THE QUANTUM CONNECTION FOR FANO VARIETIES

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Reading suggestions:

- Dubrovin, Painlévé transcendants in 2d topological field theory (1999)
- Galkin-Iritani, Gamma classes and quantum cohomology ... (2016)
- Hugtenburg, The open-closed map, u-connections and R-matrices (2024)
- Zihong Chen, On the exponential type conjecture (2024)
- S, p-adic splittings of the quantum connection (2025)

Regular singular points

Look at a linear differential equation

$$\frac{dv}{dq} + A(q)\, v(q) = 0$$

Here $A(q)$ is a complex $r \times r$ matrix, holomorphically depending on $q \in \mathbb{C}$, and with a singularity at $q=0$ (and $q=\infty$). Suppose we have a simple non-resonant pole at $q=0$

$$A(q) = A_{-1} q^{-1} + A_0 + A_1 q + \cdots$$

and no two eigenvalues of A_{-1} differ by an integer. Then there is a unique holomorphic

$$\Phi(q) \in GL_r(\mathbb{C}), \qquad \Phi(0) = \mathbb{1}$$

such that

no other terms

$$\Phi(q)^{-1}\left(\frac{d}{dq} + A(q)\right)\Phi(q) = \frac{d}{dq} + A_{-1} q^{-1}$$

The equation on the right has sol'ns

$$v(q) = q^{-A_{-1}} v_0 = \exp(-A_{-1}\log(q))\, v_0$$

our original equation is solved by

$$v(q) = \Phi(q)\exp(-A_{-1}\log(q))\, v_0$$

In particular, the monodromy of solutions (going once around $q=0$) is conjugate to $\exp(-2\pi i A_{-1})$

For algebraic geometers: M^n is a smooth complex projective Fano variety.

For symplectic geometers: M^{2n} is a closed monotone symplectic manifold.

The cup product on $H^*(M)$ has a deformation, the quantum product

$$x *_q y = x \smile y + q(\ldots) + q^2(\ldots) + \ldots$$

counts rational curves $C \to M$ with $\int_C c_1(M) = 1$, hence q^1. There are only finitely many nonzero terms: one way to see this is to note that this is a graded product, $\deg(q) = 2$.

We are mostly interested in

$$c_1(M) *_q : H^*(M)[q^{\pm 1}] \longrightarrow H^*(M)[q^{\pm 1}]$$

<u>Example</u> $M = \mathbb{CP}^n$

$$c_1(M) *_q = (n+1) \begin{pmatrix} 0 & & & & q^{n+1} \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & 1 & \ddots & \\ & & & 1 & 0 \end{pmatrix}$$

This has eigenvalues $q\,(n+1)\sqrt[n+1]{1}$.

$$\boxed{\text{Question: Is there a bound on the Jordan block size of } c_1(M) *_{q=1} \cdot \text{, depending only on } \dim(M)\text{?}}$$

The quantum connection is a closely related object,

$$\nabla_{\partial_q} = \frac{\partial}{\partial q} + q^{-1} c_1(M) *_q \quad \text{on } H^*(M)[q^{\pm 1}]$$

the residue (pole term) is the classical cup product $c_1(M) \cup$. This is nilpotent, hence non-resonant:

$$\Phi^{-1} \nabla_{\partial_q} \Phi = \frac{\partial}{\partial q} + q^{-1} c_1(M) \cup$$

Enumerative geometry gives an explicit formula (Givental)

$$\int_M x \cup \Phi(y) = \int_M xy + \sum_{\substack{\ell \\ k>0}} q^k (-1)^\ell \langle x, T_\ell(y) \rangle_k$$

Gromov-Witten invariant with gravitational descendents.

Irregular singularities Let's look at

$$\frac{d}{d\tau} + A_m \tau^m + A_{m+1} \tau^{m+1} + \dots$$

for $m \leq -2$ (we have changed the variable, since our application will be to $\tau = q^{-1}$). There is always a formal power series $\psi(\tau)$, $\psi(0) = I$,

> $\psi(\tau)^{-1} \left(\frac{d}{d\tau} + A_m \tau^m + \dots \right) \psi(\tau)$ is block diagonal for the generalized eigenspace decomposition of A_m.

Example $\nabla_{\partial_\tau} = A_{-2} \tau^{-2} + A_{-1} \tau^{-1} + \dots$ and A_{-2} has pairwise different eigenvalues λ_j. Then

$$\nabla_{\partial_\tau} \underset{\text{over } \mathbb{C}((\tau))}{\cong} \bigoplus (\partial_\tau + \lambda_j \tau^{-2} + \mu_j \tau^{-1})$$

Def ∇_{∂_τ} has <u>unramified exponential type</u> if, over $\mathbb{C}((\tau))$, it is isomorphic to

$$\bigoplus_j \left(\partial_\tau + \lambda_j \tau^{-2} \cdot I + B_j \tau^{-1}\right)$$

In terms of $\tau = q^{-1}$, this means we have an isomorphism over $\mathbb{C}((q^{-1}))$ to

$$\bigoplus_i \left(\partial_q - \lambda_j \cdot I - B_j \cdot q^{-1}\right)$$

Def For a connection which is of unramified exponential type, we can define the <u>formal monodromy</u>

$$\bigoplus_i \exp(-2\pi i B_j)$$

Example Quadratic pole, A_{-2} semi-simple $\Rightarrow$ unramified exp. type.

