COURBES ET FIBRÉS VECTORIELS EN THÉORIE DE HODGE z-ADIQUE GLOBALE

SIYAN DANIEL LI-HUERTA

Abstract. We study the global analogue of the Fargues–Fontaine curve over function fields F. We prove some foundational results about its moduli of G-bundles $\mathrm{Bun}_{G,F}$, which is a geometrization of the global Kottwitz set $B(F,G)$. For example, $\mathrm{Bun}_{G,F}$ plays the role of Igusa stacks over function fields. We use $\mathrm{Bun}_{G,F}$ to reformulate the global Langlands conjecture for G over F in terms of categorical local Langlands, refining conjectures of Arinkin–Gaitsgory–Kazhdan–Raskin–Rozenblyum–Varshavsky and Zhu. Finally, we verify this conjecture when G is commutative. Along the way, we prove a GAGA theorem for smooth proper schemes over sousperfectoid spaces, which is of independent interest.

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Introduction

The title of this paper pays homage to the foundational work of Fargues–Fontaine [25]. Starting from a nonarchimedean local field F_v with residue field $\mathbb{F}_q$ (and an auxiliary algebraically closed nonarchimedean field K over $\overline{\mathbb{F}}_q$), they constructed an adic space $X_{F_v,K}$ with remarkable properties. Since its introduction, the *Fargues–Fontaine curve* $X_{F_v,K}$ has played a fundamental role in arithmetic geometry over F_v. For example, work of Fargues–Scholze [26] shows that its moduli of G-bundles Bun_{G,F_v} is central to the Langlands correspondence for G over F_v.

When F_v is the function field $\mathbb{F}_q(\!(z)\!)$, the Fargues–Fontaine curve already appears in work of Hartl–Pink [38], so we also call it the *Hartl–Pink curve*. It can be described as follows. The product $\operatorname{Spa} F_v \times \operatorname{Spa} K$ over $\mathbb{F}_q$ exists as an adic space (in fact, it is isomorphic to the punctured open unit disk over K), and the absolute

q-Frobenius automorphism Frob_K of K acts freely and totally discontinuously on $\operatorname{Spa} F_v \times \operatorname{Spa} K$. Hence we can form the quotient adic space

$$X_{F_v,K} \coloneqq (\operatorname{Spa} F_v \times \operatorname{Spa} K)/\operatorname{Frob}_K . \tag{†}$$

The goal of this paper is to study a *global* analogue of the above constructions.

G-bundles on the global Hartl–Pink curve. Let F be a global function field with field of constants $\mathbb{F}_q$. By viewing schemes as adic spaces, one could try to define a *global Hartl–Pink curve* by naively copying (†) and attempting to form

$$\text{“}(\operatorname{Spec} F \times \operatorname{Spa} K)/\operatorname{Frob}_K\text{”}. \tag{‡}$$

However, two obstacles arise:

1) It is unclear how to interpret “$\operatorname{Spec} F \times \operatorname{Spa} K$” in a well-behaved way.
2) In any reasonable interpretation of “$\operatorname{Spec} F \times \operatorname{Spa} K$”, Frob_K will no longer act freely and totally discontinuously.

We will primarily be interested in G-bundles on (‡), so we can circumvent 2) via descent. To circumvent 1), write C for the geometrically connected smooth proper curve over $\mathbb{F}_q$ associated with F. Then $\operatorname{Spec} F = \varprojlim_U U$, where U runs over dense open subschemes of C, so formally we expect

$$\text{“}\operatorname{Spec} F \times \operatorname{Spa} K\text{”} = \text{“}\varprojlim_U U \times \operatorname{Spa} K\text{”}.$$

Now $U \times \operatorname{Spa} K$ *does* have a well-behaved interpretation: this product exists as an adic space and in fact is isomorphic to the analytification U_K^{an} of U over K. Moreover, this interpretation works well in families, which leads to the following:

Definition. Let G be a connected split[1] reductive group over F. For all affinoid perfectoid spaces $S = \operatorname{Spa}(R, R^+)$ over $\overline{\mathbb{F}}_q$,

a) Write $\mathrm{Bun}_{G,U}(S)$ for the groupoid of G-bundles $\mathscr{G}$ on U_S^{an} equipped with an isomorphism $\phi : \mathscr{G} \xrightarrow{\sim} \mathrm{Frob}_S^* \mathscr{G}$.
b) Write $\mathrm{Bun}_{G,F}(S)$ for the groupoid $\varinjlim_U \mathrm{Bun}_{G,U}(S)$.

Remarks.

1) One can recover $\mathrm{Bun}_{G,U}$ from $\mathrm{Bun}_{G,F}$; see Lemma 2.6. Therefore we will interchangeably use $\mathrm{Bun}_{G,U}$ and $\mathrm{Bun}_{G,F}$.
2) Amusingly, while we do not define *the* global Hartl–Pink curve, nonetheless we can define its moduli of G-bundles $\mathrm{Bun}_{G,F}$. This is reminiscent of the situation for *local* Hartl–Pink curves $X_{F_v,K}$, which depend on the auxiliary K.

Geometry of $\mathrm{Bun}_{G,F}$. We start with the following basic properties of $\mathrm{Bun}_{G,U}$.

Theorem A. *$\mathrm{Bun}_{G,U}$ is a small Artin v-stack over $\overline{\mathbb{F}}_q$. It is ℓ-cohomologically smooth over $\overline{\mathbb{F}}_q$, and its dualizing complex with $\mathbb{F}_\ell$-coefficients is isomorphic to $\mathbb{F}_\ell$.*

Theorem A mirrors results of Fargues–Scholze [26, Theorem IV.1.19] and Hamann–Imai [35, Proposition 3.18] in the local setting.

To state our next theorem, let us introduce the *localization map*, which plays an important role throughout this paper. Write Z for the closed complement $C \smallsetminus U$.

[1] In the introduction, we work with split G for simplicity. However, everything works for general connected reductive groups G over F, and we work in this generality in the body of the paper.

For all z in Z, there is a natural map $\operatorname{Spa} F_z \to U$, so by restricting G-bundles along these maps and using the presentation (†), we get a map of v-stacks

$$\operatorname{loc}_Z : \operatorname{Bun}_{G,U} \to \prod_{z \in Z} \operatorname{Bun}_{G,F_z}.$$

Recall that the local moduli $\operatorname{Bun}_{G,F_z}$ has an open substack $\operatorname{Bun}_{G,F_z}^{\mathrm{ss}}$ [26, Theorem III.4.5], which is the locus where the corresponding G-bundle on the curve $X_{F_z,K}$ is semistable. By taking its preimage under loc_Z, we define a *global semistable locus* $\operatorname{Bun}_{G,U}^{\mathrm{ss}}$. While $\operatorname{Bun}_{G,U}^{\mathrm{ss}}$ is defined in terms of its local analogue $\operatorname{Bun}_{G,F_z}^{\mathrm{ss}}$, we prove that it admits the following intrinsic description.

Theorem B. *$\operatorname{Bun}_{G,F}^{\mathrm{ss}}$ is an open substack of $\operatorname{Bun}_{G,F}$, and we have an isomorphism*

$$\operatorname{Bun}_{G,F}^{\mathrm{ss}} \cong \coprod_{b \in B(F,G)_{\mathrm{basic}}} *\underline{/G_b(F)},$$

where $B(F,G)$ denotes the global Kottwitz set [50]*, $B(F,G)_{\mathrm{basic}}$ denotes its subset of* basic *elements, and G_b is the inner twist of G associated with b.*

Theorem B mirrors a result of Fargues–Scholze [26, Theorem III.4.5] in the local setting. This is one incarnation of the idea that $\operatorname{Bun}_{G,F_z}$ is a geometrization of the local Kottwitz set $B(F_z,G)$, but there are other such incarnations. For example, Anschütz [3, Theorem 10] proved that the underlying topological space $|\operatorname{Bun}_{G,F_z}|$ is naturally in bijection with $B(F_z,G)$.

We similarly regard $\operatorname{Bun}_{G,F}$ as a geometrization of the global Kottwitz set. However, for reasons explained in the next subsection, we do *not* expect a natural bijection between $|\operatorname{Bun}_{G,F}|$ and $B(F,G)$ in general. Nonetheless, we prove the following description of the $\overline{\mathbb{F}}_q$-points of $\operatorname{Bun}_{G,F}$. Write $\breve{F}$ for the field $F \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$.

Theorem C. *$\operatorname{Bun}_{G,F}(\overline{\mathbb{F}}_q)$ is naturally equivalent to the groupoid quotient*

$$G(\breve{F})/G(\breve{F}),$$

where the action is given by $\operatorname{Frob}_{\overline{\mathbb{F}}_q}$-conjugation. Consequently, the set of isomorphism classes in $\operatorname{Bun}_{G,F}(\overline{\mathbb{F}}_q)$ is naturally in bijection with $B(F,G)$.

Theorem C mirrors a result of Anschütz [4, Theorem 5.3] in the local setting. Actually, we prove a generalization of Theorem C that fully describes the *reduction* of $\operatorname{Bun}_{G,F}$; see Theorem 2.10. The more general Theorem 2.10 mirrors a result of Gleason–Ivanov–Zillinger [33, Theorem 7.14.(1)] in the local setting.

Relation with shtukas. It turns out that $\operatorname{Bun}_{G,U}$ plays the role of *Igusa stacks* [73] over function fields. Igusa stacks capture the geometry of Shimura varieties, so let us recall the function field analogue of the latter. Let I be a finite set, and for all i in I, let V_i be a representation of the Langlands dual group $\widehat{G}$. Write V for the representation $\boxtimes_{i \in I} V_i$ of $\widehat{G}^I$. Associated with I and V, there is a moduli space $\operatorname{Sht}_{G,V}^I$ of *global shtukas* bounded by V [70], which is a Deligne–Mumford stack that is separated and locally of finite type over C^I.

There is also a local version of $\operatorname{Sht}_{G,V}^I$. Let $\{I_z\}_{z \in Z}$ be a partition of I, and for all z in Z, write V_z for the representation $\boxtimes_{i \in I_z} V_i$ of $\widehat{G}^{I_z}$. Associated with I_z and V_z, there is a moduli space $\operatorname{LocSht}_{G,V_z}^{I_z}$ of *local shtukas* bounded by V_z [68], which is a small v-stack over $(\operatorname{Spd} \mathcal{O}_z)^{I_z}$.

We prove the following function field analogue of Scholze's *fiber product conjecture* [73, Conjecture 1.1], which is the defining feature of Igusa stacks:

Theorem D. *There is a natural cartesian square of small v-stacks*

$$\begin{array}{ccc} (\mathrm{Sht}^I_{G,V})^{\Diamond}|_{\prod_{z\in Z}(\mathrm{Spd}\,\mathcal{O}_z)^{I_z}_{\overline{\mathbb{F}}_q}} & \xrightarrow{\pi_{\mathrm{HT}}} & \prod_{z\in Z}\big(\mathrm{LocSht}^{I_z}_{G,V_z}\big)_{\overline{\mathbb{F}}_q} \\ \downarrow & & \downarrow \mathrm{BL} \\ \mathrm{Bun}_{G,U} & \xrightarrow{\mathrm{loc}_Z} & \prod_{z\in Z}\mathrm{Bun}_{G,F_z}, \end{array}$$

where BL *denotes the product of the* Beauville–Laszlo maps $(\mathrm{LocSht}^{I_z}_{G,V_z})_{\overline{\mathbb{F}}_q}\to\mathrm{Bun}_{G,F_z}$ [26, p. 98]*, and* π_{HT} *is a function field analogue of the Hodge–Tate period map.*

Actually, we prove a generalization of Theorem D that allows level structures; see Theorem 3.10. Previously, Hartl–Viehmann [39] studied a version of the period map π_{HT} for the function field analogue of Rapoport–Zink spaces.

Remarks.

1) Theorem D implies that $\mathrm{Bun}_{G,U}$ encodes the function field analogue of *Igusa varieties*; see Proposition 4.10. Because Igusa varieties associated with non-basic elements of the Kottwitz set can be positive-dimensional in general, this explains why we do not expect a natural bijection between $|\mathrm{Bun}_{G,F}|$ and $B(F,G)$.
2) Gleason–Ivanov–Zillinger [33, Theorem 7.14.(2)] proved a natural description of the $\overline{\mathbb{F}}_q$-points of $\mathrm{LocSht}^{I_z}_{G,V_z}$. By combining this description with Theorem C and the aforementioned result of Anschütz [4, Theorem 5.3], Theorem D immediately implies the *Langlands–Rapoport conjecture* for moduli spaces of global shtukas with arbitrary (in particular, colliding) legs. This generalizes results of Arasteh Rad–Hartl [6, Theorem 3.21], who proved this when the legs are disjoint.

Relation with Langlands. In the local setting, Bun_{G,F_z} plays a central role in the Langlands correspondence for G over F_z. For example, let Λ be one of $\overline{\mathbb{Q}}_\ell$, $\overline{\mathbb{Z}}_\ell$, or $\overline{\mathbb{F}}_\ell$, and write $D(\mathrm{Bun}_{G,F_z},\Lambda)$ for the associated derived category of étale sheaves.[2] Associated with I_z and V_z, Fargues–Scholze [26] constructed a *Hecke operator*

$$T_{V_z}: D(\mathrm{Bun}_{G,F_z},\Lambda)\to D(\mathrm{Bun}_{G,F_z},\Lambda),$$

which they used to define the automorphic-to-Galois direction of the Langlands correspondence for G over F_z, up to semisimplification.

Using Theorem D, we prove that applying Hecke operators to the object

$$\mathrm{loc}_{Z,!}\Lambda\in D\big(\prod_{z\in Z}\mathrm{Bun}_{G,F_z},\Lambda\big)=\bigotimes_{z\in Z}D(\mathrm{Bun}_{G,F_z},\Lambda)$$

recovers the cohomology of the moduli of global shtukas at infinite level. Write

$$\pi_{\infty Z}:(\mathrm{Sht}^I_{G,V,\infty Z})_{\overline{\mathbb{F}}_q}\to U^I_{\overline{\mathbb{F}}_q}$$

for the moduli of global shtukas with infinite level at Z, which has an action of $G(F_Z):=\prod_{z\in Z}G(F_z)$, and write $\mathcal{S}_V$ for its intersection cohomology sheaf. Forthcoming work of Eteve–Gaitsgory–Genestier–Lafforgue will endow $\pi_{\infty Z,!}\mathcal{S}_V$ with the structure of a representation of the Weil group W^I_U on the level of derived categories.

[2]Strictly speaking, when $\Lambda\neq\overline{\mathbb{F}}_\ell$ we use (a base change of) motivic sheaves [64, 65]; see 5.1.

Theorem E. *After restricting to $\prod_{z\in Z} W_{F_z}^{I_z}$, we have a natural isomorphism*

$$\pi_{\infty Z,!}\mathcal{S}_V \cong i_1^*(T_V(\mathrm{loc}_{Z,!}\Lambda)),$$

where i_1 denotes the product of the natural open embeddings $*/\underline{G(F_z)} \hookrightarrow \mathrm{Bun}_{G,F_z}$, and T_V is the endofunctor $\bigotimes_{z\in Z} T_{V_z}$ of $\bigotimes_{z\in Z} D(\mathrm{Bun}_{G,F_z},\Lambda)$.

Remarks.

1) If one is only interested in Theorem E after taking cohomology groups, it suffices to use work of Xue [72] instead of the forthcoming work of Eteve–Gaitsgory–Genestier–Lafforgue. For example, this is the case for the next remark.
2) One can use Theorem E to recover all of the representation-theoretic results in [55]; see Remark 5.13.2).

In light of Theorem E, it is natural to ask how to interpret $\mathrm{loc}_{Z,!}\Lambda$ on the Galois side of the Langlands correspondence. Write $\mathrm{LS}_{\widehat{G},F_z}$ for the moduli of continuous $\widehat{G}$-valued representations of W_{F_z} over Λ [16, 26, 74], which is an algebraic stack locally of finite type over Λ. When the order of $\pi_0(Z(G)_{\overline{F}})$ is invertible in Λ, Fargues–Scholze [26, Conjecture X.3.5] conjecture that there is a canonical equivalence

$$\mathbb{L}_{\psi_z} : D(\mathrm{Bun}_{G,F_z},\Lambda) \xrightarrow{\sim} \mathrm{Ind}\, D^{\mathrm{qc}}_{\mathrm{coh}}(\mathrm{LS}_{\widehat{G},F_z})_{\mathrm{Nilp}},$$

where $\mathrm{Ind}\, D^{\mathrm{qc}}_{\mathrm{coh}}(\mathrm{LS}_{\widehat{G},F_z})_{\mathrm{Nilp}}$ is the derived category of *ind-coherent sheaves* on $\mathrm{LS}_{\widehat{G},F_z}$ with *nilpotent singular support*, and $\mathbb{L}_{\psi_z}$ depends on the choice of a maximal unipotent subgroup N of G along with a generic continuous character $\psi_z : N(F_z)\to\Lambda^\times$.

The following conjecture was suggested by Scholze. Let $\psi : N(\mathbb{A})\to\Lambda^\times$ be a generic continuous character trivial on $N(F)$, and for all z in Z, take $\psi_z = \psi|_{N(F_z)}$.

Conjecture F. *Assume Fargues–Scholze's conjecture, so that $\bigotimes_{z\in Z}\mathbb{L}_{\psi_z}$ yields*

$$\bigotimes_{z\in Z} D(\mathrm{Bun}_{G,F_z},\Lambda) \xrightarrow{\sim} \bigotimes_{z\in Z} \mathrm{Ind}\, D^{\mathrm{qc}}_{\mathrm{coh}}(\mathrm{LS}_{\widehat{G},F_z})_{\mathrm{Nilp}} = \mathrm{Ind}\, D^{\mathrm{qc}}_{\mathrm{coh}}\big(\prod_{z\in Z}\mathrm{LS}_{\widehat{G},F_z}\big)_{\mathrm{Nilp}}.$$

Under this equivalence, $\mathrm{loc}_{Z,!}\Lambda$ corresponds to $\mathrm{res}_{Z,*}(\omega_{\mathrm{LS}_{\widehat{G},U}})$, where $\mathrm{LS}_{\widehat{G},U}$ denotes the moduli of continuous $\widehat{G}$-valued representations of W_U over Λ [74], and

$$\mathrm{res}_Z : \mathrm{LS}_{\widehat{G},U} \to \prod_{z\in Z}\mathrm{LS}_{\widehat{G},F_z}$$

denotes the restriction map.

Remarks.

1) For $U = C$ and $\Lambda = \overline{\mathbb{Q}}_\ell$, Conjecture F specializes to asking for a natural isomorphism $C_c(G(F)\backslash G(\mathbb{A})/G(\mathbb{O}),\overline{\mathbb{Q}}_\ell) \cong \Gamma(\mathrm{LS}_{\widehat{G},C},\omega_{\mathrm{LS}_{\widehat{G},C}})$; see Example 6.11. Previously, Arinkin–Gaitsgory–Kazhdan–Raskin–Rozenblyum–Varshavsky [8] conjectured this, and recently Gaitsgory–Raskin [30] proved a weak version thereof.

   More generally, we use Theorem E to prove that Conjecture F implies a conjecture of Zhu [74, Conjecture 4.49] after restricting to $\prod_{z\in Z} W_{F_z}^{I_z}$; see Proposition 6.12.
2) Given the previous remark, we regard Conjecture F as *the* Langlands conjecture for G over F when ramification is allowed at places in Z. This fulfills a suggestion of Fargues–Scholze [27, §7] that the global Langlands conjecture ought to study an object that encodes automorphic forms on G_b for all b in $B(F,G)$.

In the number field setting, Fargues [24, section 7] has already emphasized the importance of studying the analogous object $\overline{\pi}_{\mathrm{HT},!}\Lambda$, though Shimura varieties place severe restrictions on which b in $B(\mathbb{Q},G)$ can appear.

3) Conjecture F fits well with the interpretation of Langlands as an isomorphism of 4-*dimensional topological quantum field theories*. In particular, it fits well with the recent interpretation of relative Langlands as an identification of *boundary theories* [9]; we will investigate this in future work.
4) For a version of $\mathrm{Bun}_{G,F}$ adapted to the *global Kaletha set* [20] instead of $B(F,G)$, see Appendix C, written by Peter Dillery. This builds on work of Fargues [23], who introduced a version of Bun_{G,F_z} adapted to the *local Kaletha set* [19].

We conclude by proving Conjecture F in the commutative case:

Theorem G. *When G is a (not necessarily split) torus, Conjecture F holds.*

Proof sketches. We start with part of Theorem A. We prove that $\mathrm{Bun}_{G,U}$ and $\mathrm{Bun}_{G,F}$ are small v-stacks by locally choosing a Frobenius-splitting on U and analyzing affinoid charts of U_S^{an}, which reduces the problem to a v-descent result of Scholze–Weinstein [68, Lemma 17.1.8] for vector bundles on perfectoid spaces.

Next, we turn to Theorem D. The S-points of $(\mathrm{Sht}^I_{G,V})^{\Diamond}$ involve G-bundles on the algebraic curve C_R, while one can glue in the analytic topology to show that S-points of the fiber product in Theorem D involve G-bundles on the analytification C_S^{an}. To reconcile this, we prove the following GAGA result in Appendix A:

Theorem H. *Let $S=\mathrm{Spa}(R,R^+)$ be a sousperfectoid space, and let X be a smooth proper scheme over R. Then the analytification functor*

$$\{\textit{vector bundles on } X\}\to\{\textit{vector bundles on } X_S^{\mathrm{an}}\}$$

is an equivalence of categories.

Remark. While we were finalizing this paper, Theorem H was proved independently by Wang [71, Theorem 4.3.1] along very similar lines; evidently, it is of independent interest. For example, as an application we prove that algebraic stacks of Higgs bundles agree with their analytification, answering a question of Heuer–Xu [41, Remark 8.1.2]; see Theorem A.20. This was also proved independently by Wang.

