The quantum connection: topology, analysis, arithmetic

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October 2025

(partly based on joint work with D. Pomerleano, and S. Bai–D. Pomerleano)

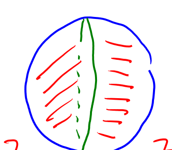
Motivation: <u>disc-counting superpotentials</u>

M^{2n} closed monotone symplectic manifold (e.g. $\mathbb{CP}^n$, $Gr_k(\mathbb{C}^N)$, flag variety)

L monotone Lagrangian torus in M
$L \cong T^n \subseteq M$, $\omega_M|L = 0$ (e.g. the circle dividing $\mathbb{CP}^1 = S^2$ into halves)

$W_L \in \mathbb{Z}[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$ disc-counting superpotential of L, e.g. $S^1 \subset \mathbb{CP}^1$ has

$$W_L(z) = z + \frac{1}{z}$$



z $\quad$ z^{-1}

<u>Ex</u> $M = \mathbb{CP}^1$, L_1 Clifford, L_2 Chekanov [Auroux]

$$W_{L_1} = z_1 + z_2 + \frac{1}{z_1 z_2} \qquad W_{L_2} = z_1 + \frac{(1+z_2)^2}{z_1^2 z_2}$$

<u>Question</u> For a given M, what superpotentials W_L can occur? It turns out that the period

$$\Pi_M = (2\pi i)^{-n} \int_{|z_1| = \dots = |z_n| = 1} e^{qW_L} \frac{dz_1}{z_1} \wedge \dots \wedge \frac{dz_n}{z_n} =$$

$$= \sum_{d \geq 0} q^d \frac{(\text{constant term of } W_L^d)}{d!}$$

depends only on M, not on L.

<u>Theorem</u> (Tonkonog)

$$\Pi_M = 1 + \sum_{d \geq 2} q^d \langle \psi_{d-2}(pt) \rangle$$

$pt \in H^{2n}(M)$; and $\langle \psi_{d-2}(\cdot) \rangle : H^*(M) \to \mathbb{Q}$ is the Gromov-Witten invariant with gravitational descendants.

Let's look at this period

$$\Pi_M = 1 + \sum_{d \geq 2} q^d \langle \psi_{d-2}(pt) \rangle \in \mathbb{Q}[[q]]$$

Ex Constant term of $(z+z^{-1})^d$ is $\binom{d}{d/2}$

so $\Pi_{\mathbb{P}^1} = \sum_{d \text{ even}} q^d \left(\frac{d}{2}!\right)^{-2}$

Archimedean viewpoint $\mathbb{Q} \subseteq \mathbb{C}$:
this series converges for all $q \in \mathbb{C}$.

Why? If there is a Lagrangian torus, it follows from writing it as $\int e^{qW_L}$. In general, it follows from the quantum differential equation (which is the main topic of this talk)

Non-archimedean viewpoint $\mathbb{Q} \subset \mathbb{Q}_p$ ($p \neq 2$ prime): the series converges for

$$|q| < p^{-1/(p-1)}$$

which is the same radius of convergence as $e^q = 1 + q + q^2/2 + \cdots$ Why? If there is a Lagrangian torus, it's because

$$\frac{(\text{constant term of } W^d)}{d!} \in \frac{1}{d!}\mathbb{Z}$$

In general, this is a bound on the denominators of $\langle \psi^{d-2}(pt) \rangle$. One can see it from Tonkonog's paper, or (again) from the quantum differential equation.

which brings us to...

The quantum differential equation (or maybe better "connection")

$$\nabla_{\partial_q} : H^*(M)[q^{\pm 1}] \longrightarrow H^*(M)[q^{\pm 1}]$$

$$\nabla_{\partial_q}(x) = \frac{dx}{dq} + q^{-1}(c_1(M) *_q x)$$
$$= \frac{dx}{dq} + q^{-1} c_1(M) \cup x + c_1(M) *^1 x + q\, c_1(M) *^2 x + q^2 \dots$$

$*^k : H^*(M) \otimes H^*(M) \longrightarrow H^{*-2k}(M)$ are the components of the quantum product (Gromov-Witten invariants). The initial one $*^0$ is just the cup product.

Ex $M = \mathbb{CP}^1$, $\nabla_{\partial_q} = \frac{dx}{dq} + 2\begin{pmatrix} 0 & q \\ q^{-1} & 0 \end{pmatrix}$.

It is sometimes useful to set $q=1$, which gives us a $\mathbb{Z}/2$-graded map

$$c_1(M) *_{q=1} : H^*(M) \longrightarrow H^*(M)$$

The eigenvalues of this play an important role in understanding the symplectic topology of M, for instance:

Prop L monotone Lagrangian torus $\Rightarrow$ the critical values of W_L are eigenvalues of $c_1(M) *_{q=1}$.

