THE EXPONENTIAL TYPE CONJECTURE FOR THE QUANTUM CONNECTION

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Based on: Pomerleano - S. '23 and upcoming Zihong Chen '24
Quantum connection

$M^n$ smooth projective Fano / $\mathbb{C}$

The quantum connection in one variable $q$

$\nabla_{\partial q} : H^*(M; \mathbb{C})[q^{\pm 1}] \to H^*(M; \mathbb{C})[q^{\pm 1}]$

$\nabla_{\partial q} (x) = \partial_q x + q^{-1}(c_1(M) \times_q x)$

$= \partial_q x + q^{-1}(c_1(M) \times_0 x) + (c_1(M) \times_1 x) + q(c_1(M) \times_2 x) + \ldots$

$c_1(M) \times_0 x = c_1(M) \cdot x$

$c_1(M) \times_k x = \text{counts rational curves } C \text{ in } M \text{ with } c_2(M) \cdot C = k, \text{ going through two fixed cycles, with multiplicity } k$ (Gromov–Witten invariants)

Example $M = \mathbb{CP}^2$, $c_1(M) = 3$ [line]

$\nabla_{\partial q} = \partial_q + q^{-1} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} + q \begin{pmatrix} 0 & 0 & 2q \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Line through every two points

Example $M = \text{cubic surface, restrict to } \{ 1, c_1(M), \text{[point]} \} \subset H^*(M)$.

$\nabla_{\partial q} = \partial_q + \begin{pmatrix} 0 & 108q & 252q^2 \\ 0 & 3q^{-1} & 36q \\ 0 & 0 & 0 \end{pmatrix}$

$\Sigma 27 \text{ lines} = g c_1(M) \in H^2(M)$

The quantum connection is Laurent polynomial, with coefficients $q^{-1}, 1, \ldots, q^n$.

Many open conjectures about its singularities ($q=0$ and $\infty$).
DIFFERENTIAL EQUATIONS PRIMER

Let's look at formal connections

$$\nabla_q = \partial_q + A \quad A \in \text{Mat}_r(\mathbb{C}(q))$$

Classically, this means looking at the system of linear differential equations

$$\frac{dx}{dq} = -Ax \quad x \in \mathbb{C}(q)^r$$

We consider such connections up to q-dependent base change

$$\tilde{\nabla}_q = G \nabla_q G^{-1} \tilde{A}$$

$$= \partial_q + GAG^{-1} - G^{-1}(\partial_q G)$$

where $G \in \text{GL}_r(\mathbb{C}(q))$ (also called formal gauge transformations).

Def $\nabla_q$ has a regular singularity if it is gauge equivalent to a connection with a pole of order $\leq 1$.

Ex $A = \begin{pmatrix} q^{-1} & 0 \\ q^{-2} & q^{-1} \end{pmatrix}$ has a quadratic pole, but is regular:

$$G = \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} \Rightarrow \tilde{A} = \begin{pmatrix} q^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

Lemma $A = A_d q^d + A_{d+1} q^{d+1} + \cdots$ with $d \leq -2$ and $A_d$ not nilpotent $\Rightarrow$ the singularity is not regular singular (unfortunately not $\Leftrightarrow$)

Proof $\nabla_q \cdots \nabla_q x = q^{-p} A^p x + \cdots$ but in the regular case, the pole order grows with $q^{-p}$.
DIFFERENTIAL EQ, CONTINUED

Theorem (Turrittin) A connection does not have a regular singularity \(\iff\) after a gauge transformation involving \(q^{1/2}\), it can be tuned into

\[
\tilde{A} = \tilde{A}_d q^d + \cdots \quad d < -1 \quad (d \in \frac{1}{r!} \mathbb{Q})
\]

with \(\tilde{A}_d\) not nilpotent.