Theorem H provides a lot of control. In addition to proving Theorem D, we use it to prove Theorem C as follows. An $\overline{\mathbb{F}}_q$-point of $\mathrm{Bun}_{G,U}$ involves a G-bundle $\mathscr{G}$ on $U^{\mathrm{an}}_{\mathrm{Spa}\,\overline{\mathbb{F}}_q(\!(t^{1/q^\infty})\!)}$ equipped with a descent datum with respect to the diagram

$$\begin{aligned}&U^{\mathrm{an}}_{\mathrm{Spa}\,\overline{\mathbb{F}}_q(\!(t^{1/q^\infty})\!)\times_{\overline{\mathbb{F}}_q}\mathrm{Spa}\,\overline{\mathbb{F}}_q(\!(t^{1/q^\infty})\!)\times_{\overline{\mathbb{F}}_q}\mathrm{Spa}\,\overline{\mathbb{F}}_q(\!(t^{1/q^\infty})\!)}\\ \rightrightarrows&U^{\mathrm{an}}_{\mathrm{Spa}\,\overline{\mathbb{F}}_q(\!(t^{1/q^\infty})\!)\times_{\overline{\mathbb{F}}_q}\mathrm{Spa}\,\overline{\mathbb{F}}_q(\!(t^{1/q^\infty})\!)}\ \rightrightarrows\ U^{\mathrm{an}}_{\mathrm{Spa}\,\overline{\mathbb{F}}_q(\!(t^{1/q^\infty})\!)}.\end{aligned}$$

A result of Gleason–Ivanov–Zillinger [33, Theorem 3.15] implies that the restriction of $\mathscr{G}$ along $\mathrm{Spa}\,F_z\to U$ arises uniquely from a G-bundle on $\breve{F}_z$. The latter are all trivial, so we can glue in the analytic topology to assume that $U=C$. Finally, we can apply Theorem H and a result of D. Kim [49, Theorem 4.9] to conclude that $\mathscr{G}$ arises uniquely from a G-bundle on $C_{\overline{\mathbb{F}}_q}$.

We now turn to Theorem B. By proving a description of pro-étale $\underline{\mathscr{O}_U}$-local systems analogous to Kedlaya–Liu's description [48, Theorem 8.5.12] of pro-étale $\underline{\mathbb{Q}_p}$-local systems, we show that $\mathrm{Bun}_{G,F}$ has a natural open substack isomorphic to $*/\underline{G(F)}$. From here, twisting and Theorem C let us show that $\mathrm{Bun}_{G,F}$ has an open

substack isomorphic to the disjoint union in Theorem B. To prove that this is all of $\mathrm{Bun}^{\mathrm{ss}}_{G,F}$, we use Beauville–Laszlo uniformization [26, Proposition III.3.1] to lift geometric points from $\mathrm{Bun}^{\mathrm{ss}}_{G,F_z}$ to $\mathrm{LocSht}^{I_z}_{G,V_z}$ for some V_z. Combining this with Theorem D yields geometric points of $\mathrm{Sht}^{I}_{G,V}$, and we conclude using a standard nonemptiness criterion for $\mathrm{Sht}^{I}_{G,V}$. Similarly, we finish proving Theorem A by using the charts provided by Beauville–Laszlo uniformization and Theorem D.

In an ideal world, Theorem E would follow immediately from Theorem D. For $\Lambda = \overline{\mathbb{F}}_\ell$, one can successfully execute this using the étale sheaf theory of [62], but for general Λ one has to compare the motivic sheaf theory of [64] with classical étale ℓ-adic sheaf theories for algebraic varieties. We develop some tools for facilitating these comparisons in Appendix B.

Finally, we prove Theorem G by using work of Langlands [53] on his conjectures for tori T to prove an explicit description of $\mathrm{LS}_{\iota T,U}$.

Outline. In §1, we study vector bundles on the global Hartl–Pink curve. In §2, we use the Tannakian description of G-bundles to convert results from §1 into part of Theorem A, part of Theorem B, and Theorem C. In §3, we introduce shtukas and prove Theorem D. In §4, we use Theorem D to finish proving Theorem A and Theorem B, as well as explain the relation with Igusa varieties. In §5, we state a forthcoming result of Eteve–Gaitsgory–Genestier–Lafforgue and prove Theorem E. Finally, in §6 we discuss Conjecture F and prove Theorem G.

In Appendix A, we prove Theorem H, which is used throughout the paper. In Appendix B, we prove basic results about the (overconvergent) motivic sheaves of [64, 65], which we use in §5 and §6. Finally, in Appendix C, written by Peter Dillery, we consider a version of $\mathrm{Bun}_{G,F}$ adapted to the global Kaletha set.

Notation. Except for in Appendix A, all rings and stacks are classical (i.e. not derived) unless otherwise specified.

By nonarchimedean field, we mean a topological field complete with respect to a rank-1 nonarchimedean valuation. For any ring homomorphism $A \to B$, write $A^\sim$ for the integral closure of the image of A in B. For any scheme X and affine group G over X, write $\mathrm{Rep}_X G$ for the category of representations of G in vector bundles on X.

We view schemes as adic spaces via the fully faithful functor sending $\mathrm{Spec}\, A$ to $\mathrm{Spa}(A, \mathbb{Z}^\sim)$ for all rings A, where A is endowed with the discrete topology [67, p. 64]. For any adic space X over $\mathrm{Spa}\,\mathbb{Z}_p$, write $X^\Diamond$ for the associated small v-sheaf over $\mathbb{F}_p$ as in [68, Lemma 18.1.1].

Unless otherwise specified, all products are over $\mathbb{F}_q$. For any prestack X on

$$\{\text{affine schemes over } \mathbb{F}_q\},$$

write $\mathrm{Frob}_X : X \to X$ for its absolute q-Frobenius endomorphism, and write X^{perf} for its limit perfection. The transition morphisms for our ind-schemes are required to be closed embeddings. Write Perf for the category of affinoid perfectoid spaces over $\mathbb{F}_p$, and for any small v-stack X over $\mathbb{F}_q$, write $\mathrm{Frob}_X : X \to X$ for its absolute q-Frobenius automorphism.

We view derived categories as stable ∞-categories, and we view all functors between derived categories as derived functors. Write Sym for the ∞-category of presentably symmetric monoidal stable ∞-categories.

For any ring A, write $D(A)$ for the derived category of A-modules, and write $D_{\mathrm{perf}}(A) \subseteq D(A)$ for the full subcategory of perfect objects. For any derived algebraic stack X, write $D_{\mathrm{qcoh}}(X)$ for its derived category of quasi-coherent sheaves, and write $D_{\mathrm{perf}}(X) \subseteq D(X)$ for the full subcategory of perfect objects. Write X^{cl} for the underlying classical algebraic stack of X.

For any topological space X, write X for the associated condensed set as in [67, Proposition 1.7].

Acknowledgments. We thank Peter Scholze for many helpful suggestions about this project; for example, he suggested Conjecture F, as well as the proof strategy of Theorem H. We also thank Peter Dillery for suggesting that a global analogue of (†) should exist, for his collaboration in an early stage of this project, and for providing Appendix C. Our intellectual debt to Laurent Fargues and Xinwen Zhu should be clear. Finally, we thank Arnaud Eteve, Ian Gleason, Linus Hamann, David Hansen, Pol van Hoften, Alexander Petrov, Juan Esteban Rodríguez Camargo, Ziquan Yang, and Zhiwei Yun for useful discussions.

During the completion of this work, the author was partially supported by NSF Grant #DMS2303195 and the Max Planck Institute for Mathematics. We thank MPIM for its hospitality and excellent working conditions.

1. Vector bundles on the global Hartl–Pink curve

In this section, we introduce and study vector bundles on the global Hartl–Pink curve. They are defined using analytifications, so we start by discussing the latter in the context of adic spaces. For example, we prove v-descent for vector bundles on analytifications. This entire discussion applies to any smooth scheme over $\mathbb{F}_q$.

We then specialize to smooth curves U over $\mathbb{F}_q$, which is our main case of interest. We prove a description of pro-étale $\underline{\mathscr{O}_U}$-local systems in terms of vector bundles on the global Hartl–Pink curve, analogous to Kedlaya–Liu's description of pro-étale $\underline{\mathbb{Q}_p}$-local systems. Finally, we prove a vector bundle version of Theorem C.

1.1. Analytification has the following interpretation in the category of adic spaces.

Definition. Let X be a scheme. Let S be an adic space over X, and let Y be a scheme over X. Write Y_S^{an}, if it exists, for the fiber product $Y \times_X S$ in the category of adic spaces.

By replacing X and Y with affine open covers, the Spec-global sections adjunction implies that Y_S^{an} equals the "fiber product" $Y \times_X S$ in the sense of [43, (3.8)].

1.2. We will use the following generalization of [55, Lemma 4.4], which describes the functor of points of analytifications without needing to sheafify in the analytic topology. Let D be a noetherian ring. Let S be an adic space over D, and let Y be a quasiprojective scheme over D.

Lemma. *For all adic spaces* $\mathrm{Spa}(A, A^+)$ *over* S*, morphisms* $\mathrm{Spa}(A, A^+) \to Y_S^{\mathrm{an}}$ *over* S *are equivalent to morphisms* $\mathrm{Spec}\, A \to Y$ *over* D*.*

Proof. Since D is noetherian and Y is quasiprojective over D, there exist finitely many homogeneous polynomials $f_1, \dots, f_l$ and $g_1, \dots, g_m$ in $D[T_0, \dots, T_d]$ such that Y is the locus in $\mathbb{P}_D^d$ where $f_a(T_0, \dots, T_d)$ vanishes for all $1 \leq a \leq l$ and $g_b(T_0, \dots, T_d)$ vanishes nowhere for all $1 \leq b \leq m$.

By the universal property of Y_S^{an}, morphisms $\mathrm{Spa}(A, A^+) \to Y_S^{\mathrm{an}}$ over S are equivalent to morphisms $\mathrm{Spa}(A, A^+) \to Y$ over D. The Spec-global sections adjunction implies that the latter are equivalent to the data of a line bundle $\mathscr{L}$ on $\mathrm{Spa}(A, A^+)$ equipped with sections $s_0, \dots, s_d$ generating $\mathscr{L}$ such that $f_a(s_0, \dots, s_d)$ vanishes for all $1 \leq a \leq l$ and $g_b(s_0, \dots, s_d)$ vanishes nowhere for all $1 \leq b \leq m$. Finally, [47, Theorem 1.4.2] shows these are equivalent to morphisms $\operatorname{Spec} A \to Y$ over D. □

1.3. The following lemma lets us compute the global sections of analytifications.

Lemma. *Let $Y = \mathrm{Spa}(A, A^+)$ be a smooth affinoid adic space over a nonarchimedean field K, let $\mathrm{Spa}(R, R^+)$ be an affinoid sousperfectoid adic space over $\operatorname{Spa} K$, and write $(A\widehat{\otimes}_K R, (A^+\widehat{\otimes}_{K^\circ} R^+)^\sim)$ for the base change of $(K, K^\circ) \to (A, A^+)$ to (R, R^+). Then $\mathrm{Spa}(A\widehat{\otimes}_K R, (A^+\widehat{\otimes}_{K^\circ} R^+)^\sim)$ is a sousperfectoid adic space.*

Proof. Since Y is smooth over K, [44, (2.2.8)] implies that there is a finite rational open cover $\{Y_i\}_{i=1}^m$ of Y that are rational open subspaces of finite étale covers of closed unit polydisks over K. For all $1 \leq i \leq m$, write $Y_i = \mathrm{Spa}(A_i, A_i^+)$. Then $A_i\widehat{\otimes}_K R$ is a rational localization of a finite étale algebra over $R\langle T_1, \dots, T_d\rangle$, so [68, Proposition 6.3.3] shows that $A_i\widehat{\otimes}_K R$ is sousperfectoid and hence strongly sheafy as in [37, Definition 4.1].

Write $(A_i\widehat{\otimes}_K R, (A_i^+\widehat{\otimes}_{K^\circ} R^+)^\sim)$ for the base change of $(K, K^\circ) \to (A_i, A_i^+)$ to (R, R^+). Then $\{\mathrm{Spa}(A_i\widehat{\otimes}_K R, (A_i^+\widehat{\otimes}_{K^\circ} R^+)^\sim)\}_{i=1}^m$ is a finite rational open cover of

$$\mathrm{Spa}(A\widehat{\otimes}_K R, (A^+\widehat{\otimes}_{K^\circ} R^+)^\sim),$$

and its associated Čech complex is obtained by applying $-\widehat{\otimes}_K R$ to the Čech complex associated with $\{Y_i\}_{i=1}^m$. Because A is sheafy, the latter has cohomology A in zeroth degree and 0 elsewhere. The exactness of $-\widehat{\otimes}_K R$ implies that the Čech complex associated with $\{\mathrm{Spa}(A_i\widehat{\otimes}_K R, (A_i^+\widehat{\otimes}_{K^\circ} R^+)^\sim)\}_{i=1}^m$ has cohomology $A\widehat{\otimes}_K R$ in zeroth degree and 0 elsewhere, so the result follows from [37, Corollary 4.5][3]. □

1.4. We will analytify smooth varieties as follows. Let $Y = \operatorname{Spec} K[T_1, \dots, T_d]/I$ be a smooth affine scheme over a nonarchimedean field K, and let $S = \mathrm{Spa}(R, R^+)$ be an affinoid sousperfectoid space over $\operatorname{Spa} K$. For all pseudouniformizers ϖ of K, write $B_{S,\varpi}$ for the topological ring $R\langle \varpi T_1, \dots, \varpi T_d\rangle/I$, and write $B_{S,\varpi}^+$ for the integral closure in $B_{S,\varpi}$ of the image of $R^+\langle \varpi T_1, \dots, \varpi T_d\rangle$.

Proposition. *The pre-adic space $Y_{S,\varpi} \coloneqq \mathrm{Spa}(B_{S,\varpi}, B_{S,\varpi}^+)$ is a sousperfectoid adic space. Moreover, Y_S^{an} equals $\varinjlim_\varpi Y_{S,\varpi}$, where ϖ runs over pseudouniformizers of K and the transition morphisms are open embeddings. Consequently, Y_S^{an} exists and is sousperfectoid.*

Proof. Write A for $K\langle \varpi T_1, \dots, \varpi T_d\rangle/I$, and write A^+ for the integral closure in A of the image of $K^\circ\langle \varpi T_1, \dots, \varpi T_d\rangle$. Since Y is smooth over K, the affinoid adic space $\mathrm{Spa}(A, A^+)$ is smooth over K. Hence the first statement follows from Lemma 1.3 and $(B_{S,\varpi}, B_{S,\varpi}^+)$ being the base change of $(K, K^\circ) \to (A, A^+)$ to (R, R^+). The second statement follows from the universal property of Y_S^{an}. □

[3]While [37] works over $\mathbb{Q}_p$, the proof of [37, Corollary 4.5] does not use this assumption.

1.5. We now specialize to smooth varieties over $\mathbb{F}_q$. Let $Y = \operatorname{Spec} \mathbb{F}_q[T_1, \dots, T_d]/I$ be a smooth affine scheme over $\mathbb{F}_q$, and let $S = \operatorname{Spa}(R, R^+)$ be an affinoid perfectoid space over $\mathbb{F}_q$. Any pseudouniformizer ϖ of R induces a morphism

$$S \to \operatorname{Spa} \mathbb{F}_q(\!(\varpi^{1/q^\infty})\!).$$

By applying Proposition 1.4 to $Y_{\mathbb{F}_q(\!(\varpi^{1/q^\infty})\!)}$, we see that Y_S^{an} exists and equals $\bigcup_\varpi Y_{S,\varpi}$, where ϖ runs over pseudouniformizers of R.

To prove v-descent for vector bundles on Y_S^{an}, we use Frobenius to construct a natural perfectoid cover as follows. Note that $(\mathrm{Frob}_Y)_S^{\mathrm{an}}$ sends $Y_{S,\varpi^{1/q}}$ to $Y_{S,\varpi}$. Write $B_{S,\varpi}^{+,\mathrm{perf}}$ for the ϖ-adic completion of $\varinjlim_i B^+_{S,\varpi^{1/q^i}}$, where i runs over non-negative integers and the transition maps are given by $(\mathrm{Frob}_Y)_S^{\mathrm{an},*}$. Write $B_{S,\varpi}^{\mathrm{perf}}$ for $B_{S,\varpi}^{+,\mathrm{perf}}[\frac{1}{\varpi}]$. Finally, write $(Y_S^{\mathrm{an}})^{\mathrm{perf}}$, if it exists, for the limit $\varprojlim_i Y_S^{\mathrm{an}}$ in the category of uniform analytic adic spaces, where the transition maps are given by $(\mathrm{Frob}_Y)_S^{\mathrm{an}}$.

Proposition. *The pre-adic space $Y_{S,\varpi}^{\mathrm{perf}} \coloneqq \operatorname{Spa}(B_{S,\varpi}^{\mathrm{perf}}, B_{S,\varpi}^{\mathrm{perf},+})$ is affinoid perfectoid. Moreover, $Y_{S,\varpi}^{\mathrm{perf}}$ equals the fiber product $Y_{S,\varpi} \times_{Y_S^{\mathrm{an}}} (Y_S^{\mathrm{an}})^{\mathrm{perf}}$ in the category of uniform analytic adic spaces, so $(Y_S^{\mathrm{an}})^{\mathrm{perf}} = \bigcup_\varpi Y_{S,\varpi}^{\mathrm{perf}}$ exists and is perfectoid. Finally, the map $B_{S,\varpi} \to B_{S,\varpi}^{\mathrm{perf}}$ of topological $B_{S,\varpi}$-modules splits.*

Proof. For the first statement, note that $(B_{S,\varpi}^{\mathrm{perf}}, B_{S,\varpi}^{+,\mathrm{perf}})$ is the colimit of

$$\{(B_{S,\varpi^{1/q^i}}, B^+_{S,\varpi^{1/q^i}})\}_i$$

in the category of uniform Tate Huber pairs. Therefore [62, Proposition 3.5] indicates that it suffices to check that $B_{S,\varpi}^{\mathrm{perf}}$ is perfect. Because S is perfectoid, Frob_S induces an isomorphism $Y_{S,\varpi} \xrightarrow{\sim} Y_{S,\varpi^{1/q}}$. Composing this isomorphism with $(\mathrm{Frob}_Y)_S^{\mathrm{an}} : Y_{S,\varpi^{1/q}} \to Y_{S,\varpi}$ yields the absolute q-Frobenius endomorphism, which implies that $B_{S,\varpi}^{\mathrm{perf}}$ is indeed perfect.

For the second statement, the above shows that $Y_{S,\varpi}^{\mathrm{perf}}$ equals the limit $\varprojlim_i Y_{S,\varpi^{1/q^i}}$ in the category of uniform analytic adic spaces. The universal property of Y_S^{an} implies that the square

$$\begin{array}{ccc} Y_{S,\varpi^{1/q}} & \xrightarrow{(\mathrm{Frob}_Y)_S^{\mathrm{an}}} & Y_{S,\varpi} \\ \downarrow & & \downarrow \\ Y_S^{\mathrm{an}} & \xrightarrow{(\mathrm{Frob}_Y)_S^{\mathrm{an}}} & Y_S^{\mathrm{an}} \end{array}$$

is cartesian, so taking $\varprojlim_i$ in the category of uniform analytic adic spaces yields the desired result.

For the third statement, the smoothness of Y implies that $\mathrm{Frob}_{Y,*}\, \mathbb{F}_q[T_1, \dots, T_d]/I$ is a finite projective $\mathbb{F}_q[T_1, \dots, T_d]/I$-module [12, Lemma 1.1.1][4]. Since Y is also affine, [12, Proposition 1.1.6] shows that Y is Frobenius-split.

[4]While [12] works over an algebraically closed field, the proofs of [12, Lemma 1.1.1] and [12, Proposition 1.1.6] do not use this assumption.

Note that the square

$$\begin{array}{ccc} \mathbb{F}_q[T_1,\dots,T_d]/I & \xrightarrow{\mathrm{Frob}_Y^*} & \mathrm{Frob}_{Y,*}\,\mathbb{F}_q[T_1,\dots,T_d]/I \\ \downarrow & & \downarrow \\ B_{S,\varpi} & \xrightarrow{(\mathrm{Frob}_Y)_S^{\mathrm{an},*}} & B_{S,\varpi^{1/q}} \end{array}$$

is a pushout in the category of rings. Therefore applying $-\otimes_{\mathbb{F}_q[T_1,\dots,T_d]/I} B_{S,\varpi}$ to an $\mathbb{F}_q[T_1,\dots,T_d]/I$-module splitting of the top arrow yields a topological $B_{S,\varpi}$-module splitting of the bottom arrow. Finally, taking the completed colimit of $\{B_{S,\varpi^{1/q^i}}\}_i$ yields a topological $B_{S,\varpi}$-module splitting of $B_{S,\varpi} \to B_{S,\varpi}^{\mathrm{perf}}$. □

1.6. We now prove v-descent for vector bundles on (open subspaces of) analytifications. Let Y be a smooth scheme over $\mathbb{F}_q$, and let V_S be an open subspace of Y_S^{an}. For all affinoid perfectoid spaces S' over S, write $V_{S'}$ for the preimage of V_S.

Theorem. *The presheaf of categories on* Perf_S *given by*

$$S' \mapsto \{\textit{vector bundles on } V_{S'}\}$$

satisfies v-descent.

Proof. By replacing Y and V_S with open covers, we can assume that

$$Y = \mathrm{Spec}\,\mathbb{F}_q[T_1,\dots,T_d]/I$$

is affine and that $V_S = \mathrm{Spa}(A, A^+)$ is affinoid. Let $S' \to S$ be an affinoid perfectoid v-cover. Since $Y_S^{\mathrm{an}} = \bigcup_\varpi Y_{S,\varpi}$, we see that V_S lies in $Y_{S,\varpi}$ for some pseudouniformizer ϖ of R. Write V_S^{perf} for the preimage of V_S in $Y_{S,\varpi}^{\mathrm{perf}}$.

Let $\mathscr{E}'$ be a vector bundle on $V_{S'}$ with descent datum α with respect to

$$V_{S'\times_S S'\times_S S'} \rightrightarrows V_{S'\times_S S'} \rightrightarrows V_{S'}.$$

By Proposition 1.5, pullback yields a vector bundle $\widetilde{\mathscr{E}}'$ on $V_{S'}^{\mathrm{perf}}$ with commuting descent data $\widetilde{\alpha}$ with respect to the adic spaces

$$V_{S'\times_S S'\times_S S'}^{\mathrm{perf}} \rightrightarrows V_{S'\times_S S'}^{\mathrm{perf}} \rightrightarrows V_{S'}^{\mathrm{perf}}$$

and β' with respect to the affinoid pre-adic spaces

$$V_{S'}^{\mathrm{perf}} \times_{V_{S'}} V_{S'}^{\mathrm{perf}} \times_{V_{S'}} V_{S',\varpi}^{\mathrm{perf}} \rightrightarrows V_{S'}^{\mathrm{perf}} \times_{V_{S'}} V_{S'}^{\mathrm{perf}} \rightrightarrows V_{S'}^{\mathrm{perf}}.$$

Proposition 1.5 indicates that the $V_{(-)}^{\mathrm{perf}}$ are perfectoid, so [68, Lemma 17.1.8] enables us to descend $(\widetilde{\mathscr{E}}', \widetilde{\alpha})$ and $(\beta', \widetilde{\alpha})$ to a vector bundle $\widetilde{\mathscr{E}}$ on V_S^{perf} with descent datum β with respect to the affinoid pre-adic spaces

$$V_S^{\mathrm{perf}} \times_{V_S} V_S^{\mathrm{perf}} \times_{V_S} V_S^{\mathrm{perf}} \rightrightarrows V_S^{\mathrm{perf}} \times_{V_S} V_S^{\mathrm{perf}} \rightrightarrows V_S^{\mathrm{perf}}.$$

Finally, Proposition 1.5 shows that the map $B_{S,\varpi} \to B_{S,\varpi}^{\mathrm{perf}}$ of topological $B_{S,\varpi}$-modules splits, so the map $A \to A\widehat{\otimes}_{B_{S,\varpi}} B_{S,\varpi}^{\mathrm{perf}}$ of topological A-modules also splits. Hence [69, Tag 08XA] and [69, Tag 08XD] enable us to descend $(\widetilde{\mathscr{E}}, \beta)$ to a vector bundle $\mathscr{E}$ on V_S, as desired. □

1.7. Next, we compute the Frobenius-invariant global sections of analytifications. Write A for the global sections of Y, endowed with the discrete topology.

