CORRIGENDUM TO: "LEFSCHETZ FIBRATIONS AND EXOTIC SYMPLECTIC STRUCTURES ON COTANGENT BUNDLES OF SPHERES"

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In the proof of [4, Lemma 1.1], we appealed to an explicit isotopy of totally real spheres, constructed in [3, Section 5]. That construction works in the lowest dimension ($n = 2$), but is wrong in general (one of the endpoints is not the desired sphere). Here, we explain a different approach, leading to a corrected version of [4, Lemma 1.1], which requires an additional assumption. Independently, while [4, Lemma 1.2] makes a statement about homotopy classes of almost complex structures, its proof only determines the isomorphism class of the tangent bundle as an abstract complex vector bundle, which is a priori a weaker statement. The argument here also fills that gap. The rest of the original paper is unaffected.

Consider $M = M_m$ as in [4], in complex dimension $n > 2$. The construction of E depends on a choice of Lagrangian sphere $S = S_{\delta_{m+1}} \subset M$. By [1], the smooth isotopy class of S depends only on $[S] \in H_n(M)$. Since S is Lagrangian, it comes with a canonical formal Legendrian structure (more precisely, a formal Legendrian structure for $\{0\} \times S \subset \mathbb{R} \times M$, as defined in [5]). Given two homologous Lagrangian spheres, we can use a smooth isotopy between them to compare their canonical formal Legendrian structures. If these coincide, the resulting manifolds E are diffeomorphic, compatibly with the homotopy classes of their almost complex structures. In general, the difference between two formal Legendrian structures for a given n-sphere is described by an element of $\pi_{n+1}(V_{n,2n+1}, U_n)$, where $V_{n,2n+1}$ is the Stiefel manifold. That homotopy group was analyzed in [5, Lemmas A.5–A.7], with the following implications for our situation (compare [5, Theorem A.4]).

Suppose that n is odd. Then,

$$\pi_{n+1}(V_{n,2n+1}, U_n) \cong \mathbb{Z} \oplus \mathbb{Z}. \tag{1}$$

A formal Legendrian structure for S gives rise to a stable complex trivialization of $TM|S$. Two such trivializations differ by an element of $\pi_n(U_\infty) \cong \mathbb{Z}$, and this is one component of (1). For the spheres $S_{\delta_{m+1}}$, all such trivializations are compatible with the stable trivialization of TM coming from the embedding $M \subset \mathbb{C}^{n+1}$ (because $S_{\delta_{m+1}}$ bounds a Lagrangian disc in $\mathbb{C}^{n+1}$). Hence, that component of (1) is zero in our case. A formal Legendrian structure on S also gives rise to a trivialization of the stabilized normal bundle $\nu_S \oplus \mathbb{R}$. Two trivializations differ by an element of $\pi_n(O_{n+1})$, and the other component of (1) is the image of that element in $\pi_n(S^n) \cong \mathbb{Z}$. In our construction, this integer is determined by the self-intersection number

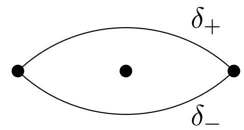

FIGURE 1.

on $H_{n+1}(E) \cong \mathbb{Z}$, which by [4, Eq. (9.3)] depends only on the homology class of $S_{\delta_{m+1}}$. It follows that the formal Legendrian structure contains no additional information. This corrects the proof of [4, Lemma 1.2].

Suppose that $n \geq 4$ is even. Then,

$$\pi_{n+1}(V_{n,2n+1}, U_n) \cong \mathbb{Z}/2. \tag{2}$$

Consider the simplest case $m = 2$, and Lagrangian spheres $S_{\delta_\pm}$ associated to paths as in Figure 1. Let's fix a smooth isotopy between them, and use that to compare their formal Legendrian structures, which leads to an element of (2). By embedding M_2 into M_m in different ways, one sees that for an isotopy of paths in $\mathbb{C}$ which crosses over an even number of marked points, there is an associated isotopy of spheres in M_m which respects the formal Legendrian structure. This proves the following:

Lemma 1.1. *Suppose that $n = 2$. Then, any choice of δ_{m+1} leads to a manifold E which is diffeomorphic to T^*S^{n+1}, and this diffeomorphism is compatible with the homotopy classes of almost complex structures. For higher even n, the same holds under the following additional assumption:*

- (∗) *δ_{m+1} can be connected to a "standard path" by an isotopy (rel endpoints) which crosses over an even number of marked points in the plane (here, the "standard paths" are the $\delta^{k,l}$ from* [4, Section 7]*).*

The following example shows the importance of assumption (∗). Consider the affine variety

$$X = \{xy^2 + z_1^2 + \cdots + z_n^2 = 1\} \subset \mathbb{C}^{n+2}. \tag{3}$$

As pointed out in [6, Example 1.5], X (with the standard Kähler form) is one of the manifolds E constructed in [4], corresponding to the choice of path as in the right-hand part of [4, Figure 2]. Note that X is a double branched cover of $T^*S^{n+1} = \{xw + z_1^2 + \cdots + z_n^2 = 1\}$, under $w = y^2$. Take the zero-section $S^{n+1} = \{x = \bar{w},\, z \in \mathbb{R}^n\} \subset T^*S^{n+1}$, and let Z be its preimage in X. Explicit computation shows that Z is an embedded sphere. The inclusion $Z \hookrightarrow X$ is a homotopy equivalence. By the h-cobordism theorem, X must be diffeomorphic to the total space of the normal bundle ν_Z. Again by explicit computation, ν_Z is the pullback of the normal bundle of the zero-section. From now on, assume that n is even. Then ν_Z is trivial (since it's classified by twice the class of the tangent bundle of a sphere, inside $ker(\pi_n(O_{n+1}) \to \pi_n(O_\infty))$, and that group is either $\mathbb{Z}/2$ or zero). Suppose that $n \neq 2, 6$.

Then, $X \cong S^{n+1} \times \mathbb{R}^{n+1}$ is not even homeomorphic to T^*S^{n+1}, by [2]. By comparing this with the argument concerning Figure 1, one sees that any isotopy from S_{δ_-} to S_{δ_+} necessarily yields a nontrivial obstruction element in (2).

It remains to consider the case $n = 6$. Then, for any choice of δ_{m+1}, the resulting E will be diffeomorphic to $T^*S^7 \cong S^7 \times \mathbb{R}^7$ (one shows this using the h-cobordism theorem, and the fact that any 7-dimensional vector bundle over S^7 is trivial). However, there are two possible homotopy classes of almost complex structures ($\pi_7(O_{14}/U_7) \cong \pi_7(O_\infty/U_\infty) \cong \mathbb{Z}/2$), and it is not clear which one will arise if $(*)$ is dropped. In particular, we still don't know what element of (2) appears there.

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