# THE EXPONENTIAL TYPE CONJECTURE FOR THE QUANTUM CONNECTION

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Based on: Pomerleano-S. '23 and upcoming
Zihong Chen '24

# QUANTUM CONNECTION

$M^n$ smooth projective Fano $/\mathbb{C}$

The quantum connection in one variable $q$

$$\boxed{\begin{aligned} &\nabla_{\partial_q} : H^*(M;\mathbb{C})[q^{\pm 1}] \longrightarrow H^*(M;\mathbb{C})[q^{\pm 1}] \\ &\nabla_{\partial_q}(x) = \partial_q x + q^{-1}(c_1(M) *_q x) \\ &\quad = \partial_q x + q^{-1}(c_1(M) *_0 x) + \\ &\qquad (c_1(M) *_1 x) + q(c_1(M) *_2 x) + \cdots \end{aligned}}$$

$$c_1(M) *_0 x = c_1(M) \smile x$$

$c_1(M) *_k x$ = counts rational curves $C$ in $M$ with $c_1(M) \cdot C = k$, going through two fixed cycles, with multiplicity $k$ (Gromov-Witten invariants)

**Ex** $M = \mathbb{CP}^2$, $c_1(M) = 3\,[\text{line}]$

$$\nabla_{\partial_q} = \partial_q + q^{-1}\begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} + q^2 \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

line through every two points

**Ex** $M$ = cubic surface, restrict to $\{1, c_1(M), [\text{point}]\} \subseteq H^*(M)$.

$$\nabla_{\partial_q} = \partial_q + \begin{pmatrix} 0 & 108\,q & 252\,q^2 \\ 9q^{-1} & 3q^{-1} & 36\,q \\ 0 & 0 & 0 \end{pmatrix}$$

$\sum$ 27 lines $= 9\,c_1(M) \in H^2(M)$

The quantum connection is Laurent polynomial, with coefficients $q^{-1}, 1, \ldots, q^n$.
Many open conjectures about its singularities ($q=0$ and $\infty$).

# DIFFERENTIAL EQUATIONS PRIMER

Let's look at formal connections

$$\boxed{\nabla_{\partial_q} = \partial_q + A \qquad A \in \mathrm{Mat}_r(\mathbb{C}((q)))}$$

Classically, this means looking at the system of linear differential equations

$$\frac{dx}{dq} = -Ax \qquad x \in \mathbb{C}((q))^r$$

We consider such connections up to $q$-dependent base change

$$\tilde{\nabla}_{\partial_q} = G \nabla_{\partial_q} G^{-1}$$

$$= \partial_q + \overbrace{G A G^{-1} - G^{-1}(\partial_q G)}^{\tilde{A}}$$

where $G \in GL_r(\mathbb{C}((q)))$ (also called formal gauge transformations).

**Def** $\nabla_{\partial_q}$ has a regular singularity if it is gauge equivalent to a connection with a pole of order $\leq 1$.

**Ex** $A = \begin{pmatrix} q^{-1} & 0 \\ q^{-2} & q^{-1} \end{pmatrix}$ has a quadratic pole, but is regular:

$$G = \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} \Rightarrow \tilde{A} = \begin{pmatrix} q^{-1} & 0 \\ q^{-1} & 0 \end{pmatrix}$$

**Lemma** $A = A_d q^d + A_{d+1} q^{d+1} + \dots$ with $d \leq -2$ and $A_d$ not nilpotent $\Rightarrow$ the singularity is not regular singular (unfortunately not $\Leftrightarrow$)

**Proof** $\underbrace{\nabla_{\partial_q} \cdots \nabla_{\partial_q}}_{p} x = q^{-pd} A_d^p x + \dots$ but in the regular case, the pole order grows with $q^{-p}$

# DIFFERENTIAL EQ, CONTINUED

**Theorem** (Turrittin) A connection does not have a regular singularity $\iff$ after a gauge transformation involving $q^{1/r!}$, it can be turned into

$$\tilde{A} = \tilde{A}_d q^d + \cdots \qquad d < -1 \quad (d \in \tfrac{1}{r!}\mathbb{Q})$$

with $\tilde{A}_d$ not nilpotent.