Proposition. *The natural map* $\underline{A}(S) \to H^0(Y_S^{\mathrm{an}}, \mathscr{O}_{Y_S^{\mathrm{an}}})^{\mathrm{Frob}_S^*=1}$ *is an isomorphism.*

Proof. By replacing Y with an open cover, we can assume that

$$Y = \operatorname{Spec} \mathbb{F}_q[T_1, \dots, T_d]/I$$

is affine. Let $\{e_m\}_m$ be an $\mathbb{F}_q$-basis of A consisting of images of monomials. For all pseudouniformizers ϖ of R, this induces an identification $\widehat{\bigoplus}_m R \cdot e_m \xrightarrow{\sim} B_{S,\varpi}$ via sending T_i to ϖT_i for all $1 \le i \le d$. Under this identification, $\mathrm{Frob}_S^* : B_{\varpi,S} \xrightarrow{\sim} B_{\varpi^q,S}$ acts coordinatewise, so $\bigoplus_m R^{\mathrm{Frob}_S^*=1} \cdot e_m \xrightarrow{\sim} B_{S,\varpi}^{\mathrm{Frob}_S^*=1}$. Now [48, Corollary 3.1.4][5] implies that $\mathrm{Cont}(|S|, \mathbb{F}_q) \xrightarrow{\sim} R^{\mathrm{Frob}_S^*=1}$, so the quasicompactness of $|S|$ shows that

$$\underline{A}(S) = \mathrm{Cont}(|S|, A) = \mathrm{Cont}\left(|S|, \bigoplus_m \mathbb{F}_q \cdot e_m\right) \xrightarrow{\sim} \bigoplus_m R^{\mathrm{Frob}_S^*=1} \cdot e_m.$$

Finally, note that the global sections of Y_S^{an} are given by $\bigcap_\varpi B_{S,\varpi}$, where ϖ runs over pseudouniformizers of R. Therefore taking $\bigcap_\varpi$ yields the claim. □

1.8. Finally, we specialize to smooth curves over $\mathbb{F}_q$, which form our main case of interest. Let C be a geometrically connected smooth proper curve over $\mathbb{F}_q$. Write F for its function field, and write $\mathbb{A}$ for its adele ring. Let U be a dense open subscheme of C, and write $\mathbb{O}_U$ for its ring of integral adeles. We write $\mathbb{O}$ for $\mathbb{O}_C$, so that $\mathbb{O}$ is the maximal compact subring of $\mathbb{A}$.

For all closed points v of C, write $\mathbb{F}_v$ for its residue field, fix an embedding $\mathbb{F}_v \to \overline{\mathbb{F}}_q$ over $\mathbb{F}_q$, write $\mathcal{O}_v$ for the completed local ring at v, and write F_v for the fraction field of $\mathcal{O}_v$. Let A_v be one of $\{\mathcal{O}_v, F_v\}$. By applying [26, Proposition II.1.1] to $\operatorname{Spa} \mathbb{F}_v \times S$, we see that $\operatorname{Spa} A_v \times S$ is a sousperfectoid adic space.

We define vector bundles on the *global* and *local Hartl–Pink curves* as follows.

Definition.

a) Write $\mathrm{Bun}_U(S)$ for the category of vector bundles $\mathscr{E}$ on U_S^{an} equipped with an isomorphism $\phi : \mathscr{E} \xrightarrow{\sim} \mathrm{Frob}_S^* \mathscr{E}$.
b) Write $\mathrm{Bun}_F(S)$ for the category $\varinjlim_U \mathrm{Bun}_U(S)$, where U runs over dense open subschemes of C.
c) Write $\mathrm{Bun}_{A_v}(S)$ for the category of vector bundles $\mathscr{E}$ on $\operatorname{Spa} A_v \times S$ equipped with an isomorphism $\phi : \mathscr{E} \xrightarrow{\sim} \mathrm{Frob}_S^* \mathscr{E}$.

Theorem 1.6 indicates that Bun_U satisfies v-descent. Proposition 1.7 implies that the transition morphisms in b) are faithful, so Bun_F also satisfies v-descent.

1.9. Usually, the local Hartl–Pink curve for F_v is defined using $q^{\deg v}$-Frobenius instead of q-Frobenius. We now relate these two definitions. Write X_{S,F_v} for the relative Fargues–Fontaine curve for F_v as in [26, Definition I.2.2].

Proposition. *The presheaf of categories* Bun_{A_v} *satisfies v-descent, and its base change to* $\mathrm{Perf}_{\mathbb{F}_v}$ *is naturally isomorphic to*

$$S \mapsto \left\{ \begin{array}{l} \text{vector bundles } \mathscr{E} \text{ on } \operatorname{Spa} A_v \times_{\mathbb{F}_v} S \text{ equipped} \\ \text{with an isomorphism } \phi : \mathscr{E} \xrightarrow{\sim} (\mathrm{Frob}_S^{\deg v})^* \mathscr{E} \end{array} \right\}.$$

[5] While [48, Corollary 3.1.4] is stated for $q = p$, the proof applies verbatim for general q.

Consequently, the base change of Bun_{F_v} *to* $\mathrm{Perf}_{\mathbb{F}_v}$ *is naturally isomorphic to*

$$S \mapsto \{\textit{vector bundles on } X_{S,F_v}\}.$$

Proof. For the first statement, let $S' \to S$ be an affinoid perfectoid v-cover. Then $\operatorname{Spa}\mathbb{F}_v \times S' \to \operatorname{Spa}\mathbb{F}_v \times S$ is also an affinoid perfectoid v-cover, so the first statement follows from [68, Proposition 19.5.3]. The second statement follows from arguing as in the proof of [55, Lemma 5.12]. Finally, the third statement follows from

$$X_{S,F_v} = (\operatorname{Spa} F_v \times_{\mathbb{F}_v} S)/\operatorname{Frob}_S^{\deg v}. \qquad \square$$

1.10. Thanks to Artin–Schreier–Witt, $\mathrm{Bun}_{\mathcal{O}_v}$ enjoys the following description.

Proposition. *The presheaf of categories* $\mathrm{Bun}_{\mathcal{O}_v}$ *is naturally isomorphic to*

$$S \mapsto \{\textit{pro-étale } \underline{\mathcal{O}_v}\textit{-local systems on } S\}.$$

Proof. After replacing $(-)[\![\pi]\!]$ with $-\widehat{\otimes}_{\mathbb{F}_q}\mathbb{F}_v[\![\pi]\!]$, where π denotes a uniformizer of $\mathcal{O}_v$, the proof proceeds as in [55, Proposition 3.7] and [55, Theorem 3.12]. $\square$

1.11. The following relationship between vector bundles on the global and local Hartl–Pink curves plays an important role in this paper. Write Z for the closed complement $C \smallsetminus U$. For all closed points u of U, we have a natural morphism of adic spaces $\operatorname{Spa}\mathcal{O}_u \to U$, and for all z in Z, we have a natural morphism of adic spaces $\operatorname{Spa} F_z \to U$. Therefore pullback yields morphisms

$$\mathrm{loc}_u : \mathrm{Bun}_U \to \mathrm{Bun}_{\mathcal{O}_u} \text{ and } \mathrm{loc}_z : \mathrm{Bun}_U \to \mathrm{Bun}_{F_z}.$$

Definition.

a) Write

$$\mathrm{loc} : \mathrm{Bun}_U \to \prod_u \mathrm{Bun}_{\mathcal{O}_u} \times \prod_z \mathrm{Bun}_{F_z}$$

for the morphism induced by the $\{\mathrm{loc}_u\}_u$ and $\{\mathrm{loc}_z\}_z$, where u runs over closed points of U, and z runs over Z.

b) Write $\mathrm{Bun}_{\mathbb{A}}$ for the presheaf of categories $\varinjlim_U \prod_u \mathrm{Bun}_{\mathcal{O}_u} \times \prod_z \mathrm{Bun}_{F_z}$, where U runs over dense open subschemes of C.

c) Write $\mathrm{loc} : \mathrm{Bun}_F \to \mathrm{Bun}_{\mathbb{A}}$ for the morphism obtained by taking $\varinjlim_U$ of a).

Proposition 1.9 implies that $\mathrm{Bun}_{\mathbb{A}}$ satisfies v-descent.

1.12. We will use the following lemma to reduce to the situation where $\operatorname{Pic} U = 1$. Because U is Dedekind, $\operatorname{Pic} U = 1$ if and only if all vector bundles on U are trivial. Since U is a smooth curve, this implies that U is affine.

Lemma. *There exists an open cover* $\{C_1, C_2\}$ *of* C *satisfying* $\operatorname{Pic} C_1 = \operatorname{Pic} C_2 = 1$ *and hence an open cover* $\{U_1, U_2\}$ *of* U *satisfying* $\operatorname{Pic} U_1 = \operatorname{Pic} U_2 = 1$.

Proof. The finitude of $\mathbb{F}_q$ implies that $\operatorname{Pic} C$ is finitely generated. Let $\mathscr{L}_1, \ldots, \mathscr{L}_g$ be generators of $\operatorname{Pic} C$, and let C_1 be a dense open subscheme of C such that $\mathscr{L}_i|_{C_1}$ is trivial for all $1 \le i \le g$. Then excision for class groups implies that $\operatorname{Pic} C_1 = 1$. Because $C \smallsetminus C_1$ is finite, weak approximation shows that the order-of-vanishing map $F^\times \to \mathbb{Z}^{\oplus(C \smallsetminus C_1)}$ is surjective, so for all $1 \le i \le g$, there exists a divisor D_i supported on C_1 such that $\mathscr{L}_i \cong \mathscr{O}_C(D_i)$. By taking $C_2 = C \smallsetminus \bigcup_{i=1}^g \operatorname{supp} D_i$, excision for class groups again implies that $\operatorname{Pic} C_2 = 1$. Finally, the analogous statements hold for $U_1 = U \cap C_1$ and $U_2 = U \cap C_2$ by excision for class groups. $\square$

1.13. We will use the following notion of local systems on S with coefficients in $\mathscr{O}_U$, which is local both in S and in U. For any $\mathscr{E}$ in $\mathrm{QCoh}(U)$, write $\underline{\mathscr{E}}$ for the constant $\mathrm{QCoh}(U)$-valued sheaf on the site $*_{\text{pro-ét}}$ induced by $\mathscr{E}$.

Definition. A *pro-étale $\underline{\mathscr{O}_U}$-local system on S* is an $\underline{\mathscr{O}_U}$-module on the site $S_{\text{pro-ét}}$ that, locally in $S_{\text{pro-ét}}$, is of the form $\underline{\mathscr{E}}$ for some vector bundle $\mathscr{E}$ on U.

When $\operatorname{Pic} U = 1$, note that this agrees with the notion of pro-étale $\underline{A}$-local systems, where A denotes the global sections of U endowed with the discrete topology.

1.14. We now prove a description of pro-étale $\underline{\mathscr{O}_U}$-local systems in terms of vector bundles on the global Hartl–Pink curve; this is the global analogue of a theorem of Kedlaya–Liu [48, Theorem 8.5.12].

We say that an object in $\mathrm{Bun}_{F_v}(S)$ *has slope zero* if, under the identification from Proposition 1.9, its image in $\mathrm{Bun}_{F_v}(\operatorname{Spa}\mathbb{F}_v \times S)$ has slope zero Harder–Narasimhan polygon as in [26, p. 74]. Note that this agrees with the usual notion when S lies over $\mathbb{F}_v$. Moreover, for all $(\mathscr{E}, \phi)$ in $\mathrm{Bun}_{G,\mathcal{O}_v}(S)$, its image in $\mathrm{Bun}_{G,F_v}(S)$ has slope zero.

Theorem. *We have an exact tensor equivalence of categories*

$$\{\textit{pro-étale } \underline{\mathscr{O}_U}\textit{-local systems on } S\} \xrightarrow{\sim} \left\{ \begin{array}{c} \textit{objects in } \mathrm{Bun}_U(S) \textit{ whose image in} \\ \mathrm{Bun}_{F_z}(S) \textit{ has slope zero for all } z \textit{ in } Z \end{array} \right\}$$

given by $\mathbb{L} \mapsto (\mathbb{L} \otimes_{\underline{\mathscr{O}_U}} \mathscr{O}_{U_S^{\mathrm{an}}}, \mathbb{L} \otimes_{\underline{\mathscr{O}_U}} (\mathrm{Frob}_S^{-1})^*)$.

Proof. By replacing U with the open cover $\{U_1, U_2\}$ from Lemma 1.12, we can assume that $\operatorname{Pic} U = 1$. Note that the above functor preserves tensor products and duals. Hence taking internal homs reduces full faithfulness to proving that, for all $\underline{A}$-local systems $\mathbb{L}$ on S, the map

$$\mathrm{Hom}_{\underline{A}}(\underline{A}, \mathbb{L}) \to \mathrm{Hom}_{\mathscr{O}_{U_S^{\mathrm{an}}}}((\mathscr{O}_{U_S^{\mathrm{an}}}, (\mathrm{Frob}_S^{-1})^*), (\mathbb{L} \otimes_{\underline{A}} \mathscr{O}_{U_S^{\mathrm{an}}}, \mathbb{L} \otimes_{\underline{A}} (\mathrm{Frob}_S^{-1})^*))$$

is a bijection. By replacing S with a pro-étale cover and using Theorem 1.6, we can assume that $\mathbb{L} = \underline{A}$. Then the result follows from Proposition 1.7.

For essential surjectivity, let $(\mathscr{E}, \phi)$ be an object in $\mathrm{Bun}_U(S)$. By full faithfulness and pro-étale descent, it suffices to prove that there exists an affinoid perfectoid pro-étale cover $S' \to S$ such that $(\mathscr{E}, \phi)|_{S'}$ is trivial. Since $\mathrm{loc}_z(\mathscr{E}, \phi)$ has slope zero for all z in Z, [26, Theorem II.2.19] yields an affinoid perfectoid pro-étale cover $S' = \operatorname{Spa}(R', R'^+) \to S$ such that, for all z in Z, the base change $\mathrm{loc}_z(\mathscr{E}, \phi)|_{S'}$ is trivial. Now $\{U_{S'}^{\mathrm{an}}\} \cup \{\operatorname{Spa}\mathcal{O}_z \times S'\}_z$ is an open cover of $C_{S'}^{\mathrm{an}}$, so we can glue $(\mathscr{E}, \phi)|_{S'}$ with the trivial object in $\mathrm{Bun}_{\mathcal{O}_z}(S')$ for all z in Z to obtain an object $(\mathscr{E}', \phi')$ in $\mathrm{Bun}_C(S')$. Theorem A.19 shows that $(\mathscr{E}', \phi')$ is the analytification of a vector bundle $\mathscr{E}'^{\mathrm{alg}}$ on $C_{R'}$ equipped with an isomorphism $\phi'^{\mathrm{alg}} : \mathscr{E}'^{\mathrm{alg}} \xrightarrow{\sim} \mathrm{Frob}_{R'}^* \mathscr{E}'^{\mathrm{alg}}$. After replacing $\operatorname{Spec} R'$ with a clopen cover, [51, Lemme 3 du paragraphe I.3] indicates that $(\mathscr{E}'^{\mathrm{alg}}, \phi'^{\mathrm{alg}})$ is isomorphic to $(\mathscr{E}^{\mathrm{alg}}|_{R'}, \mathrm{id}_{\mathscr{E}^{\mathrm{alg}}} \otimes_{\mathbb{F}_q} (\mathrm{Frob}_{R'}^{-1})^*)$ for some vector bundle $\mathscr{E}^{\mathrm{alg}}$ on C. Finally, $\operatorname{Pic} U = 1$ implies that $\mathscr{E}^{\mathrm{alg}}|_U$ is trivial. □

1.15. Let B be a perfect $\mathbb{F}_q$-algebra, endowed with the discrete topology. We conclude this section by proving that, in many situations, vector bundles on U_B are equivalent to vector bundles on "$U_{\operatorname{Spd} B}$".

What do we mean by "$U_{\operatorname{Spd} B}$"? Note that $\operatorname{Spa} B(\!(t^{1/q^\infty})\!) \to \operatorname{Spd} B$ is surjective and representable in perfectoid spaces, so we have a natural diagram of perfectoids

$$(\star)\qquad \begin{aligned} &\operatorname{Spa} B(\!(t^{1/q^\infty})\!) \times_{\operatorname{Spd} B} \operatorname{Spa} B(\!(t^{1/q^\infty})\!) \times_{\operatorname{Spd} B} \operatorname{Spa} B(\!(t^{1/q^\infty})\!) \\ &\mathrel{\substack{\rightarrow\\[-1ex]\rightarrow\\[-1ex]\rightarrow}} \operatorname{Spa} B(\!(t^{1/q^\infty})\!) \times_{\operatorname{Spd} B} \operatorname{Spa} B(\!(t^{1/q^\infty})\!) \mathrel{\substack{\rightarrow\\[-1ex]\rightarrow}} \operatorname{Spa} B(\!(t^{1/q^\infty})\!). \end{aligned}$$

Theorem. *Assume that each connected component of* $\operatorname{Spec} B$ *is a valuation ring. Then analytification induces an exact tensor equivalence from*

$$\{\textit{vector bundles on } U_B\}$$

to vector bundles on $U^{\mathrm{an}}_{\operatorname{Spa} B(\!(t^{1/q^\infty})\!)}$ *with descent datum with respect to*

$$(\star\star)\qquad \begin{aligned} &U^{\mathrm{an}}_{\operatorname{Spa} B(\!(t^{1/q^\infty})\!)\times_{\operatorname{Spd} B}\operatorname{Spa} B(\!(t^{1/q^\infty})\!)\times_{\operatorname{Spd} B}\operatorname{Spa} B(\!(t^{1/q^\infty})\!)} \\ &\mathrel{\substack{\rightarrow\\[-1ex]\rightarrow\\[-1ex]\rightarrow}} U^{\mathrm{an}}_{\operatorname{Spa} B(\!(t^{1/q^\infty})\!)\times_{\operatorname{Spd} B}\operatorname{Spa} B(\!(t^{1/q^\infty})\!)} \mathrel{\substack{\rightarrow\\[-1ex]\rightarrow}} U^{\mathrm{an}}_{\operatorname{Spa} B(\!(t^{1/q^\infty})\!)}. \end{aligned}$$

Proof. For full faithfulness, let $\mathscr{E}_1$ and $\mathscr{E}_2$ be vector bundles on U_B, write $(\mathscr{E}_1', \alpha_1)$ and $(\mathscr{E}_2', \alpha_2)$ for the corresponding vector bundles on $U^{\mathrm{an}}_{\operatorname{Spa} B(\!(t^{1/q^\infty})\!)}$ with descent data with respect to $(\star\star)$, and let $f' : \mathscr{E}_1' \to \mathscr{E}_2'$ be a morphism of vector bundles compatible with the descent data. By replacing U with an open cover, we can assume that $U = \operatorname{Spec} A$ is affine. Write $A = \mathbb{F}_q[T_1, \dots, T_d]/I$. For all affinoid perfectoid spaces $S = \operatorname{Spa}(R, R^+)$ over $\mathbb{F}_q$, Lemma 1.3 shows that

$$U_{S,1} \coloneqq \operatorname{Spa}(R\langle T_1, \dots, T_d\rangle/I, (R^+\langle T_1, \dots, T_d\rangle)^\sim)$$

is an open subspace of U^{an}_S. For all affinoid perfectoid spaces S' over S, note that the preimage of $U_{S,1}$ in $U^{\mathrm{an}}_{S'}$ equals $U_{S',1}$, so restricting $(\star\star)$ to $U_{\operatorname{Spa} B(\!(t^{1/q^\infty})\!),1}$ yields

$$\begin{aligned} &U_{\operatorname{Spa} B(\!(t^{1/q^\infty})\!)\times_{\operatorname{Spd} B}\operatorname{Spa} B(\!(t^{1/q^\infty})\!)\times_{\operatorname{Spd} B}\operatorname{Spa} B(\!(t^{1/q^\infty})\!),1} \\ &\mathrel{\substack{\rightarrow\\[-1ex]\rightarrow\\[-1ex]\rightarrow}} U_{\operatorname{Spa} B(\!(t^{1/q^\infty})\!)\times_{\operatorname{Spd} B}\operatorname{Spa} B(\!(t^{1/q^\infty})\!),1} \mathrel{\substack{\rightarrow\\[-1ex]\rightarrow}} U_{\operatorname{Spa} B(\!(t^{1/q^\infty})\!),1}. \end{aligned}$$

Since I is finitely generated, for all positive integers m, we have

$$B(\!(t_1^{1/q^\infty}, \dots, t_m^{1/q^\infty})\!)\langle T_1, \dots, T_d\rangle/I = \big(B[T_1, \dots, T_d]/I\big)(\!(t_1^{1/q^\infty}, \dots, t_m^{1/q^\infty})\!),$$

so taking global sections yields

$$(A \otimes_{\mathbb{F}_q} B)(\!(t_1^{1/q^\infty}, t_2^{1/q^\infty}, t_3^{1/q^\infty})\!) \mathrel{\substack{\leftarrow\\[-1ex]\leftarrow\\[-1ex]\leftarrow}} (A \otimes_{\mathbb{F}_q} B)(\!(t_1^{1/q^\infty}, t_2^{1/q^\infty})\!) \mathrel{\substack{\leftarrow\\[-1ex]\leftarrow}} (A \otimes_{\mathbb{F}_q} B)(\!(t_1^{1/q^\infty})\!).$$

Therefore [49, Theorem 4.9] indicates that the restriction of f' to $U_{\operatorname{Spa} B(\!(t^{1/q^\infty})\!),1}$ arises uniquely from a morphism $f : \mathscr{E}_1 \to \mathscr{E}_2$ of vector bundles on U_B. Since morphisms of vector bundles on U^{an}_S are determined by their restrictions to $U_{S,1}$, this yields the desired result.