Ex $M = \mathbb{CP}^1$, $c_1(M) *_{q=1} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ has eigenvalues ± 2; critical values of $W_L = z + z^{-1}$ are $W_L(\pm 1) = \pm 2$.

Structure of the equation:

$$\nabla_{\partial_q} : H^*(M)[q^{\pm 1}] \longrightarrow H^*(M)[q^{\pm 1}]$$

$$\nabla_{\partial_q}(x) = \frac{dx}{dq} + q^{-1}(c_1(M) \ast_q x)$$

This has a simple pole at $q=0$ and a more complicated pole at $q=\infty$. One can apply a monomial base change

$$q^{\deg/2} \nabla_{\partial_q} q^{-\deg/2} = \frac{dx}{dq} + c_1(M) \ast_{q=1} x + q^{-1} \frac{\deg}{2}$$

($\deg$ multiplies each cohomology class by its degree) and this reduces the pole at $q=\infty$ to a quadratic pole with leading term $-c_1(M) \ast_{q=1}$

The pole at $q=0$. There is a base change Φ over $\mathbb{Q}[[q]]$ such that

$$\Phi^{-1} \nabla_{\partial_q} \Phi = \frac{d}{dq} + q^{-1} c_1(M) \cup .$$

This is elementary, but the answer has geometric meaning [Givental]

$$\Phi : H^*(M)[[q]] \longrightarrow H^*(M)[[q]]$$

$$\int_M \Phi(x) \cup y = \int_M x \cup y + \text{gravitational descendant}$$

$$+ \sum_{d>0} (-1)^{?+1} q^d \langle \psi^{?}(x), y \rangle$$

In particular, if $x \in H^*(M)$ satisfies $c_1(M) \cup x = 0$, then $\Phi(x) \in H^*(M)[[q]]$ is a solution: $\nabla_{\partial_q} \Phi(x) = 0$.

For instance $x = pt \in H^{2n}(M)$ has $c_1(M) \smile pt = 0$; and then,

$$\int_M 1 \cdot \phi(pt) = \text{solution of quantum diffeq}$$

$$= 1 + \sum_{d \geq 2} (-q)^d \langle \psi^{d-1}(pt), 1 \rangle$$

$$= 1 + \sum_{d \geq 2} (-q)^d \langle \psi^{d-2}(pt) \rangle = \Pi_M(-q)$$

is what we saw in Tonkonog's theorem.

<u>The pole at $q=\infty$</u> $\mathbb{C}((q^{-1}))$ = finitely many powers of q, infinite series in q^{-1} (Laurent series around $q=\infty$).

<u>Thm</u> There is a base change over $\mathbb{C}((q^{-1}))$, $\Psi : H^*(M)((q^{-1})) \longrightarrow H^*(M)((q^{-1}))$,

$$\Psi^{-1} \nabla_{\partial_q} \Psi = \frac{d}{dq} + c_1(M) \underset{q=1}{*} \cdot + q^{-1} A$$

and the extra term A respects the decomposition into (generalized) eigen-spaces for $c_1(M) *_{q=1} \cdot$

[Zihong Chen 2024], following partial resuts of [Pomerleano-Seidel 2023].

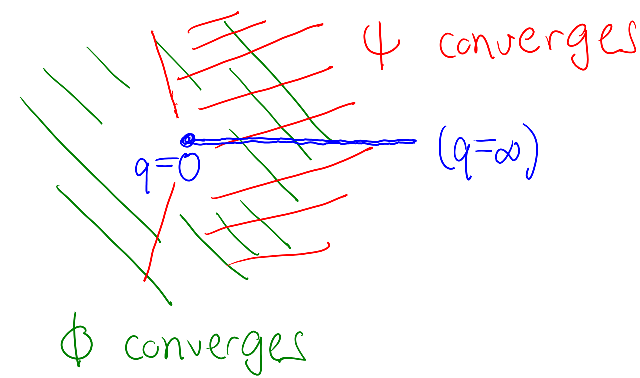
The proofs are nothing like what you might imagine!

(Archimedean) convergence

Around $q=0$: for elementary ODE reasons, Φ must be convergent over all of $q \in \mathbb{C}$ ($\Rightarrow$ same holds for Π_M, as stated before).

Around $q=\infty$: the map Ψ can't be locally convergent ($\Rightarrow$ would be a contradiction with Φ, concerning monodromy) but it can lifted in the sense of asymptotic analysis, uniquely on sectors with angle $\pi+\varepsilon$, $\varepsilon>0$ small:

Recap and Gamma conjecture



Use the eigenvalue from Ψ; transport it from $q \approx \infty$ to $q \approx 0$ along the real axis (parallel transport for connection); apply Φ^{-1} to get splitting of $H^*(M;\mathbb{C})$. This is an entirely transcendental process.