**Ex** $A = \begin{pmatrix} 0 & -q^{-2} \\ -q^{-1} & -\frac{1}{2}q^{-1} \end{pmatrix}$. This has a quadratic pole and the $q^{-2}$ part is nilpotent, so a priori we don't know if it's regular. Actually it's not, over $\mathbb{C}((q^{1/2}))$ it's equivalent to

$$\tilde{A} = \begin{pmatrix} q^{-3/2} & 0 \\ 0 & -q^{-3/2} \end{pmatrix}$$

**Lemma** (Splitting Lemma) Take

$$A = A_d q^d + A_{d+1} q^{d+1} + \cdots \qquad d \leq -2$$

Then there is a unique splitting $\mathbb{C}[[q]]^r = \bigoplus_\lambda E_\lambda$ compatible with the connection, which at $q=0$ reduces to the (gen.) eigenspace decomposition for $A_d$: there is a $\mathbb{C}[[q]]$-gauge transformation so that $\tilde{A}$ splits like $A_d$.

**Def** A connection has *unramified exponential type* if it is equivalent to

$$\bigoplus_{\substack{\text{finitely} \\ \text{many } \lambda \in \mathbb{C}}} \left(\nabla_\lambda^{reg} + \lambda \cdot q^{-2} I\right)$$

where $\nabla_\lambda^{reg}$ have a regular singularity

Ex Not unramified exponential type

BACK TO THE QUANTUM CONNECTION

$$\nabla_{\partial_q} = \partial_q + q^{-1}(c_1(M) *_0 \cdot) + \cdots$$

has a simple pole (hence regular) at $q=0$. Change to $Q = q^{-1}$

$$\nabla_{\partial_Q} = \frac{\partial q}{\partial Q} \nabla_{\partial_q} = -Q^{-2} \nabla_{\partial_q}$$
$$= \partial_Q - Q^{-1}(c_1(M) *_{Q^{-1}} \cdot)$$

apparently has an order $(n+1)$ pole pole, but is equivalent (roughly) to

multiplies $H^j(M)$ by $j/2$

$$\boxed{\tilde{\nabla}_{\partial_Q} = \partial_Q - Q^{-2}(c_1(M) * \cdot) + Q^{-1}\frac{Gr}{2}}$$

all contributions counted with weight 1

**Conj** The quantum connection has unramified exponential type at $q=\infty$

(Galkin-Iritani, Katzarkov-Kontsevich-Pantev)

Easy to check in any one example

**Ex** True if $x \mapsto c_1(M) * x$ is semisimple (e.g. projective space) Just apply the splitting lemma!

**Thm** (Pomerleano-S '23) True if $M$ admits a smooth anticanonical divisor (or "symplectic divisor")

**Thm** (Zihong Chen 24) True if the Fukaya category of $M$ satisfies Abouzaid's generation criterion.

INPUT FROM ALGEBRAIC DIFF. EQ.

Def A holonomic D-module is a vector space $N$ with an action of

$$q : N \longrightarrow N, \quad \nabla_{\partial_q} : N \longrightarrow N$$
$$[\nabla_{\partial_q}, q] = id_N$$

satisfying a finite generation and torsion condition.

Lemma $N$ holonomic $\Rightarrow$ there is $p(q) \in \mathbb{C}[q] \setminus \{0\}$ such that

$$\mathbb{C}[q, 1/p] \otimes_{\mathbb{C}[q]} N$$

is a vector bundle on $\mathbb{C} \setminus \{p=0\}$ with an algebraic connection $\nabla_{\partial_q}$

Fourier-Laplace transform of $N$: just set

$$t = \nabla_{\partial_q}, \quad \nabla_{\partial_t} = -q$$

and we get a holonomic D-module in $t$ (and vice versa)

Thm (Malgrange) Suppose $N$ is a holonomic D-module in $t$, with only regular singularities, including at $t = \infty$. Then its Fourier-Laplace transform has

- a regular singularity at $q = 0$
- unramified exponential type at $q = \infty$.

# INPUT FROM NUMBER THEORY

$k$ a field of characteristic $p>0$. Take a connection $\nabla_{\partial_q} = \partial_q + A$ over $k$. The p-curvature is

$$F_{\partial_q} = \nabla_{\partial_q}^p \in \mathrm{Mat}_r(k((q)))$$

This contains no derivatives, and it behaves in a very simple way under gauge transformations: $\tilde{F} = GFG^{-1}$.

<u>Ex</u> $p=3$, note $\partial_q^3 = 0$

$$\begin{aligned} F_{\partial_q} &= (\partial_q + A)(\partial_q + A)(\partial_q + A) \\ &= (\partial_q + A)(\partial_q^2 + A' + 2A\partial_q + A^2) \\ &= A'' + A'A - AA' + A^3 \end{aligned}$$

$A' = (\partial A/\partial q)$

$R \subset \overline{\mathbb{Q}}$ finitely generated ring (integers in a number field, with finitely many primes inverted). We can reduce mod a max ideal, $k = R/m$

**Thm** (Katz) Take a connection over $R$, and suppose that for all $m$, the reduction to $k$ has nilpotent p-curvature. Then the connection has a regular singularity.