For essential surjectivity, let $\mathscr{E}'$ be a vector bundle on $U^{\mathrm{an}}_{\operatorname{Spa} B(\!(t^{1/q^\infty})\!)}$ with descent datum α with respect to $(\star\star)$. For all z in Z, pullback yields a vector bundle $\mathscr{E}_z'$ on $\operatorname{Spa} F_z \times \operatorname{Spa} B(\!(t^{1/q^\infty})\!)$ with descent datum α_z with respect to the evaluation of $\operatorname{Spa} F_z \times (-)$ on $(\star)$, and applying [33, Theorem 3.15] and [33, Remark 3.14] to $\mathbb{F}_v \otimes_{\mathbb{F}_q} B$ shows that $(\mathscr{E}_z', \alpha_z)$ arises uniquely from a vector bundle $\mathscr{E}_z$ on $\operatorname{Spec} F_z \mathbin{\widehat{\otimes}}_{\mathbb{F}_q} B$. After replacing $\operatorname{Spec} B$ by a clopen cover, [46, Theorem 6.1] indicates that $\mathscr{E}_z$ is trivial. Now $\{U^{\mathrm{an}}_{(-)}\} \cup \{\operatorname{Spa} \mathcal{O}_z \times (-)\}_z$ is an open cover of $C^{\mathrm{an}}_{(-)}$, so we can glue $(\mathscr{E}', \alpha)$ to the trivial vector bundle on $\operatorname{Spa} \mathcal{O}_z \times \operatorname{Spa} B(\!(t^{1/q^\infty})\!)$ with descent datum

with respect to the evaluation of $\operatorname{Spa}\mathcal{O}_z \times (-)$ on $(\star)$. This yields a vector bundle $\widehat{\mathscr{E}}'$ on $C^{\mathrm{an}}_{\operatorname{Spa} B(\!(t^{1/q^\infty})\!)}$ with descent datum $\widehat{\alpha}$ with respect to

$$\begin{aligned} &C^{\mathrm{an}}_{\operatorname{Spa} B(\!(t^{1/q^\infty})\!)\times_{\operatorname{Spd} B}\operatorname{Spa} B(\!(t^{1/q^\infty})\!)\times_{\operatorname{Spd} B}\operatorname{Spa} B(\!(t^{1/q^\infty})\!)} \\ \rightrightarrows &C^{\mathrm{an}}_{\operatorname{Spa} B(\!(t^{1/q^\infty})\!)\times_{\operatorname{Spd} B}\operatorname{Spa} B(\!(t^{1/q^\infty})\!)} \rightrightarrows C^{\mathrm{an}}_{\operatorname{Spa} B(\!(t^{1/q^\infty})\!)}. \end{aligned}$$

Theorem A.19 shows that this corresponds to a vector bundle $\widehat{\mathscr{E}}'^{\,\mathrm{alg}}$ on $C_{B(\!(t_1^{1/q^\infty})\!)}$ with, for all positive integers n, a descent datum $\widehat{\alpha}_n$ with respect to

$$(\star\star\star) \qquad \begin{aligned} &C_{B[\![t_1^{1/q^\infty},t_2^{1/q^\infty},t_3^{1/q^\infty}]\!]\langle t_1^n/t_2t_3,t_2^n/t_1t_3,t_3^n/t_1t_2\rangle[1/t_1t_2t_3]} \\ \rightrightarrows &C_{B[\![t_1^{1/q^\infty},t_2^{1/q^\infty}]\!]\langle t_1^n/t_2,t_2^n/t_1\rangle[1/t_1t_2]} \rightrightarrows C_{B(\!(t_1^{1/q^\infty})\!)}. \end{aligned}$$

Let $\{C_1, C_2\}$ be an affine open cover of C. For all i in $\{1,2\}$, write A_i for the global sections of C_i, and note that A_i is free over $\mathbb{F}_q$. Hence restricting $(\star\star\star)$ to $(C_i)_{B(\!(t_1^{1/q^\infty})\!)}$, taking global sections, and taking $\varprojlim_n$ yields

$$A_i \otimes_{\mathbb{F}_q} B(\!(t_1^{1/q^\infty}, t_2^{1/q^\infty}, t_3^{1/q^\infty})\!) \Leftarrow A_i \otimes_{\mathbb{F}_q} B(\!(t_1^{1/q^\infty}, t_2^{1/q^\infty})\!) \leftarrow A_i \otimes_{\mathbb{F}_q} B(\!(t_1^{1/q^\infty})\!).$$

Therefore [49, Theorem 4.9] indicates that the restriction of $(\widehat{\mathscr{E}}', \widehat{\alpha})$ to $(C_i)_{(-),1}$ arises uniquely from a vector bundle $\widehat{\mathscr{E}}_i$ on $(C_i)_B$. Now $\{(C_1)_{(-),1}, (C_2)_{(-),1}\}$ is an open cover of $C^{\mathrm{an}}_{(-)}$ and $\{(C_1)_B, (C_2)_B\}$ is an open cover of C_B, so $(\widehat{\mathscr{E}}', \widehat{\alpha})$ arises uniquely from a vector bundle $\widehat{\mathscr{E}}$ on C_B. Finally, restrict $\widehat{\mathscr{E}}$ to U_B. □

2. G-BUNDLES

In this section, we introduce G-bundles on the global Hartl–Pink curve. After defining the moduli stack $\operatorname{Bun}_{G,F}$ thereof, we define its *localization map* to a product of local moduli stacks $\operatorname{Bun}_{G,F_z}$, which plays an important role in this paper. For example, we use the localization map to define the global semistable locus $\operatorname{Bun}^{\mathrm{ss}}_{G,F}$ in $\operatorname{Bun}_{G,F}$. Using results from §1, we prove part of Theorem A, part of Theorem B (though we postpone the official statement of Theorem B to §4), and Theorem C.

2.1. Let G be a parahoric group scheme over C as in [58, Definition 2.18]. We also write G for its base change to U, F, $\mathcal{O}_v$, or F_v for any closed point v of C.

We have the following standard Tannakian description of G-torsors. Let Y be a sousperfectoid space over C, and let $\mathscr{G}$ be an étale G-torsor on Y. For all V in $\operatorname{Rep}_C G$, write $\mathscr{G}(V)$ for the locally free $\mathscr{O}_{Y,\text{ét}}$-module $\mathscr{G} \times^G (V \otimes_{\mathscr{O}_C} \mathscr{O}_{Y,\text{ét}})$, which arises uniquely from a vector bundle $\mathscr{G}(V)$ on Y by [48, Theorem 8.2.22 (d)].

Proposition. *The above induces an equivalence of categories between*

a) étale G-torsors on Y,
b) exact tensor functors $\operatorname{Rep}_C G \to$ {*vector bundles on Y*}.

Moreover, when $Y = \operatorname{Spa}(T, T^+)$ is affinoid, all étale G-torsors on Y are representable by adic spaces, and the above are naturally equivalent to

c) étale G-torsors on $\operatorname{Spec} T$,
d) exact tensor functors $\operatorname{Rep}_C G \to$ {*finite projective T-modules*}.

Proof. By [13, Lemma 3.1], $\mathscr{O}_G$ is a filtered colimit of objects in $\operatorname{Rep}_C G$. Hence the result follows from the proof of [56, Proposition 6.10]. □

2.2. We now define the stack of G-bundles on the global and local Hartl–Pink curves. Let A_v be one of $\{\mathcal{O}_v, F_v\}$.

Definition.

a) Write $\mathrm{Bun}_{G,U}(S)$ for the groupoid of G-torsors $\mathscr{G}$ on U_S^{an} equipped with an isomorphism $\phi : \mathscr{G} \xrightarrow{\sim} \mathrm{Frob}_S^* \mathscr{G}$.
b) Write $\mathrm{Bun}_{G,F}(S)$ for the groupoid $\varinjlim_U \mathrm{Bun}_{G,U}(S)$, where U runs over dense open subschemes of C.
c) Write $\mathrm{Bun}_{G,A_v}(S)$ for the groupoid of G-torsors $\mathscr{G}$ on $\mathrm{Spa}\, A_v \times S$ equipped with an isomorphism $\phi : \mathscr{G} \xrightarrow{\sim} \mathrm{Frob}_S^* \mathscr{G}$.

Using Proposition 2.1, Theorem 1.6 indicates that $\mathrm{Bun}_{G,U}$ is a v-stack. Proposition 1.7 implies that the transition morphisms in b) are faithful, so $\mathrm{Bun}_{G,F}$ is also a v-stack. Finally, Proposition 1.9 indicates that Bun_{G,A_v} is a v-stack, and the proof of [26, Proposition III.1.3] implies that all of these v-stacks are small.

2.3. *Example.* We claim that $\mathrm{Bun}_{G,C}$ is naturally isomorphic to the constant stack over $\mathbb{F}_q$ associated with the groupoid

$$(\heartsuit) \qquad \coprod_{\alpha \in \ker^1(F,G)} G_\alpha(F) \backslash G_\alpha(\mathbb{A}) / G(\mathbb{O}),$$

where $\ker^1(F,G)$ denotes the kernel of the localization map

$$(\mathrm{loc}_v)_v : H^1(F,G) \to \prod_v H^1(F_v, G),$$

and G_α over F denotes the inner twist of G associated with α. To see this, write $\mathcal{B}_G$ for the smooth algebraic stack over $\mathbb{F}_q$ of G-bundles on C. Then Theorem A.19 identifies S-points of $\mathrm{Bun}_{G,C}$ with $\mathrm{Spec}\, R$-points of the algebraic stack $(\mathcal{B}_G)^{\mathrm{Frob}_{\mathcal{B}_G}}$ of $\mathrm{Frob}_{\mathcal{B}_G}$-fixed points of $\mathcal{B}_G$. By [70, Lemma 3.3 b)], the latter is naturally isomorphic to the constant stack over $\mathbb{F}_q$ associated with the groupoid $\mathcal{B}_G(\mathbb{F}_q)$, and [52, Remarque 12.2] identifies $\mathcal{B}_G(\mathbb{F}_q)$ with $(\heartsuit)$. Since its automorphism groups are finite, the claim follows from the natural equivalence $(\mathrm{Spec}\, R)_{\text{fét}} \xrightarrow{\sim} S_{\text{fét}}$.

2.4. We have the following trivial locus in Bun_{G,F_v} over $\mathbb{F}_q$, which descends the usual trivial locus over $\mathbb{F}_{q^{\deg v}}$ via Proposition 1.9. Write Bun^1_{G,F_v} for the substack of $\mathrm{Bun}_{G,F_v}(S)$ whose S-points consist of objects such that, for all geometric points $\overline{s}$ of S, its image in $\mathrm{Bun}_{G,F_v}(\overline{s})$ is trivial. Consider the morphism $* \to \mathrm{Bun}_{G,F_v}$ corresponding to the trivial object for all S, which factors through a morphism $* \to \mathrm{Bun}^1_{G,F_v}$. One can identify $\underline{G(F_v)}$ with $* \times_{\mathrm{Bun}^1_{G,F_v}} *$ as group v-sheaves, so descent yields a morphism $*/\underline{G(F_v)} \to \mathrm{Bun}^1_{G,F_v}$.

Proposition. *The substack* $\mathrm{Bun}^1_{G,F_v} \subseteq \mathrm{Bun}_{G,F_v}$ *is open, and* $*/\underline{G(F_v)} \to \mathrm{Bun}^1_{G,F_v}$ *is an isomorphism. Moreover,* $\mathrm{Bun}_{G,\mathcal{O}_v}$ *is naturally isomorphic to* $*/\underline{G(\mathcal{O}_v)}$*. Finally, the base change of* Bun_{G,F_v} *to* $\mathbb{F}_v$ *is naturally isomorphic to the stack of* G*-bundles on the Fargues–Fontaine curve for* F_v *as in* [26, Definition III.1.3].

Proof. After replacing $(-)[\![\pi]\!]$ with $-\widehat{\otimes}_{\mathbb{F}_q} \mathbb{F}_v[\![\pi]\!]$, where π denotes a uniformizer of $\mathcal{O}_v$, the first statement follows from the proof of [26, Theorem III.2.4], and the second statement follows from the proof of [55, Proposition 3.8] and [55, Theorem 3.12]. The third statement follows from Proposition 1.9. □

2.5. The following *localization maps* play an important role in this paper. For all closed points u in U and z in Z, pullback yields morphisms

$$\mathrm{loc}_u : \mathrm{Bun}_{G,U} \to \mathrm{Bun}_{G,\mathcal{O}_u} \text{ and } \mathrm{loc}_z : \mathrm{Bun}_{G,U} \to \mathrm{Bun}_{G,F_z}.$$

Definition.

a) Write

$$\mathrm{loc} : \mathrm{Bun}_{G,U} \to \prod_u \mathrm{Bun}_{G,\mathcal{O}_u} \times \prod_z \mathrm{Bun}_{G,F_z}$$

for the morphism induced by the $\{\mathrm{loc}_u\}_u$ and $\{\mathrm{loc}_z\}_z$, where u runs over closed points of U, and z runs over Z.

b) Write $\mathrm{Bun}_{G,\mathbb{A}}$ for the prestack $\varinjlim_U \prod_u \mathrm{Bun}_{G,\mathcal{O}_u} \times \prod_z \mathrm{Bun}_{G,F_z}$ on $\mathrm{Perf}_{\mathbb{F}_q}$, where U runs over dense open subschemes of C.

c) Write $\mathrm{loc} : \mathrm{Bun}_{G,F} \to \mathrm{Bun}_{G,\mathbb{A}}$ for the morphism obtained by taking $\varinjlim_U$ of a).

Proposition 2.4 implies that the transition morphisms in b) are faithful, so $\mathrm{Bun}_{G,\mathbb{A}}$ is a small v-stack.

2.6. When studying $\mathrm{Bun}_{G,U}$, one has the following mechanism for changing U.

Lemma. *For all dense open subschemes U' of U, the square*

$$\begin{array}{ccc} \mathrm{Bun}_{G,U} & \xrightarrow{\mathrm{loc}} & \prod_u \mathrm{Bun}_{G,\mathcal{O}_u} \times \prod_z \mathrm{Bun}_{G,F_z} \\ \downarrow & & \downarrow \\ \mathrm{Bun}_{G,U'} & \xrightarrow{\mathrm{loc}} & \prod_{u'} \mathrm{Bun}_{G,\mathcal{O}_{u'}} \times \prod_{z'} \mathrm{Bun}_{G,F_{z'}} \end{array}$$

is cartesian, where u runs over closed points of U', and z' runs over $C \smallsetminus U'$. Similarly, the square

$$\begin{array}{ccc} \mathrm{Bun}_{G,U} & \xrightarrow{\mathrm{loc}} & \prod_u \mathrm{Bun}_{G,\mathcal{O}_u} \times \prod_z \mathrm{Bun}_{G,F_z} \\ \downarrow & & \downarrow \\ \mathrm{Bun}_{G,F} & \xrightarrow{\mathrm{loc}} & \mathrm{Bun}_{G,\mathbb{A}} \end{array}$$

is cartesian.

Proof. Note that $\{U'^{\mathrm{an}}_S\} \cup \{\mathrm{Spa}\,\mathcal{O}_c \times S\}_c$ is an open cover of U^{an}_S, where c runs over $U \smallsetminus U'$. Therefore the first statement follows from gluing, and the second statement follows from taking $\varinjlim_{U'}$ of the first statement. □

2.7. We *use* the semistable locus in the stack of G-bundles on the local Hartl–Pink curve to define a global analogue as follows. Write $\mathrm{Bun}^{\mathrm{ss}}_{G,F_v}(S)$ for the full subcategory of objects in $\mathrm{Bun}_{G,F_v}(S)$ whose image in $\mathrm{Bun}_{G,F_v}(\mathrm{Spa}\,\mathbb{F}_v \times S)$, under the identification from Proposition 2.4, lies in the semistable locus as in [26, Section III.4.2]. Then [26, Theorem III.4.5] implies that $\mathrm{Bun}^{\mathrm{ss}}_{G,F_v}$ is an open substack of Bun_{G,F_v}, and the base change of $\mathrm{Bun}^{\mathrm{ss}}_{G,F_v}$ to $\mathbb{F}_v$ equals the semistable locus as in [26, Section III.4.2].

Definition.

a) Write $\mathrm{Bun}_{G,U}^{\mathrm{ss}}$ for the preimage of $\prod_u \mathrm{Bun}_{G,\mathcal{O}_u} \times \prod_z \mathrm{Bun}_{G,F_z}^{\mathrm{ss}}$ in $\mathrm{Bun}_{G,U}$ under loc.
b) Write $\mathrm{Bun}_{G,F}^{\mathrm{ss}}$ for the prestack $\varinjlim_U \mathrm{Bun}_{G,U}^{\mathrm{ss}}$ on $\mathrm{Perf}_{\mathbb{F}_q}$.

As in Definition 2.2.b), $\mathrm{Bun}_{G,F}^{\mathrm{ss}}$ is a small v-stack. Since $\mathrm{Bun}_{G,U}^{\mathrm{ss}} \subseteq \mathrm{Bun}_{G,U}$ is an open substack, we see that $\mathrm{Bun}_{G,F}^{\mathrm{ss}} \subseteq \mathrm{Bun}_{G,F}$ is an open substack as well. Proposition 2.4 shows that $\mathrm{Bun}_{G,\mathcal{O}_v} \to \mathrm{Bun}_{G,F_v}$ factors through $\mathrm{Bun}_{G,F_v}^{\mathrm{ss}}$, so for all dense open subschemes U' of U, Lemma 2.6 implies that the preimage of $\mathrm{Bun}_{G,U'}^{\mathrm{ss}}$ in $\mathrm{Bun}_{G,U}$ equals $\mathrm{Bun}_{G,U}^{\mathrm{ss}}$.

2.8. The stack of G-bundles on the global Hartl–Pink curve also has a trivial locus in the following sense. Write $\mathrm{Bun}_{G,U}^{1}$ for the substack of $\mathrm{Bun}_{G,U}$ whose S-points consist of objects such that, for all geometric points $\overline{s}$ of S, its image in $\mathrm{Bun}_{G,U}(\overline{s})$ is trivial. Consider the morphism $* \to \mathrm{Bun}_{G,U}$ corresponding to the trivial object for all S, which factors through a morphism $* \to \mathrm{Bun}_{G,U}^{1}$. Proposition 1.7 identifies $\underline{G(A)}$ with $* \times_{\mathrm{Bun}_{G,U}^{1}} *$ as group v-sheaves, so descent yields a morphism

$$*/\underline{G(A)} \to \mathrm{Bun}_{G,U}^{1}\,.$$

Theorem. *The substack* $\mathrm{Bun}_{G,U}^{1} \subseteq \mathrm{Bun}_{G,U}$ *is open, and* $*/\underline{G(A)} \to \mathrm{Bun}_{G,U}^{1}$ *is an isomorphism.*

Proof. Let $(\mathscr{G},\phi)$ be an object in $\mathrm{Bun}_{G,U}(S)$. We claim that

$$\big\{\overline{s} \in |S| \mid \text{the image of } (\mathscr{G},\phi) \text{ in } \mathrm{Bun}_{G,U}(\overline{s}) \text{ is trivial}\big\}$$

is an open subset of $|S|$. To see this, by replacing S with a pro-étale cover, we can assume that S is strictly totally disconnected. Note that the above subset lies in

$$\big\{\overline{s} \in |S| \mid \text{the image of } (\mathscr{G},\phi) \text{ in } \mathrm{Bun}_{G,F_z}(\overline{s}) \text{ is trivial for all } z \text{ in } Z\big\},$$

which is an open subset of $|S|$ by [26, Theorem III.2.4]. Therefore, by replacing S with an open subspace, we can assume that the image of $(\mathscr{G},\phi)$ in $\mathrm{Bun}_{G,F_z}(\overline{s})$ is trivial for all z in Z and geometric points $\overline{s}$ of S. For all V in $\mathrm{Rep}_C G$, this implies that the image of $(\mathscr{G}(V),\phi(V))$ in $\mathrm{Bun}_{F_z}(S)$ has slope zero for all z in Z, so Theorem 1.14 shows that $(\mathscr{G},\phi)$ corresponds to an exact tensor functor

$$\mathrm{Rep}_C G \to \{\text{pro-étale } \underline{\mathscr{O}_U}\text{-local systems on } S\}.$$

Let $\{U_1,U_2\}$ be the open cover of U from Lemma 1.12. For all i in $\{1,2\}$, write A_i for the global sections of U_i, and write A_{12} for the global sections of $U_1 \cap U_2$. Because $\mathrm{Pic}\, U_i = 1$, rank-n $\underline{\mathscr{O}_{U_i}}$-local systems are equivalent to $\underline{\mathrm{GL}_n(A_i)}$-torsors. Hence [26, Lemma III.2.6] implies that $\underline{\mathscr{O}_{U_i}}$-local systems on S are trivial, so $(\mathscr{G},\phi)$ corresponds to two exact tensor functors

$$\Big\{\mathrm{Rep}_C G \to \{\text{finite free } \underline{A_i}(S)\text{-modules}\}\Big\}_i$$

along with an isomorphism between them after postcomposing with

$$\{\text{finite free } \underline{A_i}(S)\text{-modules}\} \to \{\text{finite free } \underline{A_{12}}(S)\text{-modules}\}.$$

Since A_i is discrete, we have $\mathrm{Spec}\,\underline{A_i}(S) = \mathrm{Spec}\,\mathrm{Cont}(\pi_0(|S|),A_i) = \pi_0(|S|) \times U_i$. Therefore Proposition 2.1 and gluing show that $(\mathscr{G},\phi)$ corresponds to an étale G-torsor $\mathscr{F}$ on $\pi_0(|S|) \times U$.

For all $\overline{s}$ in $|S|$, note that $|\overline{s}| \times U \xrightarrow{\sim} \varprojlim_N \big(N \times U\big)$, where N runs over compact neighborhoods of the image of $\overline{s}$ in $\pi_0(|S|)$, and the transition morphisms are affine.

Therefore if $\mathscr{F}|_{|\overline{s}|\times U}$ is trivial, then $\mathscr{F}|_{N\times U}$ is trivial for some compact neighborhood N of the image of $\overline{s}$ in $\pi_0(|S|)$. Hence the image of $(\mathscr{G},\phi)$ in $\mathrm{Bun}_{G,U}(\widetilde{N})$ is trivial, where $\widetilde{N}$ denotes the preimage of N in S. As $\overline{s}$ varies, this yields the claim.