Example: \(A = \begin{pmatrix} 0 & -q^{-2} \\ -q^{-1} & -\frac{1}{2} q^{-1} \end{pmatrix}\). This has a quadratic pole and the \(q^{-2}\) part is nilpotent, so a priori we don't know if it's regular. Actually it's not, over \(\mathbb{C}[[q^{1/2}]]\) it's equivalent to

\[
\tilde{A} = \begin{pmatrix} q^{-3/2} & 0 \\ 0 & -q^{-3/2} \end{pmatrix}
\]

**Lemma** (Splitting Lemma) Take

\[
A = A_d q^d + A_{d+1} q^{d+1} + \cdots \quad d \leq -2
\]

Then there is a unique splitting \(\mathbb{C}[[q]]^r = \bigoplus_{\lambda} E_{\lambda}\) compatible with the connection, which at \(q=0\) reduces to the (gen.) eigenspace decomposition for \(A_d\): there is a \(\mathbb{C}[[q]]\)-gauge transformation so that \(\tilde{A}\) splits like \(A_d\).

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**Def** A connection has unramified exponential type if it is equivalent to

\[
\bigoplus \left( \nabla_{\lambda}^{\text{reg}} + \lambda \cdot q^{-2} I \right)
\]

finitely many \(\lambda\in \mathbb{C}\)

where \(\nabla_{\lambda}^{\text{reg}}\) have a regular singularity

Example Not unramified exponential type
The quantum connection has unramified exponential type at $q=\infty$.

(Ealkin-Iritani, Katzarkov-Kontsevich-Pantev)

Easy to check in any one example

Ex True if $x \mapsto c_1(M) \ast x$ is semisimple (e.g. projective space) Just apply the splitting lemma!

<table>
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<tr>
<th>Thm (Pomerleano-S '23)</th>
<th>True if $M$ admits a smooth anticanonical divisor (or &quot;symplectic divisor&quot;)</th>
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<td>Thm (Zhong Chen 24)</td>
<td>True if the Fukaya category of $M$ satisfies Abouzaid's generation criterion.</td>
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Def: A holonomic D-module is a vector space \( N \) with an action of \( q : N \to N, \nabla_{\partial q} : N \to N \) satisfying a finite generation and torsion condition.

Lemma: \( N \) holonomic \( \Rightarrow \) there is \( \phi(q) \in \mathbb{C}[q][\partial q] \) such that

\[ \mathbb{C}[q, \frac{1}{q}] \otimes_{\mathbb{C}[q]} N \]

is a vector bundle on \( \mathbb{C}\{p=0\} \) with an algebraic connection \( \nabla_{\partial q} \).

**Fourier-Laplace transform of** \( N \)

just set \( t = \partial_q, \partial_t = -q \)

and we get a holonomic D-module in \( t \) (and vice versa).

**Thm (Malgrange):** Suppose \( N \) is a holonomic D-module in \( t \), with only regular singularities, including at \( t=\infty \). Then its Fourier-Laplace transform has

- a regular singularity at \( q=0 \)
- unramified exponential type at \( q=\infty \).
INPUT FROM NUMBER THEORY

$K$ a field of characteristic $p > 0$. Take a connection $\nabla_{\partial q} = \partial q + A$ over $K$. The $p$-curvature is

$$F_{\partial q} = \nabla_{\partial q}^p \in \text{Mat}_p(K(lq))$$

This contains no derivatives, and it behaves in a very simple way under gauge transformations: $F = GFG^{-1}$.

Ex $p=3$, note $\partial_q^3 = 0$

$$F_{\partial q} = (\partial_q + A)(\partial_q + A)(\partial_q + A) = (\partial_q + A)(\partial_q^2 + A' + 2AQ + A^2) = A'' + A'A - AA' + A^3$$

$A' = (\partial A/\partial q)$

$\mathbb{R}/\mathbb{Q}$ finitely generated ring (integers in a number field, with finitely many primes inverted). We can reduce mod a max ideal, $k = \mathbb{R}/\mathbb{M}$

**Thm (Katz)** Take a connection over $\mathbb{R}$, and suppose that for all $m$, the reduction to $k$ has nilpotent $p$-curvature. Then the connection has a regular singularity.

