the interesting part is still p-torsion.

We also have Γ_m-equivariant generalizations,

$$H^*(M;R) \xrightarrow[Q\Sigma_{m,b}]{} H^*(M;R[q^{\pm1}])\otimes H^*(B\Gamma_m;R)$$

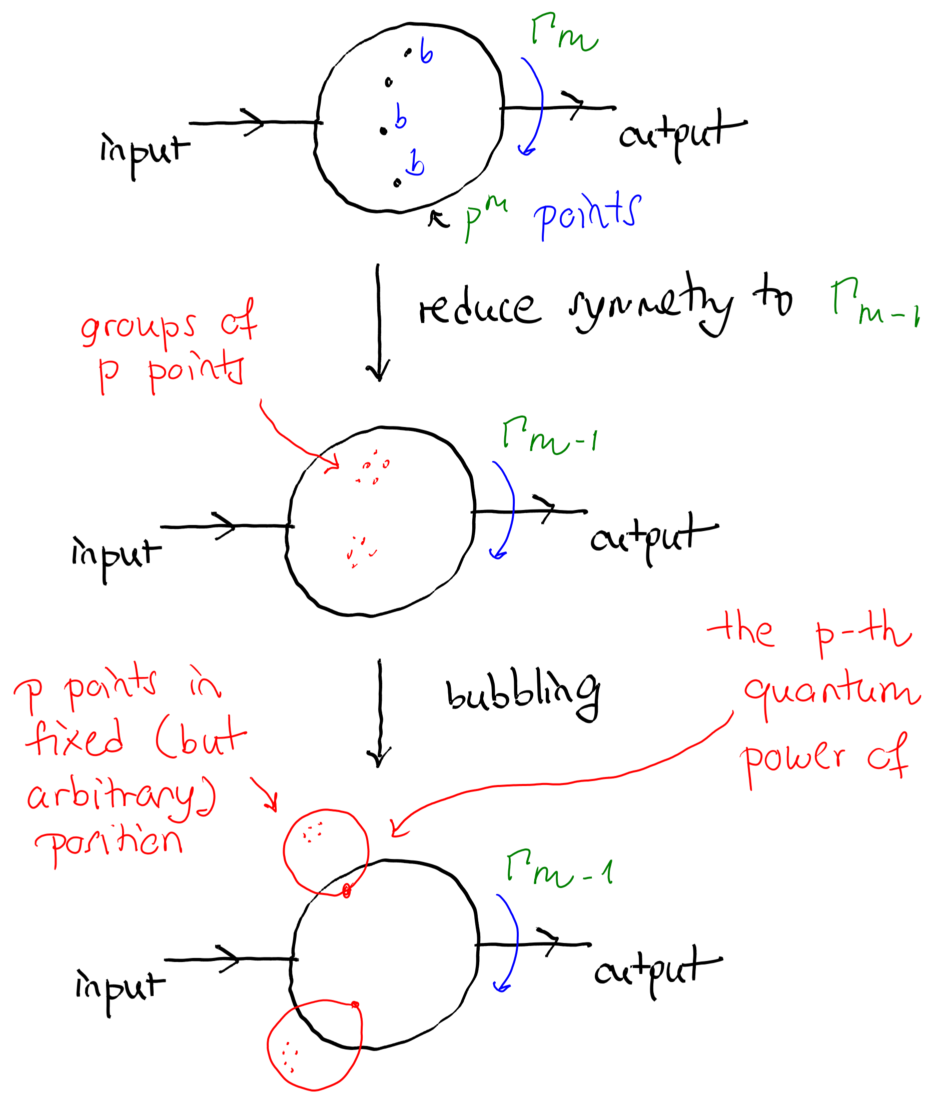
using p^m points, and the outcome is p^m-torsion. Reducing m:

$$\begin{tikzcd}
H^*(M;R) \arrow[r, "Q\Sigma_{m,b}"'] \arrow[d, equal] & H^*(M;R[q^{\pm1}])\otimes H^*(B\Gamma_m) \arrow[d] \\
H^*(M;R) \arrow[r, "Q\Sigma_{m-1,\, b^{*_q p}}"'] & H^*(M;R[q^{\pm1}])\otimes H^*(B\Gamma_{m-1})
\end{tikzcd}$$

p-fold quantum power

If $b=e_i$ is an idempotent, $b^{*_q p}=b$ and we can pass to $\varprojlim$ to get our splitting.

Picture explanation of the diagram:



Question Take other situations with an irregular singularity (e.g. $X \to \mathbb{A}^1$ and the Fourier-Laplace transform of the Gauß-Manin $\mathcal{D}$-module). Do we have similar p-adic splittings?

Question Splittings beyond the Fano case (e.g. for blowups)? The problem is that we have used Gromov-Witten theory integrally.

Keep track of the p^{-1} that can occur for curves of a given degree, what is the correct weaker statement?

Use better tech (Bai-Xu) to work integrally for general manifolds, when is this still the same structure and do the $Q\Sigma$ still have the usual properties?

Question Implications of the use of TC (e.g. splitting of QH^* over $\mathbb{F}_p$ $\Rightarrow$ splitting of the quantum connection, with t inverted, over $\mathbb{Z}_p$?)

(the end)