The claim yields the first statement. For the second statement, the above shows that $*\to \mathrm{Bun}^1_{G,U}$ is a pro-étale cover, so descent yields the desired result. □

2.9. Let us recall some notation on the Kottwitz set. In this subsection, we work over $\mathrm{Spd}\,\overline{\mathbb{F}}_q$. Let K be one of $\{F, F_v\}$, write $B(K,G)$ for the Kottwitz set as in [50, Subsection 10.2], and write $B(K,G)_{\mathrm{basic}}$ for its subset of basic elements as in [50, Definition 10.2]. For all b in $B(K,G)$, write G_b for the associated connected reductive group over K as in [50, Subsection 2.6]. Write $\mathrm{loc}_v : B(F,G)\to B(F_v,G)$ for the localization map as in [50, (10.7)], and recall that $B(F,G)$ is naturally isomorphic to the set of $\mathrm{Frob}_{\overline{\mathbb{F}}_q}$-conjugacy classes in $G(F\otimes_{\mathbb{F}_q}\overline{\mathbb{F}}_q)$ [45, Theorem 12.2].

Similarly to the local setting [26, Corollary III.4.3], in the global setting we can pass between the trivial locus and other basic loci by twisting as follows. Let b be an element in $G(U_{\overline{\mathbb{F}}_q})$ whose image in $B(F,G)$ is basic. Write $\mathrm{Bun}^b_{G,U}$ for the substack of $\mathrm{Bun}_{G,U}$ whose S-points consist of objects such that, for all geometric points $\overline{s}$ of S, its image in $\mathrm{Bun}_{G,U}(\overline{s})$ is isomorphic to $(G,(\mathrm{Frob}_S^{-1})^*\circ b)$. Twisting G by $\mathrm{ad}\, b$ over $U_{\overline{\mathbb{F}}_q}$ yields a parahoric group scheme G_b over U, and because b is basic, [34, (5.2)] shows that the generic fiber of G_b agrees with the above notion.

Proposition. *This induces a natural isomorphism* $\mathrm{Bun}_{G,U}\xrightarrow{\sim}\mathrm{Bun}_{G_b,U}$ *such that the image of* $\mathrm{Bun}^b_{G,U}$ *equals* $\mathrm{Bun}^1_{G_b,U}$.

Proof. For all objects $(\mathscr{G},\phi)$ in $\mathrm{Bun}_{G,U}(S)$, we have a natural isomorphism

$$\underline{\mathrm{Isom}}(\mathscr{G},G)\xrightarrow{\sim}\underline{\mathrm{Isom}}(\mathrm{Frob}_S^*\mathscr{G},\mathrm{Frob}_S^*G)=\mathrm{Frob}_S^*\,\underline{\mathrm{Isom}}(\mathscr{G},G),$$

where the left arrow denotes precomposing with $\phi^{-1}:\mathrm{Frob}_S^*\mathscr{G}\xrightarrow{\sim}\mathscr{G}$ and postcomposing with $(\mathrm{Frob}_S^{-1})^*\circ b : G\xrightarrow{\sim}\mathrm{Frob}_S^*G$. Since $(G_b)_{U_{\overline{\mathbb{F}}_q}}=G_{U_{\overline{\mathbb{F}}_q}}$ and we work over $\mathrm{Spd}\,\overline{\mathbb{F}}_q$, we see $\underline{\mathrm{Isom}}(-,G)$ induces the desired isomorphism $\mathrm{Bun}_{G,U}\xrightarrow{\sim}\mathrm{Bun}_{G_b,U}$. □

2.10. For any small v-stack X, write X^{red} its *reduction*, i.e. the perfect v-stack given by $\mathrm{Spec}\,B\mapsto\mathrm{Hom}(\mathrm{Spd}\,B,X)$ for all perfect $\mathbb{F}_p$-algebras B [32, Proposition 3.7]. We conclude by proving the following generalization of Theorem C.

Theorem. *The perfect v-stack* $\mathrm{Bun}^{\mathrm{red}}_{G,U}$ *over* $\mathbb{F}_q$ *is the v-sheafification of*

$$(*)\qquad \mathrm{Spec}\,B\mapsto\left\{\begin{array}{l} G\text{-torsors }\mathscr{G}\text{ on }U_B\text{ equipped with}\\ \text{an isomorphism }\phi:\mathscr{G}\xrightarrow{\sim}\mathrm{Frob}_B^*\mathscr{G}\end{array}\right\}.$$

Moreover, when each connected component of $\mathrm{Spec}\,B$ *is a valuation ring, no sheafification is needed.*

Proof. By using [11, Lemma 6.2] and taking perfections, we see that $\mathrm{Spec}\,B$ as in the second statement form a basis for the v-topology on {affine perfect schemes}. Hence it suffices to prove that the restriction of $\mathrm{Bun}^{\mathrm{red}}_{G,U}$ to such $\mathrm{Spec}\,B$ is naturally equivalent to $(*)$. Because $\mathrm{Spa}\,B(\!(t^{1/q^\infty})\!)\to\mathrm{Spd}\,B$ is surjective and representable in perfectoid spaces, this follows from Proposition 2.1 and Theorem 1.15. □

3. Shtukas and the fiber product conjecture

Our goal in this section is to prove Theorem D. First, we recall the algebraic stack of global shtukas, and we explain how to convert it into a v-stack. Next, we recall the v-stack of local shtukas. Finally, we define the function field analogue of the Hodge–Tate period map, and we prove Theorem D.

3.1. We start with some more notation concerning G over F. Fix a separable closure $\overline{F}$ of F, and write W_F for the associated absolute Weil group of F. Then global class field theory yields an isomorphism $F^\times\backslash\mathbb{A}^\times \xrightarrow{\sim} W_F^{\mathrm{ab}}$ that sends uniformizers to geometric Frobenii. Let T be a maximal subtorus of G over F, let $B \subseteq G_{\overline{F}}$ be a Borel subgroup containing $T_{\overline{F}}$, and write $X_*^+(T)$ for the associated set of dominant cocharacters of $T_{\overline{F}}$. Write $\pi_1(G)$ for the cocharacter group of $T_{\overline{F}}$ quotiented by the subgroup generated by the coroots.

3.2. Let us recall the stack of global shtukas. Let I be a finite set, and let $\mu_\bullet = (\mu_i)_{i\in I}$ be in $X_*^+(T)^I$. For all i in I, write C_i for the normalization of C in the field of definition of μ_i. For all affine schemes $\operatorname{Spec} B$ over $\mathbb{F}_q$ and morphisms

$$x : \operatorname{Spec} B \to C_i$$

over $\mathbb{F}_q$, write $\Gamma_x \subseteq C_B$ for the graph of the corresponding morphism $\operatorname{Spec} B \to C$ over $\mathbb{F}_q$. Let N be a finite closed subscheme of C, and write N_i for the preimage of N in C_i.

Definition. Write $\mathrm{Sht}^I_{G,\mu_\bullet,N}|_{\prod_{i\in I} C_i\smallsetminus N_i}$ for the prestack on $\{\text{affine schemes over } \mathbb{F}_q\}$ whose $\operatorname{Spec} B$-points parametrize data consisting of

i) for all i in I, a morphism $x_i : \operatorname{Spec} B \to C_i \smallsetminus N_i$ over $\mathbb{F}_q$,
ii) a G-torsor $\mathscr{G}$ on C_B,
iii) an isomorphism of G-torsors

$$\phi : \mathscr{G}|_{C_B\smallsetminus\sum_{i\in I}\Gamma_{x_i}} \xrightarrow{\sim} \mathrm{Frob}_B^*\,\mathscr{G}|_{C_B\smallsetminus\sum_{i\in I}\Gamma_{x_i}}$$

whose relative position along $\{\Gamma_{x_i}\}_{i\in I}$ is bounded by $\mu_\bullet$,
iv) an isomorphism of G-torsors $\psi : \mathscr{G}|_{N_B} \xrightarrow{\sim} G$ such that the square

$$\begin{array}{ccc} \mathscr{G}|_{N_B} & \xrightarrow{(\phi)_N} & \mathrm{Frob}_B^*\,\mathscr{G}|_{N_B} \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\psi} \\ G & \xrightarrow{(\mathrm{Frob}_B^*)^{-1}} & \mathrm{Frob}_B^*\,G \end{array}$$

commutes.

Write $\mathrm{Sht}^{I,\circ}_{G,\mu_\bullet,N}|_{\prod_{i\in I} C_i\smallsetminus N_i}$ for the open substack of $\mathrm{Sht}^I_{G,\mu_\bullet,N}|_{\prod_{i\in I} C_i\smallsetminus N_i}$ whose $\operatorname{Spec} B$-points consist of $(\{x_i\}_{i\in I}, \mathscr{G}, \phi, \psi)$ such that the isomorphism ϕ has relative position along $\{\Gamma_{x_i}\}_{i\in I}$ equal to $\mu_\bullet$.

Recall that $\mathrm{Sht}^I_{G,\mu_\bullet,N}|_{\prod_{i\in I} C_i\smallsetminus N_i}$ is a Deligne–Mumford stack that is separated and locally of finite type over $\prod_{i\in I} C_i \smallsetminus N_i$ [55, Proposition 5.6], and its open substack $\mathrm{Sht}^{I,\circ}_{G,\mu_\bullet,N}|_{\prod_{i\in I} C_i\smallsetminus N_i}$ is smooth over $\prod_{i\in I} C_i \smallsetminus N_i$.

For all finite closed subschemes $N' \supseteq N$ of C, write $K_N^{N'}$ for the kernel of $G(\mathscr{O}_{N'}) \to G(\mathscr{O}_N)$. Recall that $\mathrm{Sht}^I_{G,\mu_\bullet,N'}|_{\prod_{i\in I} C_i\smallsetminus N_i'} \to \mathrm{Sht}^I_{G,\mu_\bullet,N}|_{\prod_{i\in I} C_i\smallsetminus N_i'}$ is finite Galois with group $K_N^{N'}$ [55, Proposition 5.5].

3.3. For convenience, we package the Harder–Narasimhan stratification for global shtukas into a single parameter t as follows. By [5, Proposition 2.2(b)], there exists an SL_h-bundle $\mathscr{V}$ on C along with a closed embedding $\iota : G^{\mathrm{ad}} \to \underline{\mathrm{Aut}}(\mathscr{V})$ of group schemes over C such that $\underline{\mathrm{Aut}}(\mathscr{V})/G^{\mathrm{ad}}$ satisfies [5, (2.1)]. Let t be a positive integer.

Definition. Write $\mathrm{Sht}^{I,\leq t}_{G,\mu_\bullet,N}|_{\prod_{i\in I} C_i \smallsetminus N_i}$ for the open substack of

$$\mathrm{Sht}^{I}_{G,\mu_\bullet,N}|_{\prod_{i\in I} C_i \smallsetminus N_i}$$

whose $\operatorname{Spec} B$-points consist of $(\{x_i\}_{i\in I}, \mathscr{G}, \phi, \psi)$ such that the SL_h-torsor $\iota_*(\mathscr{G}^{\mathrm{ad}})$ on C_B has Harder–Narasimhan polygon bounded by $t2\rho^\vee$, where $2\rho^\vee$ denotes the sum of positive coroots in SL_h.

Recall that, when $\deg N$ is large enough, $\mathrm{Sht}^{I,\leq t}_{G,\mu_\bullet,N}|_{\prod_{i\in I} C_i \smallsetminus N_i}$ is a disjoint union of quasiprojective schemes over $\prod_{i\in I} C_i \smallsetminus N_i$ [55, Proposition 5.6].

3.4. We now convert the moduli of global shtukas into a v-stack.

Definition. Write $(\mathrm{Sht}^{I}_{G,\mu_\bullet,N}|_{\prod_{i\in I} C_i \smallsetminus N_i})^\diamond$ and $(\mathrm{Sht}^{I,\leq t}_{G,\mu_\bullet,N}|_{\prod_{i\in I} C_i \smallsetminus N_i})^\diamond$ for the prestack on $\mathrm{Perf}_{\mathbb{F}_q}$ whose S-points are given by

$$(\mathrm{Sht}^{I}_{G,\mu_\bullet,N}|_{\prod_{i\in I} C_i \smallsetminus N_i})(R) \text{ and } (\mathrm{Sht}^{I,\leq t}_{G,\mu_\bullet,N}|_{\prod_{i\in I} C_i \smallsetminus N_i})(R),$$

respectively.

3.5. **Proposition.**

1) *When* $\deg N$ *is large enough,* $(\mathrm{Sht}^{I,\leq t}_{G,\mu_\bullet,N}|_{\prod_{i\in I} C_i \smallsetminus N_i})^\diamond$ *equals* $(-)^\diamond$ *of the adic space* $\mathrm{Sht}^{I,\leq t}_{G,\mu_\bullet,N}|_{\prod_{i\in I} C_i \smallsetminus N_i}$ *over* $\mathbb{F}_q$.
2) *For any increasing family* $\{N_a\}_{a\geq 1}$ *of finite closed subschemes of* C *such that* $\deg N_a \to \infty$ *as* $a \to \infty$, *the morphism*

$$\varprojlim_{a\geq 1} (\mathrm{Sht}^{I}_{G,\mu_\bullet,N_a}|_{\prod_{i\in I} C_i \smallsetminus (N_a)_i})^\diamond \to \varprojlim_{a\geq 1} \Big(\prod_{i\in I} C_i \smallsetminus (N_a)_i\Big)^\diamond$$

is separated, representable in locally spatial diamonds, and locally of finite dim. trg.
3) *The prestack* $(\mathrm{Sht}^{I}_{G,\mu_\bullet,N}|_{\prod_{i\in I} C_i \smallsetminus N_i})^\diamond$ *is a small v-stack,*

$$(\mathrm{Sht}^{I,\leq t}_{G,\mu_\bullet,N}|_{\prod_{i\in I} C_i \smallsetminus N_i})^\diamond \subseteq (\mathrm{Sht}^{I}_{G,\mu_\bullet,N}|_{\prod_{i\in I} C_i \smallsetminus N_i})^\diamond$$

is an open substack, and for all finite closed subschemes $N' \supseteq N$ *of* C,

$$(\mathrm{Sht}^{I}_{G,\mu_\bullet,N'}|_{\prod_{i\in I} C_i \smallsetminus N'_i})^\diamond \to (\mathrm{Sht}^{I}_{G,\mu_\bullet,N}|_{\prod_{i\in I} C_i \smallsetminus N'_i})^\diamond$$

is finite Galois with group $K_N^{N'}$.

Proof. Part 1) follows from Lemma 1.2 and 3.3. For part 2), it follows from part 1) that, for all positive integers t, the morphism

$$\varprojlim_{a\geq 1} (\mathrm{Sht}^{I,\leq t}_{G,\mu_\bullet,N_a}|_{\prod_{i\in I} C_i \smallsetminus (N_a)_i})^\diamond \to \varprojlim_{a\geq 1} \Big(\prod_{i\in I} C_i \smallsetminus (N_a)_i\Big)^\diamond$$

is separated, representable in locally spatial diamonds, and locally of finite dim. trg. Recall that $\mathrm{Sht}^{I}_{G,\mu_\bullet,N}|_{\prod_{i\in I} C_i \smallsetminus N_i}$ is the increasing union $\bigcup_{t\geq 1} \mathrm{Sht}^{I,\leq t}_{G,\mu_\bullet,N}|_{\prod_{i\in I} C_i \smallsetminus N_i}$. Since the preimage of $\mathrm{Sht}^{I,\leq t}_{G,\mu_\bullet,N}|_{\prod_{i\in I} C_i \smallsetminus N'_i}$ in $\mathrm{Sht}^{I}_{G,\mu_\bullet,N'}|_{\prod_{i\in I} C_i \smallsetminus N'_i}$ equals

$$\mathrm{Sht}^{I,\leq t}_{G,\mu_\bullet,N'}|_{\prod_{i\in I} C_i \smallsetminus N'_i}$$

and $(-)^\diamond$ preserves open embeddings, taking $\bigcup_{t\geq 1}$ yields the desired result.

For part 3), let v be a closed point of $C \smallsetminus N'$, and let t be a positive integer. For all positive integers m, combining the natural equivalence $(\operatorname{Spec} R)_{\text{fét}} \xrightarrow{\sim} S_{\text{fét}}$ with 3.2 implies that S-points of $(\operatorname{Sht}^{I,\leq t}_{G,\mu_\bullet,N}|_{\prod_{i\in I} C_i \smallsetminus (N+v)_i})^\diamondsuit$ parametrize K_N^{N+mv}-torsors S' on S equipped with a K_N^{N+mv}-equivariant morphism

$$S' \to (\operatorname{Sht}^{I,\leq t}_{G,\mu_\bullet,N+mv}|_{\prod_{i\in I} C_i \smallsetminus (N+v)_i})^\diamondsuit.$$

By letting m be large, part 1) indicates that $(\operatorname{Sht}^{I,\leq t}_{G,\mu_\bullet,N}|_{\prod_{i\in I} C_i \smallsetminus (N+v)_i})^\diamondsuit$ is a small v-stack, the morphism

$$(\operatorname{Sht}^{I,\leq t}_{G,\mu_\bullet,N'}|_{\prod_{i\in I} C_i \smallsetminus (N'+v)_i})^\diamondsuit \to (\operatorname{Sht}^{I,\leq t}_{G,\mu_\bullet,N}|_{\prod_{i\in I} C_i \smallsetminus (N'+v)_i})^\diamondsuit$$

is finite Galois with group $K_N^{N'}$, and varying t yields open embeddings. Hence taking $\bigcup_{t\geq 1}$ shows the desired statements after restricting to $\prod_{i\in I} C_i \smallsetminus (N+v)_i$. Finally, because $\prod_{i\in I} C_i \smallsetminus N_i$ is covered by open subspaces of the form $\prod_{i\in I} C_i \smallsetminus (N+v)_i$, taking $\bigcup_v$ yields the desired result. □

3.6. Conversely, we can recover the moduli of global shtukas (up to perfection) from its associated v-stack:

Corollary.

1) *When N is large enough,* $(\operatorname{Sht}^{I,\leq t}_{G,\mu_\bullet,N}|_{\prod_{i\in I} C_i \smallsetminus N_i})^{\diamondsuit,\mathrm{red}}$ *is naturally isomorphic to the perfect scheme* $(\operatorname{Sht}^{I,\leq t}_{G,\mu_\bullet,N}|_{\prod_{i\in I} C_i \smallsetminus N_i})^{\mathrm{perf}}$ *over* $\mathbb{F}_q$.
2) *The perfect v-stack* $(\operatorname{Sht}^{I}_{G,\mu_\bullet,N}|_{\prod_{i\in I} C_i \smallsetminus N_i})^{\diamondsuit,\mathrm{red}}$ *is naturally isomorphic to*

$$(\operatorname{Sht}^{I}_{G,\mu_\bullet,N}|_{\prod_{i\in I} C_i \smallsetminus N_i})^{\mathrm{perf}}.$$

Proof. For part 1), Proposition 3.5.1) shows that $(\operatorname{Sht}^{I,\leq t}_{G,\mu_\bullet,N}|_{\prod_{i\in I} C_i \smallsetminus N_i})^\diamondsuit$ equals $(-)^\diamondsuit$ of the adic space $(\operatorname{Sht}^{I,\leq t}_{G,\mu_\bullet,N}|_{\prod_{i\in I} C_i \smallsetminus N_i})^{\mathrm{perf}}$ over $\mathbb{F}_q$.[6] Therefore the desired result follows from [32, Theorem 2.32].

For part 2), let v be a closed point of C. For all perfect $\mathbb{F}_p$-algebras B, the functor Spd induces a natural equivalence $(\operatorname{Spec} B)_{\text{fét}} \xrightarrow{\sim} (\operatorname{Spd} B)_{\text{fét}}$ [49, Corollary 5.4], so Proposition 3.5.3) implies that, for all positive integers m, the $\operatorname{Spec} B$-points of $(\operatorname{Sht}^{I,\leq t}_{G,\mu_\bullet,N}|_{\prod_{i\in I} C_i \smallsetminus (N+v)_i})^{\diamondsuit,\mathrm{red}}$ parametrizes K_N^{N+mv}-torsors $\operatorname{Spec} B'$ on $\operatorname{Spec} B$ equipped with a K_N^{N+mv}-equivariant morphism

$$\operatorname{Spec} B' \to (\operatorname{Sht}^{I,\leq t}_{G,\mu_\bullet,N+mv}|_{\prod_{i\in I} C_i \smallsetminus (N+v)_i})^{\diamondsuit,\mathrm{red}}.$$

By letting m be large, part 1) and 3.2 indicate that $(\operatorname{Sht}^{I,\leq t}_{G,\mu_\bullet,N}|_{\prod_{i\in I} C_i \smallsetminus (N+v)_i})^{\diamondsuit,\mathrm{red}}$ is naturally isomorphic to $(\operatorname{Sht}^{I,\leq t}_{G,\mu_\bullet,N}|_{\prod_{i\in I} C_i \smallsetminus (N+v)_i})^{\mathrm{perf}}$. Finally, taking $\bigcup_v \bigcup_{t\geq 1}$ yields the desired result. □

3.7. Next, let us recall the moduli of local shtukas. For all closed points v of C, fix a separable closure $\overline{F}_v$ of F_v, and write W_{F_v} for the associated absolute Weil group of F_v. Write $\mathbb{C}_v$ for the completion of $\overline{F}_v$. Fix an embedding $\overline{F} \to \overline{F}_v$ over $F \to F_v$, which induces a homomorphism $W_{F_v} \to W_F$.

Let $\{I_z\}_z$ be a partition of I, where z runs over Z. For all z in Z, write μ_z for the projection of $\mu_\bullet$ to $X_*^+(T)^{I_z}$, and for all i in I_z, write z_i for the closed point of C_i over z induced by $\overline{F} \to \overline{F}_z$. For all affinoid perfectoid spaces $S = \operatorname{Spa}(R, R^+)$

[6] We warn that this is *not* glued from $\operatorname{Spa} B$ as $\operatorname{Spec} B$ runs over affine open subspaces of the perfect scheme $(\operatorname{Sht}^{I,\leq t}_{G,\mu_\bullet,N}|_{\prod_{i\in I} C_i \smallsetminus N_i})^{\mathrm{perf}}$. Instead, this is glued from $\operatorname{Spa}(B, \mathbb{Z}^\sim)$.

over $\mathbb{F}_z$ and morphisms $x : S \to \operatorname{Spa} \mathcal{O}_{z_i}$ over $\mathbb{F}_z$, write $\Gamma_x \subseteq \operatorname{Spa} \mathcal{O}_z \times_{\mathbb{F}_z} S$ for the graph of the corresponding morphism $S \to \operatorname{Spa} \mathcal{O}_z$ over $\mathbb{F}_z$. Write $\prod_{i \in I_z} \operatorname{Spa} \mathcal{O}_{z_i}$ for the product of the $\operatorname{Spa} \mathcal{O}_{z_i}$ over $\mathbb{F}_z$, and write $\prod_{i \in I_z} \operatorname{Spa} F_{z_i}$ for the product of the $\operatorname{Spa} F_{z_i}$ over $\mathbb{F}_z$. Let n_z be a positive integer.