There is also a global version

**Proof** Not regular $\Rightarrow$ after a gauge transformation in $q^{1/2}$, we have

$$\tilde{A} = A_d q^d + \ldots, \quad d < -1$$

$\Rightarrow \tilde{F}_{\partial q} = \tilde{A}_d^p q^{pd} + \ldots$
INPUT FROM HOMOLOGICAL ALGEBRA

* A dg algebra (or category) over $\mathbb{C}$. (Negative) cyclic homology $\text{HC}_*(A)$ is a module over $\mathbb{C}[[u]]$, where $u$ is a formal variable (of degree 2). If $A$ depends on an additional parameter $t$, then $\text{HC}_*(A_t)$ comes with a connection $\nabla_{u t}$ (Getzler-Gauss-Manin). One can simplify things by passing to the periodic version

$$\text{HP}_*(A) = \mathbb{C}[[u]] \otimes_{\mathbb{C}[[u]]} \text{HC}_*/A$$

This is a noncommutative version of de Rham cohomology and its connection.

Take $A$ over $\mathbb{C}[t, \frac{1}{p(t)}]$ (punctured affine line) and assume

PROPER: $\text{H}^*(A)$ finite rank over $\mathbb{C}[t, \frac{1}{p(t)}]$  

HOMOLOGICALLY SMOOTH OVER $\mathbb{C}[t, \frac{1}{p(t)}]$:
noncommutative replacement for standard algebro-geometric smoothness

Then $\text{HP}_*(A)$ is (in each degree) a vector bundle on the punctured line

**Thm (Petrov-Vaintrob-Vologodski)** The Getzler-Gauss-Manin connection has only regular singularities (incl. at $\infty$)

(Hard, uses reduction to char. $p$ and the global version of Katz’ theorem)
We would like to apply Fourier-Laplace, but $q$ invertible on $H^\cdot(M)[q^\pm 1] \Rightarrow \Delta_t$ invertible on the Fourier-Laplace transform, not a geometrically natural situation. Recall we have DCM smooth anticanonical divisor.

**STEP 1.** Extend the quantum connection to a holonomic D-module on

$$H^\cdot(M)[q] \oplus \bigoplus_{w \geq 1} H^\cdot(D) z^w$$

(Done using deformed symplectic cohomology; presumably equivalent to punctured relative Gromov-Witten theory.)

**STEP 2.** Fourier-Laplace transform, then remove finitely many values of $t = \Delta q$, means $\otimes_{t \in \mathbb{C}[t]} \mathbb{C}[t, \frac{1}{t}]$.

**STEP 3.** Realize the outcome as Getzler-Gauss-Manin for a smooth and proper dg category over $\mathbb{C}[t, \frac{1}{t}]$.

↑

(categorical Fourier-Laplace transform)

**STEP 4.** Apply Petrov-Vainroth-Vologodski to get regular singularities for CCM, then apply Malgrange's theorem to get a conclusion for the quantum connection!)
CATEGORICAL FOURIER-LAPLACE

Let $A$ be a differential graded algebra, and $W \in A^0$, $dW = 0$, be a central element. Two interpretations:

(t) Action of $W$ makes $A$ into a dga over $C[t]$. Better use free resolution:

$$A_t = \{ A[t] \xrightarrow{t-W} A[t] \}$$

(degrees shifted down by 1)

(q) Formal variable, $q$ of degree 2,

$$A_q = (A[[q]], qW)$$

is a curved dga over $C[[q]]$ (all associated constructions should be $q$-completed)

Recall $HC_x$ cyclic homology, write $CC_x$ for the underlying chain complex. We use $HC_x(A_t)$ formed over $C[t]$, $HC_x(A_q)$ formed over $C[[q]]$ and the "negative $q$-power" $H_x(CC_x(A_q) \otimes \frac{C(q)}{C[[q]]})$