For the quantum connection,

$$\nabla_{\partial_q} = \frac{\partial}{\partial q} + q^{-1} c_1(M) *_q$$

$$\Big\downarrow \; \tau = q^{-1}, \text{ some rescaling}$$

$$\nabla_{\partial_\tau} = \frac{\partial}{\partial \tau} - \tau^{-2} c_1(M) *_1 + \tau^{-1} \frac{Gr - n}{2}$$

on $H^*(M)[\tau^{\pm 1}]$, and Gr multiplies each cohomology class with its degree

Example $M = \mathbb{CP}^n$

$$\nabla_{\partial_\tau} = \frac{\partial}{\partial \tau} - (n+1) \begin{pmatrix} 0 & & & \tau^{-2} \\ \tau^{-2} & 0 & & \\ & \ddots & \ddots & \\ & & \tau^{-2} & 0 \end{pmatrix} + \begin{pmatrix} -\frac{n}{2}\tau^{-1} & & 0 \\ & \ddots & \\ 0 & & \frac{n}{2}\tau^{-1} \end{pmatrix}$$

This has unramified exponential type.

Theorem The quantum connection has unramified exponential type at $q=\infty$ $(\tau=0)$, and its formal monodromy is quasiunipotent (eigenvalues are roots of 1)

This is unrelated to the (geometric) monodromy, which is unipotent

(Zihong Chen, 2024; under an additional assumption on M, Pomerleano - S 2023)

Both proofs involve working with mod p cohomology and p-curvature (Katz' regularity criterion)

What more can we say about the formal monodromy?

- (Under the additional assumption that M contains a smooth anti-canonical divisor), the Jordan blocks of the formal monodromy have size $\leq n+1$ ($\leq n$ for eigenvalues $\neq 1$) (Pomerleano - S 2023)

- Suppose the Jordan blocks of $c_1(M) *_1 : H^*(M) \to H^*(M)$ are of size $\leq d$. Then the same holds for the formal monodromy (Pomerleano - S unpublished); there is a more precise statement involving nilpotent orbit closures.

Relation with singularity theory:

Let $L \subset M$ be a monotone Lagrangian torus. By counting holomorphic discs $u: (D^2, \partial D^2) \longrightarrow (M, L)$ with various boundary homology classes in $H_1(L) \cong \mathbb{Z}^n$, one defines the superpotential

$$W_L : H^1(L; \mathbb{C}^*) \longrightarrow \mathbb{C}$$

$$W_L \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$$

Theorem (Jack Smith) Suppose W_L has isolated critical points. Then the quantum cohomology ring of M $(q=1)$ contains the Jacobian ring

$$\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] / (\partial_{x_1} W_L, \partial_{x_2} W_L, \dots)$$

as a direct summand.

On this summand, $c_1(M) \underset{1}{*}$ acts by multiplication with W_L.

Example Clifford torus $L \subseteq \mathbb{CP}^n$,

$$W_L(x_1, \dots, x_n) = x_1 + \dots + x_n + \frac{1}{x_1 \cdots x_n}$$

has critical values $(n+1)\sqrt[n+1]{1}$.

Cor (of Jack Smith's theorem) The quantum connection on $H^*(M)[[\tau]]$ contains a direct summand $\cong$ to

$$\bigoplus_x -\tau^{-2}\lambda + \nabla_x^{GM}$$

x are the critical points, $\lambda = W_L(x)$

∇_x^{GM} (a version of) the Gauss-Manin connection of the critical point.

better: Fourier-Laplace transform

Quantum cohomology

- Is there a bound on the Jordan block size of $c_1(M) \ast_1$?

- (Chen) Formally near $\tau = 0$,
$\nabla_{\partial_\tau} \cong \bigoplus_j \lambda_j \tau^{-2} I + (\text{simple pole})$
formal monodromy is quasi-unipotent

- (Pomerleano-S) Under an added assumption, formal monodromy has Jordan blocks of size $\leq n+1$

- Suppose the Jordan blocks of $c_1(M) \ast_{q=1}$ are of size $\leq d$. Then the same holds for the formal monodromy (Pomerleano-S.)

Singularity theory

- (Briançon-Skoda) $f \in \mathbb{C}\{z_1, \dots, z_n\}$ with $f(0)=0$ isolated critical point $\Rightarrow f^n = 0 \in$ Jacobian ring

- Gauss-Manin connection is regular singular (isomorphic to simple pole) and has quasi-unipotent monodromy

- Jordan blocks of Gauss-Manin monodromy are of size $\leq n$

- Suppose $f^d = 0 \in \mathbb{C}\{x_1, \dots, x_n\} / (\partial_{x_1} f, \dots)$
Then the Jordan blocks are of size $\leq d$ (Varchenko, Scherk)

P-adic behaviour We have seen that for

$$\nabla_{\partial_\tau} = \partial_\tau + A_m \tau^m + \cdots \qquad m \leq -2$$

we can change basis so that all terms respect the generalized eigenspace decomposition of A_m. A better way to say that is that the eigenspace decomposition

$$\mathbb{C}^r \cong \bigoplus_\lambda E_\lambda$$

has a canonical extension

$$\mathbb{C}^r[[\tau]] \cong \bigoplus E_{\lambda,\tau}$$

compatible with ∇_{∂_τ}. This only holds at the formal level.