Definition. [7]

a) Write $\operatorname{LocSht}^{I_z}_{G,\mu_z} |_{\prod_{i \in I_z} \operatorname{Spd} \mathcal{O}_{z_i}}$ for the prestack on $\operatorname{Perf}_{\mathbb{F}_z}$ whose S-points parametrize data consisting of
   i) for all i in I_z, a morphism $x_i : S \to \operatorname{Spa} \mathcal{O}_{z_i}$ over $\mathbb{F}_z$,
   ii) a G-torsor $\mathscr{G}$ on $\operatorname{Spa} \mathcal{O}_z \times_{\mathbb{F}_z} S$,
   iii) an isomorphism of G-torsors

$$\phi : \mathscr{G}|_{\operatorname{Spa} \mathcal{O}_z \times_{\mathbb{F}_z} S \smallsetminus \sum_{i \in I_z} \Gamma_{x_i}} \xrightarrow{\sim} (\operatorname{Frob}_S^{\deg z})^* \mathscr{G}|_{\operatorname{Spa} \mathcal{O}_z \times_{\mathbb{F}_z} S \smallsetminus \sum_{i \in I_z} \Gamma_{x_i}}$$

   whose relative position along $\{\Gamma_{x_i}\}_{i \in I_z}$ is bounded by μ_z,

b) Write $\operatorname{LocSht}^{I_z}_{G,\mu_z,n_z} |_{\prod_{i \in I_z} \operatorname{Spd} F_{z_i}}$ for the prestack on $\operatorname{Perf}_{\mathbb{F}_z}$ whose S-points parametrize data consisting of an S-point $(\{x_i\}_{i \in I_z}, \mathscr{G}, \phi)$ of $\operatorname{LocSht}^{I_z}_{G,\mu_z} |_{\prod_{i \in I_z} \operatorname{Spd} F_{z_i}}$ equipped with
   iv) an isomorphism of G-torsors $\psi : \mathscr{G}|_{n_z z \times_{\mathbb{F}_z} S} \xrightarrow{\sim} G$ such that the square

$$\begin{array}{ccc} \mathscr{G}|_{n_z z \times_{\mathbb{F}_z} S} & \xrightarrow{(\phi)_{n_z z}} & (\operatorname{Frob}_S^{\deg z})^* \mathscr{G}|_{n_z z \times_{\mathbb{F}_z} S} \\ \downarrow \psi & & \downarrow \psi \\ G & \xrightarrow{(\operatorname{Frob}_S^*)^{-\deg z}} & G \end{array}$$

   commutes.

Write $\operatorname{LocSht}^{I_z,\circ}_{G,\mu_z,n_z} |_{\prod_{i \in I_z} \operatorname{Spd} F_{z_i}}$ for the open substack of $\operatorname{LocSht}^{I_z}_{G,\mu_z,n_z} |_{\prod_{i \in I_z} \operatorname{Spd} F_{z_i}}$ consisting of $(\{x_i\}_{i \in I_z}, \mathscr{G}, \phi, \psi)$ such that the isomorphism ϕ has relative position along $\{\Gamma_{x_i}\}_{i \in I_z}$ equal to $\mu_\bullet$.

Using [68, Proposition 19.5.3] and the proof of [26, Proposition III.1.3], we see that $\operatorname{LocSht}^{I_z}_{G,\mu_z} |_{\prod_{i \in I_z} \operatorname{Spd} \mathcal{O}_{z_i}}$ and $\operatorname{LocSht}^{I_z}_{G,\mu_z,n_z} |_{\prod_{i \in I_z} \operatorname{Spd} F_{z_i}}$ are small v-stacks.

3.8. In the local and global settings, shtukas induce bundles on the Hartl–Pink curve in the following manner. Recall that choosing a uniformizer of $\mathcal{O}_z$ identifies $\operatorname{Spa} \mathcal{O}_z \times_{\mathbb{F}_z} S$ with the open unit disk over S. For all objects $(\{x_i\}_{i \in I_z}, \mathscr{G}, \phi)$ in

$$(\operatorname{LocSht}^{I_z}_{G,\mu_z} |_{\prod_{i \in I_z} \operatorname{Spd} \mathcal{O}_{z_i}})(S),$$

the quasicompactness of $|S|$ shows that there exists a closed disk $D \subseteq \operatorname{Spa} \mathcal{O}_z \times_{\mathbb{F}_z} S$ over S centered at the origin such that D contains Γ_{x_i} for all i in I_z. Hence ϕ restricts to an isomorphism $\mathscr{G}|_{\operatorname{Spa} F_z \times_{\mathbb{F}_z} S \smallsetminus D} \xrightarrow{\sim} (\operatorname{Frob}_S^{\deg z})^* \mathscr{G}|_{\operatorname{Spa} F_z \times_{\mathbb{F}_z} S \smallsetminus D}$. Via continuation by Frobenius and Proposition 2.4, this corresponds to an object in $\operatorname{Bun}_{G,F_z}(S)$, so altogether this construction yields a morphism

$$\operatorname{LocSht}^{I_z}_{G,\mu_z} |_{\prod_{i \in I_z} \operatorname{Spd} \mathcal{O}_{z_i}} \to \operatorname{Bun}_{G,F_z} .$$

[7] This definition does not agree with the notation in [55], for which we apologize. This definition is the one that *should* be denoted by LocSht, since it parametrizes local shtukas. The definition in [55] additionally parametrizes a framing (up to isogeny), so it is akin to a Rapoport–Zink space.

Write $\prod_{i\in I}\operatorname{Spa}\mathcal{O}_{z_i}$ for $\prod_{z\in Z}\big(\prod_{i\in I_z}\operatorname{Spa}\mathcal{O}_{z_i}\big)$, and write $\prod^\circ_{i\in I}\operatorname{Spa}\mathcal{O}_{z_i}$ for its open subspace $\big(\prod_{i\in I}\operatorname{Spa}\mathcal{O}_{z_i}\big)\times_{\prod_{i\in I}C_i}\big(\prod_{i\in I}C_i\smallsetminus N_i\big)$. Form the fiber product

$$\begin{array}{ccc}(\mathrm{Sht}^I_{G,\mu_\bullet,N}\mid_{\prod^\circ_{i\in I}\operatorname{Spa}\mathcal{O}_{z_i}})^\diamondsuit & \longrightarrow & \prod_{i\in I}{}^\circ\operatorname{Spa}\mathcal{O}_{z_i}\\ \downarrow & & \downarrow\\ (\mathrm{Sht}^I_{G,\mu_\bullet,N}\mid_{\prod_{i\in I}C_i\smallsetminus N_i})^\diamondsuit & \longrightarrow & \big(\prod_{i\in I}C_i\smallsetminus N_i\big)^\diamondsuit.\end{array}$$

By arguing as above for all z in Z, we also obtain a morphism

$$(\mathrm{Sht}^I_{G,\mu_\bullet,N}\mid_{\prod^\circ_{i\in I}\operatorname{Spa}\mathcal{O}_{z_i}})^\diamondsuit\to\mathrm{Bun}_{G,U}\,.$$

3.9. We now describe the analogue of the Hodge–Tate period map for global shtukas. Write n_v for the multiplicity of v in N, and write $K_v^{n_v}$ for the kernel of $G(\mathcal{O}_v)\to G(\mathscr{O}_{n_v v})$. Write $\mathrm{LocSht}^I_{G,\mu_\bullet,N}\mid_{\prod^\circ_{i\in I}\operatorname{Spd}\mathcal{O}_{z_i}}$ for the product

$$\prod_{\substack{z\in Z\\ n_z=0}}\mathrm{LocSht}^{I_z}_{G,\mu_z}\mid_{\prod_{i\in I_z}\operatorname{Spd}\mathcal{O}_{z_i}}\times\prod_{\substack{z\in Z\\ n_z\geq 1}}\mathrm{LocSht}^{I_z}_{G,\mu_z,n_z}\mid_{\prod_{i\in I_z}\operatorname{Spd}F_{z_i}},$$

and write $K_{N\cap U}$ for the kernel of $G(\mathbb{O}_U)\to G(\mathscr{O}_{N\cap U})$.

Let $(\{x_i\}_{i\in I},\mathscr{G},\phi,\psi)$ be an object in $(\mathrm{Sht}^I_{G,\mu_\bullet,N}\mid_{\prod^\circ_{i\in I}\operatorname{Spa}\mathcal{O}_{z_i}})^\diamondsuit(S)$. For all z in Z, arguing as in the proof of [55, Lemma 5.12] shows that

$$(\{x_i\}_{i\in I_z},\mathscr{G}|_{\operatorname{Spa}\mathcal{O}_z},\phi|_{\operatorname{Spa}\mathcal{O}_z})$$

corresponds to an object in $(\mathrm{LocSht}^{I_z}_{G,\mu_z}\mid_{\prod_{i\in I_z}\operatorname{Spd}\mathcal{O}_{z_i}})(S)$. If n_z is positive, then $(\psi)_{n_z z}$ corresponds to a lift of this object to $(\mathrm{LocSht}^{I_z}_{G,\mu_z,n_z}\mid_{\prod_{i\in I_z}\operatorname{Spd}F_{z_i}})(S)$, so altogether we obtain a morphism

$$(\mathrm{Sht}^I_{G,\mu_\bullet,N}\mid_{\prod^\circ_{i\in I}\operatorname{Spa}\mathcal{O}_{z_i}})^\diamondsuit\to\mathrm{LocSht}^I_{G,\mu_\bullet,N}\mid_{\prod^\circ_{i\in I}\operatorname{Spd}\mathcal{O}_{z_i}}$$

over $\prod^\circ_{i\in I}\operatorname{Spd}\mathcal{O}_{z_i}$. For all closed points u of U, Proposition 2.4 indicates that

$$(\mathscr{G}|_{\operatorname{Spa}\mathcal{O}_u},\phi|_{\operatorname{Spa}\mathcal{O}_u})$$

corresponds to an object in $(*/\underline{G(\mathcal{O}_u)})(S)$. If u lies in N, then $(\psi)_{n_u u}$ corresponds to a lift of this object to $(*/\underline{K_u^{n_u}})(S)$, so altogether we also obtain a morphism

$$(\mathrm{Sht}^I_{G,\mu_\bullet,N}\mid_{\prod^\circ_{i\in I}\operatorname{Spa}\mathcal{O}_{z_i}})^\diamondsuit\to */\underline{K_{N\cap U}}.$$

3.10. Finally, we arrive at Theorem D.

Theorem. *The square of small v-stacks*

$$\begin{array}{ccc}(\mathrm{Sht}^I_{G,\mu_\bullet,N}\mid_{\prod^\circ_{i\in I}\operatorname{Spa}\mathcal{O}_{z_i}})^\diamondsuit & \longrightarrow & */\underline{K_{N\cap U}}\times\mathrm{LocSht}^I_{G,\mu_\bullet,N}\mid_{\prod^\circ_{i\in I}\operatorname{Spd}\mathcal{O}_{z_i}}\\ \downarrow & & \downarrow\\ \mathrm{Bun}_{G,U} & \xrightarrow{\mathrm{loc}} & */\underline{G(\mathbb{O}_U)}\times\prod_{z\in Z}\mathrm{Bun}_{G,F_z}\end{array}$$

is cartesian.

Proof. For all z be in Z and closed disks $D_z \subseteq \operatorname{Spa}\mathcal{O}_z \times_{\mathbb{F}_z} S$ over S centered at the origin, write $D_z^{\cup}$ for the closed subspace

$$\coprod_{r\in\mathbb{Z}/\deg z} \operatorname{Frob}_S^{-r}(D_z) \subseteq \coprod_{r\in\mathbb{Z}/\deg z} \operatorname{Spa}\mathcal{O}_z \times_{\mathbb{F}_z,r} S \xrightarrow{\sim} \operatorname{Spa}\mathcal{O}_z \times S,$$

where $\operatorname{Spa}\mathcal{O}_z \times_{\mathbb{F}_z,r} S$ denotes the product of $\operatorname{Spa}\mathcal{O}_z$ and

$$S \longrightarrow \operatorname{Spa}\mathbb{F}_z \xrightarrow{\operatorname{Frob}_{\mathbb{F}_z}^r} \operatorname{Spa}\mathbb{F}_z$$

over $\mathbb{F}_z$, and the isomorphism is from [55, Lemma 5.11]. Then

$$\Big\{U_S^{\mathrm{an}} \smallsetminus \bigcup_{z\in Z} D_z^{\cup}\Big\} \cup \{\operatorname{Spa}\mathcal{O}_z \times S\}_{z\in Z}$$

is an open cover of C_S^{an}, so we can glue an object in $\operatorname{Bun}_{G,U}(S)$ to an object in $(*/\underline{K_{N\cap U}} \times \operatorname{LocSht}^I_{G,\mu_\bullet,N}|_{\prod_{i\in I}^{\circ}\operatorname{Spd}\mathcal{O}_{z_i}})(S)$ to obtain

i) an S-point $\{x_i\}_{i\in I}$ of $\prod_{i\in I}^{\circ}\operatorname{Spa}\mathcal{O}_{z_i}$,
ii) a G-torsor $\mathscr{G}^{\mathrm{an}}$ on C_S^{an},
iii) an isomorphism of G-torsors

$$\phi^{\mathrm{an}} : \mathscr{G}|_{C_S^{\mathrm{an}}\smallsetminus\sum_{i\in I}\Gamma_{x_i}} \xrightarrow{\sim} \operatorname{Frob}_S^*\mathscr{G}^{\mathrm{an}}|_{C_S^{\mathrm{an}}\smallsetminus\sum_{i\in I}\Gamma_{x_i}}$$

whose relative position along $\{\Gamma_{x_i}\}_{i\in I}$ is bounded by $\mu_\bullet$,
iv) an isomorphism of G-torsors $\psi^{\mathrm{an}} : \mathscr{G}^{\mathrm{an}}|_{N_S^{\mathrm{an}}} \xrightarrow{\sim} G$ such that the square

$$\begin{array}{ccc} \mathscr{G}^{\mathrm{an}}|_{N_S^{\mathrm{an}}} & \xrightarrow{(\phi^{\mathrm{an}})_N} & \operatorname{Frob}_S^*\mathscr{G}^{\mathrm{an}}|_{N_S^{\mathrm{an}}} \\ \downarrow{\scriptstyle \psi^{\mathrm{an}}} & & \downarrow{\scriptstyle \psi^{\mathrm{an}}} \\ G & \xrightarrow{(\operatorname{Frob}_S^{-1})^*} & \operatorname{Frob}_S^* G \end{array}$$

commutes.

Finally, Proposition 2.1 and Theorem A.19 show that the above corresponds to an object in $(\operatorname{Sht}^I_{G,\mu_\bullet,N}|_{\prod_{i\in I}^{\circ}\operatorname{Spa}\mathcal{O}_{z_i}})^{\Diamond}(S)$. □

4. Geometric consequences

In this section, we harvest some of the fruits of Theorem D. We start by deducing an algebraic analogue of Theorem D via taking reductions, and we explain how this implies the Langlands–Rapoport conjecture for global shtukas. Next, we use the charts provided by Theorem D and Beauville–Laszlo uniformization to prove Theorem A and Theorem B. Finally, we conclude by explaining the relation with Igusa varieties.

4.1. For all z in Z, write $\prod_{i\in I_z} z_i$ for the product of the z_i over $\mathbb{F}_z$. Let us recall the following algebraic analogues of $\operatorname{Bun}_{G,F_z}$ and $\operatorname{LocSht}^{I_z}_{G,\mu_z}|_{\prod_{i\in I_z}\operatorname{Spd}\mathcal{O}_{z_i}}$.

Definition.

a) Write $\mathscr{I}\mathrm{soc}_{G,F_z}$ for the prestack on {affine perfect schemes over $\mathbb{F}_q$} whose $\operatorname{Spec} B$-points parametrize data consisting of G-torsors $\mathscr{G}$ on $\operatorname{Spec} F_z\widehat{\otimes}_{\mathbb{F}_q}B$ equipped with an isomorphism $\phi : \mathscr{G} \xrightarrow{\sim} \operatorname{Frob}_B^*\mathscr{G}$.
b) Write $\mathscr{L}\mathrm{ocSht}^{I_z}_{G,\mu_z}|_{\prod_{i\in I_z} z_i}$ for the prestack on {affine perfect schemes over $\mathbb{F}_z$} whose $\operatorname{Spec} B$-points parametrize data consisting of

i) for all i in I_z, a morphism $x_i : \operatorname{Spec} B \to z_i$ over $\mathbb{F}_z$,
ii) a G-torsor $\mathscr{G}$ on $\operatorname{Spec} \mathcal{O}_z \widehat{\otimes}_{\mathbb{F}_z} B$,
iii) an isomorphism of G-torsors $\phi : \mathscr{G}|_{\operatorname{Spec} F_z \widehat{\otimes}_{\mathbb{F}_z} B} \xrightarrow{\sim} (\mathrm{Frob}_B^{\deg z})^* \mathscr{G}|_{\operatorname{Spec} F_z \widehat{\otimes}_{\mathbb{F}_z} B}$ whose relative position along $\{\Gamma_{x_i}\}_{i \in I_z}$ is bounded by μ_z,

Note that [46, Proposition 5.9] and [11, Remark 4.2] imply that $\mathscr{I}\mathrm{soc}_{G,F_z}$ and $\mathscr{L}\mathrm{ocSht}^{I_z}_{G,\mu_z}|_{\prod_{i\in I_z} z_i}$ are perfect v-stacks.

4.2. Write $\prod_{i\in I} z_i$ for $\prod_{z\in Z} \big(\prod_{i\in I_z} z_i\big)$. By taking reductions, we recover the following algebraic analogue of Theorem 3.10:

Theorem. *Assume that N and Z are disjoint. Then there is a natural cartesian square of perfect v-stacks*

$$\begin{array}{ccc} (\mathrm{Sht}^I_{G,\mu_\bullet,N}|_{\prod_{i\in I} z_i})^{\mathrm{perf}} & \longrightarrow & */\underline{K_N} \times \prod_{z\in Z} \mathscr{L}\mathrm{ocSht}^{I_z}_{G,\mu_z}|_{\prod_{i\in I_z} z_i} \\ \downarrow & & \downarrow \\ \mathrm{Bun}^{\mathrm{red}}_{G,U} & \longrightarrow & */\underline{G(\mathbb{O}_U)} \times \prod_{z\in Z} \mathscr{I}\mathrm{soc}_{G,F_z}. \end{array}$$

Proof. Because $(-)^{\mathrm{red}}$ preserves limits, base changing the top arrow of Theorem 3.10 to $\prod_{i\in I} z_i$ and taking $(-)^{\mathrm{red}}$ yields a cartesian square of perfect v-stacks. The classifying stacks are preserved by [49, Corollary 5.4], the top left is identified by Corollary 3.6.2), the top right is identified by [33, Theorem 7.14.(2)], and the bottom right is identified by [33, Theorem 7.14.(1)]. □

4.3. *Remarks.*

1) By evaluating Theorem 4.2 on $\overline{\mathbb{F}}_q$ and using the description of $\mathrm{Bun}^{\mathrm{red}}_{G,U}(\overline{\mathbb{F}}_q)$ from Theorem 2.10, we immediately deduce the Langlands–Rapoport conjecture for moduli spaces of global shtukas with arbitrary (in particular, colliding) legs.
2) If one takes the v-sheafification of $(*)$ as the definition of $\mathrm{Bun}^{\mathrm{red}}_{G,U}$, one can directly prove Theorem 4.2 using Beauville–Laszlo gluing. However, without using Theorem 2.10, it is unclear why no sheafification is needed when evaluating $\mathrm{Bun}^{\mathrm{red}}_{G,U}$ on $\operatorname{Spec} B$ for which each connected component is a valuation ring.

4.4. Next, we recall some facts about affine Grassmannians. **For the rest of this paper, we work over $\overline{\mathbb{F}}_q$.** For all z in Z, write Gr_G for the B_{dR}-affine Grassmannian over $\operatorname{Spd} \breve{F}_z$ as in [68, Definition 20.2.1].[8] When $I_z = \{i\}$ is a singleton, recall from [68, p. 184] the closed affine Schubert variety $\mathrm{Gr}_{G,\mu_z}|_{\operatorname{Spd} \breve{F}_{z_i}}$ in $\mathrm{Gr}_G|_{\operatorname{Spd} \breve{F}_{z_i}}$ and the open affine Schubert variety $\mathrm{Gr}^\circ_{G,\mu_z}|_{\operatorname{Spd} \breve{F}_{z_i}}$ in $\mathrm{Gr}_{G,\mu_z}|_{\operatorname{Spd} \breve{F}_{z_i}}$. The proof of [68, Proposition 23.3.1] shows that $\mathrm{Gr}_{G,\mu_z}|_{\operatorname{Spd} \breve{F}_{z_i}}$ is naturally isomorphic to

$$\varprojlim_{m\geq 1} \mathrm{LocSht}^{I_z}_{G,\mu_z,m}|_{\operatorname{Spd} \breve{F}_{z_i}}.$$

[8] While [68] works over $\mathbb{Q}_p$, the definition of the B_{dR}-affine Grassmannian and its basic properties hold over any nonarchimedean local field. Indeed, this is implicitly used in [26].

The proof of [26, Proposition VI.2.3] implies that

$$\mathrm{Gr}_G\,|_{\mathrm{Spa}\,\mathbb{C}_z} = \bigcup_{\mu_z \in X_*^+(T)} \mathrm{Gr}_{G,\mu_z}\,|_{\mathrm{Spa}\,\mathbb{C}_z},$$

[26, Proposition III.3.1] indicates that the natural morphism $\mathrm{Gr}_G\,|_{\mathrm{Spa}\,\breve{F}_z} \to \mathrm{Bun}_{G,F_z}$ is surjective in the pro-étale topology, and the proof of [26, Theorem IV.1.19] shows

$$\coprod_{\mu_z \in X_*^+(T)} \big(\mathrm{Gr}^{\circ}_{G,\mu_z}\,|_{\mathrm{Spd}\,\breve{F}_{z_i}}\big)/\underline{G(F_z)} \to \mathrm{Bun}_{G,F_z} \times \mathrm{Spd}\,\breve{F}_z$$

is a v-cover that is separated, representable in locally spatial diamonds, and cohomologically smooth.

4.5. Write $\mathrm{loc}_Z : \mathrm{Bun}_{G,U} \to \prod_{z\in Z} \mathrm{Bun}_{G,F_z}$ for the morphism $(\mathrm{loc}_z)_{z\in Z}$, which plays an important role in §5 and §6. It enjoys the following finitude properties.

Proposition. *When Z is nonempty, $\mathrm{loc}_Z : \mathrm{Bun}_{G,U} \to \prod_{z\in Z} \mathrm{Bun}_{G,F_z}$ is compactifiable, representable in locally spatial diamonds, and locally of finite* dim. trg. *Consequently, $\mathrm{Bun}_{G,U}$ is an Artin v-stack.*

When Z is empty, Example 2.3 shows that $\mathrm{loc}_\varnothing : \mathrm{Bun}_{G,C} \to *$ is *not* representable in locally spatial diamonds but that $\mathrm{Bun}_{G,C}$ remains an Artin v-stack.