Thm There is an isomorphism

$$H_x(CC_x(A_q) \otimes \frac{C(q)}{C[[q]]}) \cong HC_{x+2}(A_t)$$

action of $q$ \hspace{1cm} action of $t$

Getzler-Gauss-Manin $\Delta_{A_q}$ \hspace{1cm} Getzler-Gauss-Manin $\Delta_{A_t}$

In our geometric application, $A$ is the wrapped Fukaya category of $M \setminus D$, and $A_q$ comes from compactifying to $\mathbb{M}$. 
ZHONG CHEN'S PROOF (ROUGH SKETCH)

We actually consider a form of the quantum connection with added formal variable \( u \) (graded if \( 1q = 2, 1u = 2 \))

\[
\nabla u \partial_q = u \partial_q + q^{-1} (c_1(M) \cdot \partial_q) - c_1(M) \cdot \partial_u + \ldots
\]

After reduction to char(\( k \)) = \( p \),

\[
F u \partial_q = \nabla u \partial_q^p
= q^{-p} (c_1(M))^p - u^{p-1} c_1(M) + \ldots
\]

This reminds us of something: Steenrod operations on cohomology,

\[
\mathcal{P}: H^*(M; k) \rightarrow H^*(M; k)[u, \theta]
\]

\( 1\theta = 1 \)

then \( \mathcal{P}(c_1(M)) = c_1(M)^p - u^{p-1} c_1(M) \)

There is a \( q \)-deformation, quantum Steenrod operations (Fukaya, Wilkins) For each \( b \in H^*(M; k) \),

\[
Q \Sigma_b : H^*(M; k)[u, \theta; q] \rightarrow H^*(M; k)[u, \theta; q]
\]

\( Q \Sigma_b(x) = \mathcal{P}(b) \cdot x + O(q) \)

![Diagram]

\[ \mathbb{P} \text{ points} \rightarrow \mathbb{Z}/p \text{ output} \]

**Lemma** (S-Wilkins) \( Q \Sigma_b \) commutes with \( \nabla u \partial_q \), hence with \( F u \partial_q \).

**Lemma** \( Q \Sigma c_1(M) - q^p F u \partial_q \) is nilpotent. So if \( Q \Sigma c_1(M) \) were nilpotent, we could apply Katz' criterion (but that's not what happens at all)
CONCLUSION OF CHEN's PROOF

where does $q=\infty$ appear in this formalism? It's built in:

$$H^*(M)[q^{\pm 1}][u] \cong H^*(M)[q^{\pm 1}][u/q]$$

In each degree $j$, $\cong H^j \text{mod } 2(M)[[q]]$ and the connection $\nabla_{\partial_q}$ has a quadratic pole at $q=0$ ($\sim q=\infty$)

with that in mind, we can apply the splitting lemma:

$$H^*(M)[q^{\pm 1}][u] \cong \bigoplus_{\lambda} E_{\lambda}$$

over all eigenvalues $\lambda$ of $q^{-1}c_1(M)_q$

This can also be done over a suitable $\mathcal{R}\mathcal{CQ}$, hence reduced to $k$

Using Fukaya categories and assuming Abouzaid's generation criterion, one has:

**Thm** (Chen) (i) $\Phi \Sigma_b$ preserves the decomposition into $E_{\lambda} \otimes_k k$

(ii) If $b$ lies in the (generalized) $\mu$-eigenspace of $q^i\partial_q^*$, and $\lambda \neq \mu$, then $\Phi \Sigma_b$ acts trivially on $E_{\mu} \otimes_k k$.

On each $H^i$ we can consider $\nabla \Phi \otimes_q -\lambda$ ($\sim \nabla_{\partial_q} + \lambda q^{-2}$) with p-curvature $F_{\Phi \otimes_q} - \lambda p$. This is nilpotent $\iff \Phi \Sigma_{c_1(M)} - \lambda q$ nilpotent, but that will follow from (ii).

Hence (Katz) $\nabla_{\partial_q} + \lambda q^{-2}|E_{\lambda}$ has a regular singularity $\Rightarrow$ we get a splitting as in unramified exp. type!