There is also a global version

**Proof** Not regular $\Rightarrow$ after a gauge transformation in $q^{1/r!}$, we have

$$\tilde{A} = \tilde{A}_d q^d + \dots \qquad d < -1$$

$$\Rightarrow \tilde{F}_{\partial_q} = \tilde{A}_d^p q^{pd} + \dots$$

not nilpotent

# INPUT FROM HOMOLOGICAL ALGEBRA

$\mathcal{A}$ a dg algebra (or category) over $\mathbb{C}$. (Negative) cyclic homology $HC_*(\mathcal{A})$ is a module over $\mathbb{C}[[u]]$, where $u$ is a formal variable (of degree 2). If $\mathcal{A}$ depends on an additional parameter $t$, then $HC_*(\mathcal{A}_t)$ comes with a connection $\nabla_{u\partial_t}$ (Getzler-Gauss-Manin). One can simplify things by passing to the periodic version

$$HP_*(\mathcal{A}) = \mathbb{C}((u)) \otimes_{\mathbb{C}[[u]]} HC_*(\mathcal{A})$$

This is a noncommutative version of de Rham cohomology and its connection.

Take $\mathcal{A}$ over $\mathbb{C}[t, 1/p(t)]$ (punctured affine line) and assume

PROPER: $H^*(\mathcal{A})$ finite rank over $\mathbb{C}[t, 1/f(t)]$

HOMOLOGICALLY SMOOTH OVER $\mathbb{C}[t, 1/f(t)]$: noncommutative replacement for standard algebro-geometric smoothness

Then $HP_*(\mathcal{A})$ is (in each degree) a vector bundle on the punctured line

$$\boxed{\begin{array}{l}\text{\underline{Thm} (Petrov-Vaintrob-Vologodski) The}\\ \text{Getzler-Gauss-Manin connection has}\\ \text{only regular singularities (incl. at } \infty)\end{array}}$$

(Hard, uses reduction to char. $p$ and the global version of Katz' theorem)

# POMERLEANO–S. PROOF (OUTLINE ONLY)

We would like to apply Fourier–Laplace, but $q$ invertible on $H^*(M)[q^{\pm 1}]$ $\Rightarrow$ $\nabla_{\partial_t}$ invertible on the Fourier–Laplace transform, not a geometrically natural situation. Recall we have DCM smooth anticanonical divisor.

STEP 1 Extend the quantum connection to a holonomic D-module on

$$H^*(M)[q] \oplus \bigoplus_{w \geq 1} H^*(D)\, z^w$$

(Done using deformed symplectic cohomology; presumably equivalent to punctured relative Gromov–Witten theory)

STEP 2 Fourier–Laplace transform, then remove finitely many values of $t = \nabla_{\partial_q}$, means $\otimes_{\mathbb{C}[t]} \mathbb{C}[t, 1/p(t)]$

STEP 3 Realize the outcome as Getzler–Gauss–Manin for a smooth and proper dg category over $\mathbb{C}[t, 1/p(t)]$

categorical Fourier–Laplace transform

STEP 4 Apply Petrov–Vaintrob–Vologodski to get regular singularities for GGM, then apply Malgrange's theorem to get a conclusion for the quantum connection!

# CATEGORICAL FOURIER-LAPLACE

Let $\mathcal{A}$ be a differential graded algebra, and $W \in \mathcal{A}^0$, $dW = 0$, be a central element. Two interpretations:

(t) Action of $W$ makes $\mathcal{A}$ into a dga over $\mathbb{C}[t]$. Better use free resolution:

$$\mathcal{A}_t = \{ \mathcal{A}[t] \xrightarrow{t-W} \mathcal{A}[t] \}$$

(degrees shifted down by 1)

(q) Formal variable $q$ of degree 2,

$$\mathcal{A}_q = (\mathcal{A}[[q]], qW)$$

is a curved dga over $\mathbb{C}[[q]]$ (all associated constructions should be $q$-completed)

Recall $HC_*$ cyclic homology, write $CC_*$ for the underlying chain complex. We use $HC_*(\mathcal{A}_t)$ formed over $\mathbb{C}[t]$, $HC_*(\mathcal{A}_q)$ formed over $\mathbb{C}[[q]]$ and the "negative q-power" $H_*\left(CC_*(\mathcal{A}_q) \otimes_{\mathbb{C}[[q]]} \frac{\mathbb{C}((q))}{\mathbb{C}[[q]]}\right)$

**Thm** There is an isomorphism

$$H_*\left(CC_*(\mathcal{A}_q) \otimes_{\mathbb{C}[[q]]} \frac{\mathbb{C}((q))}{\mathbb{C}[[q]]}\right) \cong HC_{*+2}(\mathcal{A}_t)$$

| | |
|---|---|
| action of $q$ | action of $t$ |
| Getzler-Gauss-Manin $\nabla_{\partial_q}$ | Getzler-Gauss-Manin $\nabla_{\partial_t}$ |

In our geometric application, $\mathcal{A}$ is the wrapped Fukaya category of $M \setminus D$, and $\mathcal{A}_q$ comes from compactifying to $M$.