Conjecture This splitting of $H^*(M)$ is compatible with the $\mathbb{Q}$-vector subspace

$$\hat\Gamma(M) \smile H^*(M;\mathbb{Q})$$

where $\hat\Gamma(M) \in 1 + H^{>0}(M;\mathbb{C})$ is the characteristic class associated to $\hat\Gamma(z) = \Gamma(1+z)$

[Galkin-Golyshev-Iritani], [Katzarkov-Kontsevich-Pantev]. We write formally

$$1 + c_1(M) + c_2(M) + \cdots = \prod_{k=1}^{n} (1+\delta_i)$$

and then

Taylor expansion of $\hat\Gamma$.

$$\hat\Gamma(M) = \prod_{k=1}^{n} \hat\Gamma(1+\delta_i)$$

Why Chern classes? Why $\hat\Gamma$?

The non-archimedean situation

Background: differential equations over $\mathbb{Q}_p$

Ex $\frac{dx}{dq} = x$ is solved by $x = e^q$, which has p-adic convergence radius $p^{-1/(p-1)}$

Definition A Frobenius structure is a base change Θ which relates a diffeq (in one variable q) with its pullback via $q \mapsto -q^p/p$. The Frobenius is overconvergent if Θ has radius of convergence $> p^{-1/(p-1)}$

Ex [Dwork] For $\frac{dx}{dq} = x$, Frobenius

$$\Theta(q) = e^{q + q^p/p}$$ has greater radius of convergence than e^q

The quantum connection over $\mathbb{Q}_p$

Near $q=0$, the (Givental) map $\Phi(q)$ is locally p-adically convergent for elementary reasons.

Conjecture [Bai-Pomerleano-S. 25] $\Phi(q)$ has convergence radius $\geq p^{-1/(p-1)}$

Near $q=\infty$, the map $\Psi(q)$ again converges locally (unlike over $\mathbb{C}$!)

> **Theorem** [S. 25] The splitting over $\mathbb{Q}_p((q^{-1}))$ coming from the eigenvalue decomposition of $c_1(M) *_{q=1}$ (part of what Ψ gives you), $p \gg 0$, converges on the closed disc $|q| \geq 1$

Ex Quantum connection for $M = \mathbb{C}P^1$, written in $\tau = 1/q$

$$\nabla_{\partial_\tau} = \frac{d}{d\tau} - \tau^{-2} c_1(M) *_{q=1} \circ \; + \tau^{-1} \frac{\deg}{2}$$

The splitting matrices $\Pi_\pm$ for the eigenvalues ± 2 involve

$$h(\tau) = 1 + \sum_{d>0} \tau^{2d} \binom{2d-1}{d}^2 \frac{(2d)!}{2^{8d-2}}$$

very non-convergent over $\mathbb{C}$; but has convergence radius 1 over $\mathbb{Q}_p$, $p>2$, because $\binom{2d-1}{d}^2 (2d)!$ is an integer and $|\frac{1}{2}| = 1$. In fact the coefficients converge to 0 p-adically, since $\binom{2d-1}{d}^2 (2d)!$ is divisible by higher and higher powers of p.

The p-adic Gamma function

Remember the (ordinary) Gamma function "interpolates factorials", meaning that $\hat{\Gamma}(z) = \Gamma(1+z)$ satisfies

$$\hat{\Gamma}(n) = n! \qquad n = 0, 1, 2, \ldots$$

There is no direct p-adic analog, but we have the Morita function

$$\Gamma_p : \mathbb{Z}_p \longrightarrow \mathbb{Z}_p^\times$$

$$\Gamma_p(n) = (-1)^n \prod_{\substack{i=1 \\ p \nmid i}}^{n} i \qquad n = 0, 1, 2, \ldots$$

This has a Taylor expansion $\Gamma_p(z) = 1 + a_1 z + \cdots$, hence a characteristic class $\hat{\Gamma}_p(M) \in H^*(M; \mathbb{Q}_p)$

Conjecture [Bai-Pomerleano-S. 25]

Take the quantum connection for any monotone symplectic manifold M. Let

$$\Theta : H^*(M; \mathbb{Q}_p)[[q]] \longrightarrow H^*(M; \mathbb{Q}_p)[[q]]$$

be the unique Frobenius structure whose $q=0$ term is

$$x \longmapsto p^{-\frac{\deg}{2}} \Gamma_p(M) \smile x \in H^*(M; \mathbb{Q}_p)$$

Then $\Theta(q)$ is overconvergent.

Remarks

- Known in some cases (like the ordinary Γ-conjecture); e.g. $M = \mathbb{CP}^1$ is [Dwork 1974]
- No analysis involved (unlike the ordinary Γ-conjecture)