Example $H^*(\mathbb{CP}^1) \cong \mathbb{C}^2$, $\lambda = \pm 2$, and the projections to the pieces are $P_\pm = (I \pm H)/2$, where

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}$$

$$H_{21} = 1 + \sum_{j>0} \tau^{2j} \binom{2j-1}{j}^2 \frac{(2j)!}{2^{8j-2}},$$

$$H_{11} = \frac{1}{4} \tau \partial_\tau (\tau H_{21}), \quad H_{22} = -H_{11},$$

$$H_{12} = H_{21} - \frac{1}{2} \tau^2 \partial_\tau (H_{11})$$

p-adic field

Switch from $\mathbb{C}$ to K: if ∇_{∂_τ} is locally convergent in τ, and A_m has eigenvalues in K, we get a splitting which is locally convergent in τ.

Suppose (for simplicity) all eigenvalues λ are integers. Choose $N \in \mathbb{N}$ so that the difference between any two λ divides N. Our splitting has coefficients in $\mathbb{Z}[\frac{1}{N}]$ (elementary). If we choose p coprime to N, the splitting is over $\mathbb{Z}_p$, since $\frac{1}{N} \in \mathbb{Z}_p$. So, the splitting converges p-adically for $\|\tau\| < 1$.

<u>Example</u> For $M = \mathbb{CP}^1$,

$$H_{21} = 1 + \sum_{j>0} \tau^{2j} \binom{2j-1}{j}^2 \frac{(2j)!}{2^{8j-2}},$$

$N = 4$, so all we need is $p \neq 2$.

<u>Theorem</u> (S 2025) Projections converge p-adically for $\|\tau\| \leq 1$ (coefficients are divisible by higher and higher powers of p)

<u>Observation</u> For $M = \mathbb{CP}^1$, we actually have overconvergence, $\|\tau\| < p^{2/(p-1)}$

Could this be true more generally?

I've used the splitting at $q = \infty$ ($\tau = 0$) to introduce p-adic ideas into the discussion of quantum connections – but this is only one aspect, it's an emerging research direction...

What we don't know ... about the formal structure near $q=\infty$ $(\tau=0)$

There are distinguished numbers $\mu \in \mathbb{Q}$ such that $e^{2\pi i \mu}$ are the eigenvalues of the formal monodromy. I call these exponents (spectrum is another common name for the collection of such numbers).

Example $M=\mathbb{CP}^n$. all exponents are zero.

Example $M\subset\mathbb{CP}^3$ cubic surface, $\pm\frac{2}{3}$ are exponents (and there are more)

Conjecture The non-integer exponents satisfy $|\mu|<n$

... about the local meromorphic structure near $q=\infty$:

It contains an additional piece of data, the Stokes matrices

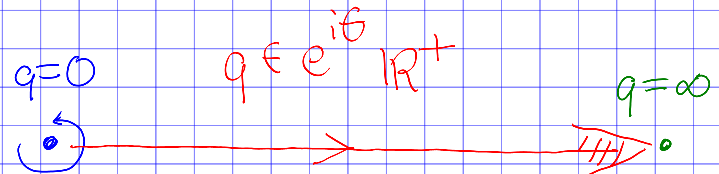
S_+ = upper triangular (1 on diagonal)

S_- = lower triangular (1 on diagonal)

and $S_+ \cdot S_-$ measures the difference between the geometric monodromy (unipotent) and the formal monodromy at ∞ (which we described in our previous results).

Conjecture (Dubrovin) In a suitable basis, $S_\pm$ are integer matrices.

... about the relation between the singularities at $q=0$ and $q=\infty$. This is the subject of the Gamma-conjectures (Galkin-Golyshev-Iritani), which partly build on Dubrovin. Recall the picture



unipotent monodromy

$$\cong$$

$$\frac{\partial}{\partial q} + q^{-1}(c_1(M)\cup)$$

formal splitting can be lifted to meromorphic splitting on sectors

Take any class $x \in H^*(M)$, the associated (multivalued) flat section

$$\xi = e^{-\log(q)\, c_1(M)}\, x, \qquad \left(\frac{\partial}{\partial q} + q^{-1} c_1(M)\cup\right)\xi = 0$$

Transform that into a flat section of ∇_{∂_q}, then look at the asymptotic near $q=\infty$ along a ray.

<u>Conjecture</u> In that comparison, the splitting at $q=\infty$ is compatible with the lattice in $H^*(M;\mathbb{C})$ given by

$$(2\pi)^{-n/2}\, \hat{\Gamma}_X\, (2\pi i)^{Gr/2}\, ch(K(M))$$

a certain characteristic class involving the Γ-function

topological K-theory