Proof. Let m be a positive integer. In Theorem 3.10, take $I_z = \{i\}$ to be a singleton for all z in Z, and take N to be mZ. Then taking $\varprojlim_{m\geq 1}$ yields a cartesian square

$$\begin{array}{ccc} \varprojlim_{m\geq 1} (\mathrm{Sht}^I_{G,\mu_\bullet,mZ}\,|_{\prod_{i\in I}\mathrm{Spa}\,\breve{F}_{z_i}})^\Diamond & \longrightarrow & \varprojlim_{m\geq 1} \mathrm{LocSht}^I_{G,\mu_\bullet,mZ}\,|_{\prod_{i\in I}\mathrm{Spd}\,\breve{F}_{z_i}} \\ \downarrow & & \downarrow \\ \mathrm{Bun}_{G,U} & \xrightarrow{\quad\mathrm{loc}_Z\quad} & \prod_{z\in Z} \mathrm{Bun}_{G,F_z}. \end{array}$$

Proposition 3.5.2) indicates that $\varprojlim_{m\geq 1}(\mathrm{Sht}^I_{G,\mu_\bullet,mZ}\,|_{\prod_{i\in I}\mathrm{Spa}\,\breve{F}_{z_i}})^\Diamond$ is a locally spatial diamond that is separated and locally of finite dim. trg over $\prod_{i\in I}\mathrm{Spd}\,\breve{F}_{z_i}$. By 4.4 and [68, Proposition 20.2.3], we see that $\varprojlim_{m\geq 1}\mathrm{LocSht}^I_{G,\mu_\bullet,mZ}\,|_{\prod_{i\in I}\mathrm{Spd}\,\breve{F}_{z_i}}$ is also a locally spatial diamond that is separated and locally of finite dim. trg over $\prod_{i\in I}\mathrm{Spd}\,\breve{F}_{z_i}$, so the top arrow is separated, representable in locally spatial diamonds by [62, Proposition 13.4 (ii)], and locally of finite dim. trg.

Next, 4.4 implies that the morphism

$$\coprod_{\mu_\bullet \in X_*^+(T)^I} \varprojlim_{m\geq 1} \mathrm{LocSht}^I_{G,\mu_\bullet,mZ}\,|_{\prod_{z\in Z}\mathrm{Spa}\,\mathbb{C}_z} \to \prod_{z\in Z}\mathrm{Bun}_{G,F_z}$$

is surjective in the pro-étale topology. Hence the bottom arrow is separated by [62, Proposition 10.11 (ii)], representable in locally spatial diamonds by [62, Proposition 13.4 (iv)] and locally of finite dim. trg, as desired.

Finally, [26, Proposition IV.1.8 (i)] and [26, Theorem IV.1.19] show $\prod_{z\in Z}\mathrm{Bun}_{G,F_z}$ is an Artin v-stack, so the same holds for $\mathrm{Bun}_{G,U}$ [26, Proposition IV.1.8 (iii)]. □

4.6. At this point, we can finish the proof of Theorem A. For any finite set Q of closed points of C, write $\mathcal{O}_Q$ for $\prod_{v\in Q}\mathcal{O}_v$, and write F_Q for $\prod_{v\in Q}F_v$. Write 2ρ in $X^*(T)$ for the sum of all positive roots, and write $d_{\mu_\bullet}$ for $\sum_{i\in I}\langle 2\rho,\mu_i\rangle$.

Proposition. *The Artin v-stack* $\mathrm{Bun}_{G,U}$ *is cohomologically smooth over* $\operatorname{Spd}\overline{\mathbb{F}}_q$*, and its dualizing complex over* $\operatorname{Spd}\overline{\mathbb{F}}_q$ *with* $\mathbb{F}_\ell$*-coefficients is isomorphic to* $\mathbb{F}_\ell$*.*

Proof. When Z is empty, this follows immediately from Example 2.3. When Z is nonempty, take $I_z=\{i\}$ to be a singleton for all z in Z. Then 4.4 and the cartesian square from the proof of Proposition 4.5 induce a cartesian square

$$\begin{array}{ccc}\Big[\varprojlim_{m\geq 1}(\mathrm{Sht}^{I,\circ}_{G,\mu_\bullet,mZ}\mid_{\prod_{i\in I}\operatorname{Spa}\breve{F}_{z_i}})^{\Diamond}\Big]\Big/\underline{G(F_Z)} & \longrightarrow & \Big[\prod_{z\in Z}\mathrm{Gr}^{\circ}_{G,\mu_z}\mid_{\operatorname{Spd}\breve{F}_{z_i}}\Big]\Big/\underline{G(F_Z)}\\ \downarrow g & & \downarrow f\\ \mathrm{Bun}_{G,U}\times\prod_{z\in Z}\operatorname{Spd}\breve{F}_{z_i} & \xrightarrow{\mathrm{loc}_Z} & \prod_{z\in Z}\big(\mathrm{Bun}_{G,F_z}\times\operatorname{Spd}\breve{F}_{z_i}\big).\end{array}$$

Now 4.4 indicates that f is separated, representable in locally spatial diamonds, and cohomologically smooth, and the proof of [26, Theorem IV.1.19] indicates that $f^!\mathbb{F}_\ell$ is isomorphic to $\mathbb{F}_\ell(d_{\mu_\bullet})[2d_{\mu_\bullet}]$. Hence the same holds for g and $g^!\mathbb{F}_\ell$.

Because $\mathrm{Sht}^{I,\circ}_{G,\mu_\bullet,mZ}\mid_{\prod_{i\in I}C_i\smallsetminus Z_i}$ is smooth over $\operatorname{Spec}\mathbb{F}_q$, Proposition 3.5 implies

$$(\circ)\qquad \Big[\varprojlim_{m\geq 1}(\mathrm{Sht}^{I,\circ}_{G,\mu_\bullet,mZ}\mid_{\prod_{i\in I}\operatorname{Spa}\breve{F}_{z_i}})^{\Diamond}\Big]\Big/\underline{G(F_Z)}$$

admits a separated, representable in locally spatial diamonds, and cohomologically smooth v-cover from a locally spatial diamond V such that V is cohomologically smooth over $\operatorname{Spd}\overline{\mathbb{F}}_q$. Moreover, the proof of [15, Lemma 8.3.4] shows that the dualizing complex of $(\circ)$ over $\prod_{i\in I}\operatorname{Spd}\breve{F}_{z_i}$ with $\mathbb{F}_\ell$-coefficients is isomorphic to $\mathbb{F}_\ell(d_{\mu_\bullet})[2d_{\mu_\bullet}]$.

Next, 4.4 implies that the morphism

$$\coprod_{\mu_\bullet\in X_*^+(T)^I}\Big[\varprojlim_{m\geq 1}(\mathrm{Sht}^{I,\circ}_{G,\mu_\bullet,mZ}\mid_{\prod_{i\in I}\operatorname{Spa}\breve{F}_{z_i}})^{\Diamond}\Big]\Big/\underline{G(F_Z)}\to\mathrm{Bun}_{G,U}\times\prod_{z\in Z}\operatorname{Spd}\breve{F}_{z_i}$$

is a v-cover. Therefore the above shows that $\mathrm{Bun}_{G,U}\times\prod_{z\in Z}\operatorname{Spd}\breve{F}_{z_i}$ is cohomologically smooth over $\prod_{z\in Z}\operatorname{Spd}\breve{F}_{z_i}$ and that its dualizing complex over $\prod_{z\in Z}\operatorname{Spd}\breve{F}_{z_i}$ with $\mathbb{F}_\ell$-coefficients is isomorphic to $\mathbb{F}_\ell$. By [26, Proposition IV.1.8 (ii)], this implies that $\mathrm{Bun}_{G,U}$ is cohomologically smooth over $\operatorname{Spd}\overline{\mathbb{F}}_q$, and by [62, Theorem 19.5 (ii)], this implies that its dualizing complex over $\operatorname{Spd}\overline{\mathbb{F}}_q$ with $\mathbb{F}_\ell$-coefficients is isomorphic to $\mathbb{F}_\ell$. □

4.7. We will use the following analogue of (perfect) Igusa varieties for shtukas. For all z in Z, let b_z be an element in $B(F_z,G)$, and write $b_\bullet$ for $(b_z)_{z\in Z}$.

Definition. Write $\mathrm{Ig}_G^{b_\bullet}$ for the prestack on $\{\text{affine schemes over }\overline{\mathbb{F}}_q\}$ whose $\operatorname{Spec}B$-points parametrize data consisting of

i) a G-torsor $\mathscr{G}$ on U_B,
ii) an isomorphism of G-torsors $\phi:\mathscr{G}\xrightarrow{\sim}\mathrm{Frob}_B^*\mathscr{G}$,

iii) for all z in Z, an isomorphism of G-torsors $\psi_z : \mathscr{G}|_{\operatorname{Spec} F_z \widehat{\otimes}_{\mathbb{F}_z} B} \xrightarrow{\sim} G$ such that

$$\begin{array}{ccc} \mathscr{G}|_{\operatorname{Spec} F_z \widehat{\otimes}_{\mathbb{F}_z} B} & \xrightarrow{(\phi^{\deg z})_{F_z}} & (\mathrm{Frob}_B^{\deg z})^* \mathscr{G}|_{\operatorname{Spec} F_z \widehat{\otimes}_{\mathbb{F}_z} B} \\ \downarrow{\scriptstyle \psi_z} & & \downarrow{\scriptstyle \psi_z} \\ G & \xrightarrow{(\mathrm{Frob}_B^*)^{-\deg z} \circ b_z} & G \end{array}$$

commutes.

For any $\mu_\bullet = (\mu_z)_{z \in Z}$ in $X_*^+(T)^Z$, write $\mathrm{Ig}^{b_\bullet}_{G,\mu_\bullet}$ for the closed subprestack of $\mathrm{Ig}^{b_\bullet}_G$ whose $\operatorname{Spec} B$-points consist of $(\mathscr{G}, \phi, \{\psi_z\}_{z \in Z})$ such that the relative position of ϕ along $\{z_B\}_{z \in Z}$ is bounded by $\mu_\bullet$.

Write d for the least common multiple of $\{\deg z\}_{z \in Z}$. Then Definition 4.7.iii) implies that the morphism $\mathrm{Frob}^d_{\mathrm{Ig}^{b_\bullet}_{G,\mu_\bullet}} : \mathrm{Ig}^{b_\bullet}_{G,\mu_\bullet} \to (\mathrm{Frob}^d_{\mathbb{F}_q})^* \mathrm{Ig}^{b_\bullet}_{G,\mu_\bullet}$ is naturally equivalent to the isomorphism sending $(\mathscr{G}, \phi, \{\psi_z\}_{z \in Z}\}) \mapsto (\mathscr{G}, \phi, \{\psi_z \circ (\phi^{(d/\deg z)})_{F_z}\}_{z \in Z})$. In particular, $\mathrm{Ig}^{b_\bullet}_{G,\mu_\bullet}$ and $\mathrm{Ig}^{b_\bullet}_G = \varinjlim_{\mu_\bullet \in X_*^+(T)^Z} \mathrm{Ig}^{b_\bullet}_{G,\mu_\bullet}$ are perfect.

4.8. **Proposition.** *When Z is nonempty, $\mathrm{Ig}^{b_\bullet}_{G,\mu_\bullet}$ is a filtered open union of schemes that are cofiltered limits of disjoint unions of quasiprojective schemes over $\overline{\mathbb{F}}_q$. In particular, $\mathrm{Ig}^{b_\bullet}_G$ is an ind-scheme.*

When Z is empty, Example 2.3 shows that $\mathrm{Ig}^{()}_G$ is the constant stack over $\overline{\mathbb{F}}_q$ associated with the groupoid (♡). In particular, $\mathrm{Ig}^{()}_G$ is *not* an ind-scheme.

Proof. By Beauville–Laszlo gluing, a $\operatorname{Spec} B$-point of $\mathrm{Ig}^{b_\bullet}_G$ is equivalent to the data:

i') a G-torsor $\mathscr{G}$ on C_B,

ii') an isomorphism of G-torsors $\phi : \mathscr{G}|_{U_B} \xrightarrow{\sim} \mathrm{Frob}_B^* \mathscr{G}|_{U_B}$,

iii') for all z in Z, an isomorphism of G-torsors $\psi_z : \mathscr{G}|_{\operatorname{Spec} \mathcal{O}_z \widehat{\otimes}_{\mathbb{F}_z} B} \xrightarrow{\sim} G$ such that

$$\begin{array}{ccc} \mathscr{G}|_{\operatorname{Spec} F_z \widehat{\otimes}_{\mathbb{F}_z} B} & \xrightarrow{(\phi^{\deg z})_{F_z}} & (\mathrm{Frob}_B^{\deg z})^* \mathscr{G}|_{\operatorname{Spec} F_z \widehat{\otimes}_{\mathbb{F}_z} B} \\ \downarrow{\scriptstyle (\psi_z)_{F_z}} & & \downarrow{\scriptstyle (\psi_z)_{F_z}} \\ G & \xrightarrow{(\mathrm{Frob}_B^*)^{-\deg z} \circ b_z} & G \end{array}$$

commutes.

Next, write $\mathcal{B}_{G,\infty Z}$ for the prestack on $\{\text{affine schemes over } \overline{\mathbb{F}}_q\}$ whose $\operatorname{Spec} B$-points parametrize data consisting of

i') a G-torsor $\mathscr{G}$ on C_B,

iii") for all z in Z, an isomorphism of G-torsors $\psi_z : \mathscr{G}|_{\operatorname{Spec} \mathcal{O}_z \widehat{\otimes}_{\mathbb{F}_z} B} \xrightarrow{\sim} G$.

Recall that $\mathcal{B}_{G,\infty Z}$ is a filtered open union of schemes that are cofiltered limits of disjoint unions of quasiprojective schemes [5, p. 15]. Given a $\operatorname{Spec} B$-point $(\mathscr{G}, \{\psi_z\}_{z \in Z})$ of $\mathcal{B}_{G,\infty Z}$, the additional data of

ii") an isomorphism of G-torsors $\phi : \mathscr{G}|_{U_B} \xrightarrow{\sim} \mathrm{Frob}_B^* \mathscr{G}|_{U_B}$ whose relative position along $\{z_B\}_{z \in Z}$ is bounded by $\mu_\bullet$

corresponds to a projective scheme over $\operatorname{Spec} B$ [5, Proposition 3.12], and the commutativity condition of iii') corresponds to a closed subscheme. Therefore the morphism $\mathrm{Ig}^{b_\bullet}_{G,\mu_\bullet} \to \mathcal{B}_{G,\infty Z}$ that sends $(\mathscr{G}, \phi, \{\psi_z\}_{z \in Z}) \mapsto (\mathscr{G}, \{\psi_z\}_{z \in Z})$ is schematic and projective, which yields the desired result. □

4.9. Similarly to Definition 3.4, we now convert Igusa varieties into v-stacks.

Definition. Write $(\mathrm{Ig}_G^{b_\bullet})^\diamond$ and $(\mathrm{Ig}_{G,\mu_\bullet}^{b_\bullet})^\diamond$ for the prestack on $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$ whose S-points are given by $\mathrm{Ig}_G^{b_\bullet}(R)$ and $\mathrm{Ig}_{G,\mu_\bullet}^{b_\bullet}(R)$, respectively.

When Z is empty, Example 2.3 shows that $(\mathrm{Ig}_G^{b_\bullet})^\diamond$ is a small v-stack. When Z is nonempty, Lemma 1.2 and Proposition 4.8 imply that

$$(\mathrm{Ig}_{G,\mu_\bullet}^{b_\bullet})^\diamond \text{ and } (\mathrm{Ig}_G^{b_\bullet})^\diamond = \varinjlim_{\mu_\bullet \in X_*^+(T)^Z} (\mathrm{Ig}_{G,\mu_\bullet}^{b_\bullet})^\diamond$$

are small v-sheaves.

4.10. As suggested by Theorem 3.10, the fibers of loc_Z are given by Igusa varieties in the following sense. We get a morphism $(\mathrm{Ig}_G^{b_\bullet})^\diamond \to \mathrm{Bun}_{G,U}$ by sending

$$(\mathscr{G}, \phi, \{\psi_z\}_z) \mapsto (\mathscr{G}^{\mathrm{an}}, \phi^{\mathrm{an}}).$$

The element b_z induces a morphism $b_z : * \to \mathrm{Bun}_{G,F_z}$, so as z varies we also get a morphism $b_\bullet : * \to \prod_{z\in Z} \mathrm{Bun}_{G,F_z}$.

Proposition. *The square of small v-stacks*

$$\begin{array}{ccc} (\mathrm{Ig}_G^{b_\bullet})^\diamond & \longrightarrow & * \\ \downarrow & & \downarrow{\scriptstyle b_\bullet} \\ \mathrm{Bun}_{G,U} & \xrightarrow{\mathrm{loc}_Z} & \prod_{z\in Z} \mathrm{Bun}_{G,F_z} \end{array}$$

is cartesian.

Proof. In Theorem 3.10, take $I_z = \{i\}$ to be a singleton for all z in Z, take N to be $\varnothing$, and base change the top arrow along the morphism $\mathrm{Spd}\,\overline{\mathbb{F}}_q \to \prod_{i\in I} z_i \to \prod_{i\in I} \mathrm{Spd}\,\mathcal{O}_{z_i}$ corresponding to the embeddings $\mathbb{F}_z \to \overline{\mathbb{F}}_q$. This yields a cartesian square

$$\begin{array}{ccc} (\mathrm{Sht}_{G,\mu_\bullet}^I |_{\overline{\mathbb{F}}_q})^\diamond & \longrightarrow & \mathrm{LocSht}_{G,\mu_\bullet}^I |_{\mathrm{Spd}\,\overline{\mathbb{F}}_q} \\ \downarrow & & \downarrow \\ \mathrm{Bun}_{G,U} & \xrightarrow{\mathrm{loc}_Z} & \prod_{z\in Z} \mathrm{Bun}_{G,F_z}\,. \end{array}$$

Note that the morphism $b_\bullet : * \to \prod_{z\in Z} \mathrm{Bun}_{G,F_z}$ naturally lifts to a morphism

$$b_\bullet : * \to \mathrm{LocSht}_{G,\mu_\bullet}^I |_{\mathrm{Spd}\,\overline{\mathbb{F}}_q},$$

so it suffices to prove that the square of small v-stacks

$$\begin{array}{ccc} (\mathrm{Ig}_{G,\mu_\bullet}^{b_\bullet})^\diamond & \longrightarrow & * \\ \downarrow & & \downarrow{\scriptstyle b_\bullet} \\ (\mathrm{Sht}_{G,\mu_\bullet}^I |_{\overline{\mathbb{F}}_q})^\diamond & \longrightarrow & \mathrm{LocSht}_{G,\mu_\bullet}^I |_{\mathrm{Spd}\,\overline{\mathbb{F}}_q} \end{array}$$

is cartesian. An S-point in the fiber product parametrizes data consisting of

i) a G-torsor $\mathscr{G}$ on C_R,

ii) an isomorphism of G-torsors $\phi : \mathscr{G}|_{U_R} \xrightarrow{\sim} \mathrm{Frob}_R^* \mathscr{G}|_{U_R}$ whose relative position along $\{z_R\}_{z\in Z}$ is bounded by $\mu_\bullet$,
iii) for all z in Z, an isomorphism of G-torsors $\psi_z^{\mathrm{an}} : \mathscr{G}|_{\mathrm{Spa}\,\mathcal{O}_z \times_{\mathbb{F}_z} S} \xrightarrow{\sim} G$ such that

$$\begin{array}{ccc} \mathscr{G}|_{\mathrm{Spa}\,F_z \times_{\mathbb{F}_z} S} & \xrightarrow{(\phi^{\deg z})_{F_z}} & (\mathrm{Frob}_S^{\deg z})^* \mathscr{G}|_{\mathrm{Spa}\,F_z \times_{\mathbb{F}_z} S} \\ \downarrow{\scriptstyle \psi_z^{\mathrm{an}}} & & \downarrow{\scriptstyle \psi_z^{\mathrm{an}}} \\ G & \xrightarrow{(\mathrm{Frob}_S^*)^{-\deg z}} & G. \end{array}$$

Note that everything is independent of R^+, so we can assume that R^+ equals R°. Because products of points as in [32, Definition 1.2] form a basis for the v-topology [32, Example 1.1], we can assume that S is a product of points. Then [33, Corollary 1.9] implies that the above corresponds to an object in $(\mathrm{Ig}^b_{G,\mu_\bullet})^\diamondsuit(S)$. □

4.11. Let us recall some more notation concerning the Kottwitz set. Let K be one of $\{F, F_v\}$, write $\mathbb{X}_K$ for the W_K-module given by

$$\mathbb{X}_K \coloneqq \begin{cases} \mathbb{Z} & \text{if } K = F_v, \\ \mathrm{Div}(\overline{F})_0 \text{ as in [34, p. 75]} & \text{if } K = F, \end{cases}$$

and write $\kappa : B(K,G) \to (\pi_1(G) \otimes_{\mathbb{Z}} \mathbb{X}_K)_{W_K}$ for the Kottwitz map as in [50, Subsection 11.5]. Recall that κ restricts to a bijection $B(K,G)_{\mathrm{basic}} \xrightarrow{\sim} (\pi_1(G) \otimes_{\mathbb{Z}} \mathbb{X}_K)_{W_K}$ [50, Proposition 13.1, Proposition 15.1].

Recall the global semistable locus from Definition 2.7, which was defined *in terms of* its local analogue. We now prove Theorem B, which says that it admits the following *intrinsic* description. For all b in $B(F,G)_{\mathrm{basic}}$, write $\mathrm{Bun}^b_{G,F}$ for the substack of $\mathrm{Bun}_{G,F}$ whose S-points consist of objects such that, for all geometric points $\overline{s}$ of S, its image in $\mathrm{Bun}_{G,F}(\overline{s})$ is isomorphic to $(G, (\mathrm{Frob}_S^*)^{-1} \circ b)$.

Theorem. *The substack* $\mathrm{Bun}^b_{G,F} \subseteq \mathrm{Bun}_{G,F}$ *is open and isomorphic to* $*/\underline{G_b(F)}$. *Moreover, the open substack* $\mathrm{Bun}^{\mathrm{ss}}_{G,F} \subseteq \mathrm{Bun}_{G,F}$ *equals* $\coprod_{b\in B(F,G)_{\mathrm{basic}}} \mathrm{Bun}^b_{G,F}$.