ZIHONG CHEN'S PROOF (ROUGH SKETCH)

We actually consider a form of the quantum connection with added formal variable $u$ (graded if $|q| = 2$, $|u| = 2$)

$$\nabla_{u\partial_q} = u\partial_q + q^{-1}(c_1(M) *_q \cdot) \leftarrow c_1(M) \smile \cdot + \cdots$$

After reduction to $\mathrm{char}(k) = p$,

$$F_{u\partial_q} = \nabla_{u\partial_q}^p$$
$$= q^{-p}\left(c_1(M)^p - u^{p-1} c_1(M)\right) + \cdots$$

This reminds us of something: Steenrod operations on cohomology,

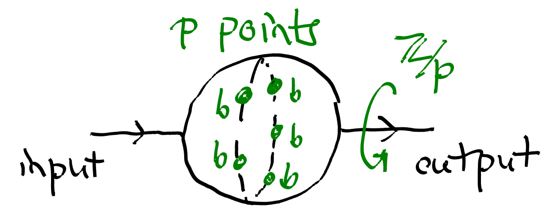
$$\mathcal{P} : H^*(M;k) \longrightarrow H^*(M;k)[u,\theta]$$
$$|\theta| = 1$$

then $\mathcal{P}(c_1(M)) = c_1(M)^p - u^{p-1} c_1(M)$

There is a q-deformation, quantum Steenrod operations (Fukaya, Wilkins)

For each $b \in H^*(M;k)$,

$$Q\Sigma_b : H^*(M;k)[u,\theta,q] \circlearrowleft$$
$$Q\Sigma_b(x) = \mathcal{P}(b) \smile x + O(q)$$




<u>Lemma</u> (S-Wilkins) $Q\Sigma_b$ commutes with $\nabla_{u\partial_q}$, hence with $F_{u\partial_q}$.

<u>Lemma</u> $Q\Sigma_{c_1(M)} - q^p F_{u\partial_q}$ is nilpotent

So if $Q\Sigma_{c_1(M)}$ were nilpotent, we could apply Katz' criterion (but that's not what happens at all)

# CONCLUSION OF CHEN'S PROOF

Where does $q=\infty$ appear in this formalism? It's built in:

$$H^*(M)[q^{\pm 1}][[u]] \cong H^*(M)[q^{\pm 1}][[\overset{Q}{\overset{\parallel}{u/q}}]]$$

In each degree $j$, $\cong H^{j \bmod 2}(M)[[Q]]$ and the connection $\nabla_{\partial_Q}$ has a quadratic pole at $Q=0$ $(\sim q=\infty)$

With that in mind, we can apply the Splitting Lemma:

$$H^*(M)[q^{\pm 1}][[u]] \cong \bigoplus_\lambda E_\lambda$$

over all eigenvalues $\lambda$ of $q^{-1} c_1(M) *_q$

This can also be done over a suitable $R \subset \overline{\mathbb{Q}}$, hence reduced to $k$

Using Fukaya categories and assuming Abouzaid's generation criterion, one has:

<u>Thm</u> (Chen) (i) $Q\Sigma_b$ preserves the decomposition into $E_\lambda \otimes_R k$

(ii) If $b$ lies in the (generalized) $\mu$-eigenspace of $q^{-1}c_1 *_q$, and $\lambda \neq \mu$, then $Q\Sigma_b$ acts trivially on $E_\mu \otimes_R k$.

On each $H_\lambda$ we can consider

$\nabla_{u\partial_q} - \lambda$ $(\Leftrightarrow \nabla_{\partial_Q} + \lambda Q^{-2})$ with p-curvature $F_{u\partial_q} - \lambda^p$. This is nilpotent $\Leftrightarrow$ $Q\Sigma_{c_1(M) - \lambda q}$ nilpotent, but that will follow from (ii).

Hence (Katz) $\nabla_{\partial_Q} + \lambda Q^{-2} \mid E_\lambda$ has a regular singularity $\Rightarrow$ we get a splitting as in unramified exp. type!