Proof. Theorem 2.8 implies that $\mathrm{Bun}^1_{G,F} \subseteq \mathrm{Bun}_{G,F}$ is open and naturally isomorphic to $*/\underline{G(F)}$. Proposition 2.9 yields an isomorphism $\mathrm{Bun}_{G,F} \xrightarrow{\sim} \mathrm{Bun}_{G_b,F}$ of v-stacks such that the image of $\mathrm{Bun}^b_{G,F}$ equals $\mathrm{Bun}^1_{G_b,F}$, so applying the above to G_b shows that $\mathrm{Bun}^b_{G,F} \subseteq \mathrm{Bun}_{G,F}$ is open and isomorphic to $*/\underline{G_b(F)}$. Theorem 2.10 identifies the isomorphism classes in $\mathrm{Hom}_{\overline{\mathbb{F}}_q}(\mathrm{Spd}\,\overline{\mathbb{F}}_q, \mathrm{Bun}_{G,F})$ with $B(F,G)$, which indicates that the $\mathrm{Bun}^b_{G,F}$ are disjoint for distinct b. Hence $\coprod_{b\in B(F,G)_{\mathrm{basic}}} \mathrm{Bun}^b_{G,F}$ is an open substack of $\mathrm{Bun}_{G,F}$, and it evidently lies in $\mathrm{Bun}^{\mathrm{ss}}_{G,F}$.

For the reverse inclusion, let $\overline{s} = \mathrm{Spa}(K, K^+)$ be a geometric point of $\mathrm{Bun}^{\mathrm{ss}}_{G,U}$, where U is some dense open subscheme of C. Then [26, Theorem III.4.5] implies that, for all z in Z, the image of $\overline{s}$ in Bun_{G,F_z} lies in $\mathrm{Bun}^{b_z}_{G,F_z}$ for some b_z in $B(F_z,G)_{\mathrm{basic}}$. Take $I_z = \{i\}$ to be a singleton, and take μ_z to be a lift of $\kappa(b_z)$ to $X_*^+(T)$. Then 4.4 and the proof of [61, Proposition A.9][9] indicate that the image of $\overline{s}$ in Bun_{G,F_z} lifts to a $\mathrm{Spa}(K,K^+)$-point of $\varprojlim_{m\geq 1} \mathrm{LocSht}^{I_z}_{G,\mu_z,mz}|_{\mathrm{Spd}\,F_{z_i}}$. As z varies, the cartesian square from the proof of Proposition 4.5 shows that $\overline{s}$ lifts to

[9] While [61, Appendix A] works over p-adic local fields and assumes that the cocharacter is minuscule, this is not used in the proof of [61, Proposition A.9].

a $\mathrm{Spa}(K, K^+)$-point of $\varprojlim_{m\geq 1}(\mathrm{Sht}^I_{G,\mu_\bullet,mZ}\,|_{\prod_{i\in I}\mathrm{Spa}\,F_{z_i}})^\Diamond$, which yields a $\mathrm{Spec}\,K$-point $(\mathscr{G},\phi)$ of $\mathrm{Sht}^I_{G,\mu_\bullet,\varnothing}\,|_{\prod_{i\in I}C_i}$.

Recall that the algebraic stack $\mathcal{B}_G$ over $\mathbb{F}_q$ of G-bundles on C satisfies $\pi_0(\mathcal{B}_G) = \pi_1(G)_{W_F}$. Since absolute q-Frobenius induces the identity on topological spaces, $\mathscr{G}$ and $\mathrm{Frob}_K^*\mathscr{G}$ have the same image in $\pi_0(\mathcal{B}_G)$. Now ϕ exhibits $\mathrm{Frob}_K^*\mathscr{G}$ as a modification of $\mathscr{G}$ of relative position $\sum_{z\in Z}\mu_z$, so the image of $\mathrm{Frob}_K^*\mathscr{G}$ in $\pi_0(\mathcal{B}_G)$ equals the image of $\mathscr{G}$ in $\pi_0(\mathcal{B}_G)$ plus the image of $\sum_{z\in Z}\mu_z$ in $\pi_1(G)_{W_F}$. Therefore the image of $\sum_{z\in Z}\mu_z$ in $\pi_1(G)_{W_F}$ is trivial. By [50, Proposition 15.6], this shows that there exists a b in $B(F,G)_{\mathrm{basic}}$ such that $\mathrm{loc}_z(b) = b_z$ for all z in Z and $\mathrm{loc}_u(b) = 1$ for all closed points u of U.

Let U' be a dense open subscheme of U such that b has a representative in $G(U'_{\overline{\mathbb{F}}_q})$, and fix such a representative. By setting $b_c = 1$ for all c in $U \smallsetminus U'$, we may replace U with U'. Then Proposition 2.9 yields an isomorphism $\mathrm{Bun}_{G,U} \xrightarrow{\sim} \mathrm{Bun}_{G_b,U}$ such that, for all z in Z, the image of $\overline{s}$ in Bun_{G_b,F_z} lies in $\mathrm{Bun}^1_{G_b,F_z}$. After using fpqc descent to extend G_b to a parahoric group scheme over C, Proposition 2.4 implies that $\mathrm{Bun}_{G_b,\mathcal{O}_z} \to \mathrm{Bun}^1_{G_b,F_z}$ is surjective in the pro-étale topology for all z in Z. Hence the image of $\overline{s}$ in Bun_{G_b,F_z} lifts to a $\mathrm{Spa}(K,K^+)$-point of $\mathrm{Bun}_{G_b,\mathcal{O}_z}$, and as z varies, Lemma 2.6 shows that $\overline{s}$ lifts to a $\mathrm{Spa}(K,K^+)$-point $(\mathscr{G}_b,\phi_b)$ of $\mathrm{Bun}_{G_b,C}$.

Proposition 2.1 and Theorem A.19 indicate that $(\mathscr{G}_b,\phi_b)$ is the analytification of a G_b-torsor $\mathscr{G}_b^{\mathrm{alg}}$ on C_K equipped with an isomorphism $\phi_b^{\mathrm{alg}} : \mathscr{G}_b^{\mathrm{alg}} \xrightarrow{\sim} \mathrm{Frob}_K^*\mathscr{G}_b^{\mathrm{alg}}$. Applying [70, Lemma 3.3 b)] to $\mathcal{B}_{G_b}$ shows that $(\mathscr{G}_b^{\mathrm{alg}},\phi_b^{\mathrm{alg}})$ is isomorphic to

$$(\overline{\mathscr{G}}_b^{\mathrm{alg}}, \mathrm{id}_{\overline{\mathscr{G}}_b^{\mathrm{alg}}} \otimes_{\mathbb{F}_q} (\mathrm{Frob}_K^{-1})^*)$$

for some G_b-torsor $\overline{\mathscr{G}}_b^{\mathrm{alg}}$ on C. Write b' for the image of $(\overline{\mathscr{G}}_b^{\mathrm{alg}})_F$ under

$$H^1(F,G_b) \hookrightarrow B(F,G_b)_{\mathrm{basic}} \xrightarrow{\sim} B(F,G)_{\mathrm{basic}},$$

where the left arrow denotes the injection from [50, (10.5)], and the right arrow denotes the bijection from [34, Lemma 5.3]. After shrinking U such that b' has a representative in $G(U_{\overline{\mathbb{F}}_q})$, we see that $\overline{s}$ lies in $\mathrm{Bun}^{b'}_{G,U}$, as desired. □

5. Sheaf-theoretic consequences

Our goal in this section is to prove Theorem E, which heavily relies on material from Appendix B. We start by briefly recalling the theory of motivic sheaves and explaining why we need it. Next, we describe a general mechanism for constructing (derived) Hecke actions, which we use in both §5 and §6. We then state a forthcoming result of Eteve–Gaitsgory–Genestier–Lafforgue, which is necessary to even formulate Theorem E.

At this point, we can prove Theorem E. As an application, we explain how Theorem E can be used to recover all of the representation-theoretic results of [55]; for this, it suffices to use existing work of Xue instead of the forthcoming result of Eteve–Gaitsgory–Genestier–Lafforgue.

5.1. We begin by recalling the sheaf theories that we will use. Let $L/\mathbb{Q}_\ell$ be a finite extension containing a fixed $\sqrt{q}$. Write $\mathcal{O}_L$ for the ring of integers of L, and write $\mathbb{F}_\lambda$ for the residue field of $\mathcal{O}_L$. Let Λ be one of $\{L, \mathcal{O}_L, \mathbb{F}_\lambda\}$.

Recall from B.3 the $\mathbb{Z}[\frac{1}{q}]$-linear 6-functor formalism $D_{\mathrm{mot}}(-)$ on

$$\{\text{small v-stacks over } \overline{\mathbb{F}}_q\},$$

and recall from B.4 the Λ-linear 6-functor formalism

$$D(-,\Lambda) \coloneqq D_{\mathrm{mot}}(-) \otimes_{D_{\mathrm{mot}}(*)} D(\Lambda).$$

We use (this base change of) the theory of motivic sheaves because it enjoys !-pushforwards and encodes "non-completed" sheaves even when $\Lambda \neq \mathbb{F}_\lambda$ (e.g. see Corollary B.4). However, to facilitate comparisons with classical étale ℓ-adic sheaf theories for algebraic varieties, we will also use the various 3-functor formalisms from Definition B.5, Proposition B.7, and B.8.

If one is only interested in the $\Lambda = \mathbb{F}_\lambda$ case, one can instead take

$$D(-,\mathbb{F}_\lambda) \coloneqq D_{\text{ét}}(-,\mathbb{F}_\lambda)$$

and just use the étale sheaf theory of [62]. In fact, since all the sheaves that we use are overconvergent as in [26, Proposition IV.2.4], the theory of motivic sheaves specializes to this when $\Lambda = \mathbb{F}_\lambda$; see B.3.

5.2. We will use the following mechanism for constructing actions of (derived) endomorphism algebras on both the automorphic and spectral sides. Let $(\mathcal{C}, E)$ be a geometric setup as in [42, Definition 2.1.1], and let $D(-)$ be a Λ-linear 3-functor formalism on $(\mathcal{C}, E)$. Consider a diagram in $\mathcal{C}$

$$\begin{array}{ccccc} Y_O & \xrightarrow{g_O} & O \times X & \xrightarrow{p_1} & O \\ \downarrow{\scriptstyle j_Y} & & \downarrow{\scriptstyle j_X} & & \downarrow{\scriptstyle j} \\ Y_F & \xrightarrow{g_F} & F \times X & \xrightarrow{p_1} & F \\ & & \downarrow{\scriptstyle p_2} & & \\ & & X, & & \end{array}$$

where the squares are cartesian, and the morphisms are all !-able. Write f for the morphism $Y_O \to X$, and let A be an object in $D(Y_F)$.

Lemma. *The object $f_! j_Y^* A$ in $D(X)$ is naturally a module for the $\mathbb{E}_1$-algebra $\mathrm{End}_{D(F)}(j_!\mathbf{1})$ over Λ.*

Proof. Since $f = p_2 \circ g_F \circ j_Y$, the projection formula and proper base change yield

$$\begin{aligned} f_! j_Y^* A &= p_{2,!} g_{F,!} j_{Y,!} j_Y^* A = p_{2,!} g_{F,!}(A \otimes j_{Y,!}\mathbf{1}) = p_{2,!} g_{F,!}(A \otimes j_{Y,!} g_O^* p_1^* \mathbf{1}) \\ &= p_{2,!} g_{F,!}(A \otimes g_F^* p_1^* j_! \mathbf{1}) = p_{2,!}(p_1^* j_! \mathbf{1} \otimes g_{F,!} A). \end{aligned}$$

Because $j_!\mathbf{1}$ is naturally a module for the $\mathbb{E}_1$-algebra $\mathrm{End}_{D(F)}(j_!\mathbf{1})$ over Λ, the same holds for $f_! j_Y^* A$ by applying the functor $p_{2,!}(p_1^* - \otimes g_{F,!} A)$. □

5.3. For places in U, we have the following (derived) Hecke algebra action on $\mathrm{loc}_{Z,!}\Lambda$. Let Q be a finite nonempty set of closed points of U, and write

$$\mathrm{loc}_Q : \mathrm{Bun}_{G,U} \to */\underline{G(\mathcal{O}_Q)}$$

for the morphism $(\mathrm{loc}_u)_{u\in Q}$. Then Lemma 2.6 implies that the square

$$\begin{array}{ccc} \mathrm{Bun}_{G,U} & \xrightarrow{(\mathrm{loc}_Q,\mathrm{loc}_Z)} & */\underline{G(\mathcal{O}_Q)} \times \prod_{z\in Z} \mathrm{Bun}_{G,F_z} \\ \downarrow & & \downarrow \\ */\underline{G(F_Q)} \times_{\prod_{u\in Q} \mathrm{Bun}_{G,F_u}} \mathrm{Bun}_{G,U\smallsetminus Q} & \longrightarrow & */\underline{G(F_Q)} \times \prod_{z\in Z} \mathrm{Bun}_{G,F_z} \end{array}$$

is cartesian. Proposition 4.5 implies that the bottom arrow is !-able, so the same holds for the top arrow. Therefore Lemma 5.2 and Corollary B.4 endow $\mathrm{loc}_{Z,!}\Lambda$ in $D(\prod_{z\in Z} \mathrm{Bun}_{G,F_z}, \Lambda)$ with the structure of a module for the $\mathbb{E}_1$-algebra

$$\mathrm{End}_{G(F_Q)}(\text{c-Ind}_{G(\mathcal{O}_Q)}^{G(F_Q)} \Lambda) = \bigotimes_{u\in Q} \mathrm{End}_{G(F_u)}(\text{c-Ind}_{G(\mathcal{O}_u)}^{G(F_u)} \Lambda)$$

over Λ, which we remind the reader can have cohomology in nonzero degrees when $\Lambda \neq L$. Taking the colimit over Q shows that $\mathrm{loc}_{Z,!}\Lambda$ is naturally a module for the $\mathbb{E}_1$-algebra $\bigotimes_u \mathrm{End}_{G(F_u)}(\text{c-Ind}_{G(\mathcal{O}_u)}^{G(F_u)} \Lambda)$ over Λ, where u runs over closed points of U.

5.4. Let us recall some notation on Langlands dual groups. **For the rest of this paper, assume that G is reductive over U.** Write W_U for the absolute Weil group of U with respect to $\overline{F}$, write $\widetilde{F}/F$ for the finite Galois extension such that $\mathrm{Gal}(\widetilde{F}/F)$ equals the image of the W_U-action on $X_*^+(T)$, and write $\widetilde{C}$ for the normalization of C in $\widetilde{F}$. Write $\widetilde{U}$ for the preimage of U in $\widetilde{C}$, and write $\widetilde{Z}$ for the closed complement $\widetilde{C} \smallsetminus \widetilde{U}$. Write $\widehat{G}$ for the dual group of G over Λ, and write ${}^L G$ for $\widehat{G} \rtimes \underline{\mathrm{Gal}(\widetilde{F}/F)}$. Similarly, for all closed points v of C, write $\widetilde{F}_v/F_v$ for the finite Galois extension such that $\mathrm{Gal}(\widetilde{F}_v/F_v)$ equals the image of the W_{F_v}-action on $X_*^+(T)$, and write ${}^L G_v$ for $\widehat{G} \rtimes \underline{\mathrm{Gal}(\widetilde{F}_v/F_v)}$.

Using ${}^L G$, we re-index stacks of global shtukas as follows. Recall from 3.2 the Deligne–Mumford stack $\mathrm{Sht}^I_{G,\mu_\bullet,N}|_{\prod_{i\in I} C_i \smallsetminus N_i}$. Let $\dot{V}$ be an object in $\mathrm{Rep}_\Lambda({}^L G)^I$, write $\widetilde{N}$ for the preimage of N in $\widetilde{C}$, and note that the disjoint union

$$\coprod_{\substack{\mu_\bullet \in X_*^+(T)^I \\ \text{arising in } \dot{V}|_{\widehat{T}^I}}} \mathrm{Sht}^I_{G,\mu_\bullet,N}|_{(\widetilde{C}\smallsetminus\widetilde{N})^I}$$

naturally descends to a Deligne–Mumford stack $\mathrm{Sht}^I_{G,\dot{V},N}|_{(C\smallsetminus N)^I}$ that is separated and locally of finite type over $(C \smallsetminus N)^I$. Write $\pi_N : \mathrm{Sht}^I_{G,\dot{V},N}|_{(C\smallsetminus N)^I} \to (C \smallsetminus N)^I$ for the structure morphism.

5.5. Briefly, write Sht for $\mathrm{Sht}^I_{G,\dot{V},N}|_{(C\smallsetminus N)^I_{\overline{\mathbb{F}}_q}}$. Using 3.2, Proposition 3.5, and finite étale descent, we can extend the Λ-linear 3-functor formalisms from Proposition B.7 and B.8 to the mildly stacky setting of $\pi_N : \mathrm{Sht} \to (C \smallsetminus N)^I_{\overline{\mathbb{F}}_q}$. More precisely, we

have a natural diagram

$$\begin{array}{ccc} \operatorname{Ind} D_{\mathrm{mot}}^{(C\smallsetminus N)^I_{\overline{\mathbb{F}}_q}\setminus \prod_{i\in I}\operatorname{Spa}\mathbb{C}_{z_i}}\big((\mathrm{Sht}\,|_{\prod_{i\in I}\operatorname{Spa}\mathbb{C}_{z_i}})^{\diamondsuit}\big) & \xrightarrow{\Upsilon} & D_{\mathrm{mot}}\big((\mathrm{Sht}\,|_{\prod_{i\in I}\operatorname{Spa}\mathbb{C}_{z_i}})^{\diamondsuit}\big) \\ \Big\downarrow{\scriptstyle r_{\ell,(\mathrm{Sht}\,|_{\prod_{i\in I}\operatorname{Spa}\mathbb{C}_{z_i}})^{\diamondsuit}}} & & \\ \operatorname{Ind} D_{\text{ét}}^{(C\smallsetminus N)^I_{\overline{\mathbb{F}}_q}\setminus \prod_{i\in I}\operatorname{Spa}\mathbb{C}_{z_i}}\big((\mathrm{Sht}\,|_{\prod_{i\in I}\operatorname{Spa}\mathbb{C}_{z_i}})^{\diamondsuit},\Lambda\big) & \xleftarrow{\rho} & D(\mathrm{Sht},\mathbb{Z}_\ell) \end{array}$$

such that !-pushforward along π_N yields a morphism of diagrams to

$$\begin{array}{ccc} \operatorname{Ind} D_{\mathrm{mot}}^{(C\smallsetminus N)^I_{\overline{\mathbb{F}}_q}\setminus \prod_{i\in I}\operatorname{Spa}\mathbb{C}_{z_i}}\big(\prod_{i\in I}\operatorname{Spa}\mathbb{C}_{z_i}\big) & \xrightarrow{\Upsilon} & D_{\mathrm{mot}}\big(\prod_{i\in I}\operatorname{Spa}\mathbb{C}_{z_i}\big) \\ \Big\downarrow{\scriptstyle r_{\ell,\prod_{i\in I}\operatorname{Spa}\mathbb{C}_{z_i}}} & & \\ \operatorname{Ind} D_{\text{ét}}^{(C\smallsetminus N)^I_{\overline{\mathbb{F}}_q}\setminus \prod_{i\in I}\operatorname{Spa}\mathbb{C}_{z_i}}\big(\prod_{i\in I}\operatorname{Spa}\mathbb{C}_{z_i},\Lambda\big) & \xleftarrow{\rho} & D((C\smallsetminus N)^I,\mathbb{Z}_\ell). \end{array}$$

5.6. We now state a forthcoming result of Eteve–Gaitsgory–Genestier–Lafforgue. Roughly, it asserts that the cohomology of stacks of global shtukas satisfies the conclusion of Drinfeld's lemma even on the level of derived categories. On the level of cohomology groups, this is known by existing work of Xue; see Remark 5.7 below.

For any separated finite type scheme X over $\overline{\mathbb{F}}_q$, write $D_{\mathrm{lis}}(X,\Lambda)\subseteq D_{\mathrm{cons}}(X,\Lambda)$ for the full subcategory of lisse objects. Since $\mathrm{Frob}_{U\smallsetminus N}$ is a universal homeomorphism, the functor

$$(\mathrm{Frob}_{U\smallsetminus N})^*_{\overline{\mathbb{F}}_q}: D_{\mathrm{cons}}((U\smallsetminus N)_{\overline{\mathbb{F}}_q},\Lambda)\to D_{\mathrm{cons}}((U\smallsetminus N)_{\overline{\mathbb{F}}_q},\Lambda)$$

restricts to an equivalence $(\mathrm{Frob}_{U\smallsetminus N})^*_{\overline{\mathbb{F}}_q}: D_{\mathrm{lis}}((U\smallsetminus N)_{\overline{\mathbb{F}}_q},\Lambda)\xrightarrow{\sim} D_{\mathrm{lis}}((U\smallsetminus N)_{\overline{\mathbb{F}}_q},\Lambda)$.

Write $\mathcal{S}_{\dot V}$ for the object in $D_{\mathrm{cons}}(\mathrm{Sht}^I_{G,\dot V,N}\,|_{(U\smallsetminus N)^I_{\overline{\mathbb{F}}_q}},\Lambda)$ associated with $\dot V$ via geometric Satake. Unless otherwise specified, all Lurie tensor products are over $D(\Lambda)$.

Theorem (Eteve–Gaitsgory–Genestier–Lafforgue)**.** *The object $\pi_{N,!}\mathcal{S}_{\dot V}$ in*

$$D((U\smallsetminus N)^I_{\overline{\mathbb{F}}_q},\Lambda)$$

lies in the image of the fully faithful external tensor product functor

$$\Big[\operatorname{Ind} D_{\mathrm{lis}}((U\smallsetminus N)_{\overline{\mathbb{F}}_q},\Lambda)\Big]^{\otimes I}\hookrightarrow D((U\smallsetminus N)^I_{\overline{\mathbb{F}}_q},\Lambda)$$

and naturally lifts to an object of $\Big[\big(\operatorname{Ind} D_{\mathrm{lis}}((U\smallsetminus N)_{\overline{\mathbb{F}}_q},\Lambda)\big)^{(\mathrm{Frob}_{U\smallsetminus N})^*_{\overline{\mathbb{F}}_q}}\Big]^{\otimes I}$.

5.7. *Remark.* Work of Xue [72, Theorem 3.2.3][10] already implies the first statement in Theorem 5.6, as well as the second statement after taking cohomology groups [72, Corollary 3.2.6]. In particular, if one is only interested in Theorem 5.11 below after taking cohomology groups, one does not need to appeal to Theorem 5.6. For example, this is the case for Corollary 5.12 below.

[10]While [72, Theorem 3.2.3] assumes that G is split, its proof does not use this. See [72, §4].

5.8. For places in U, we have the following (derived) Hecke algebra action on the cohomology of stacks of global shtukas. Let Q be a finite set of closed points of U. Write

$$\pi_{\infty(Q\cup Z)} : \varprojlim_{m\geq 1} \mathrm{Sht}^I_{G,\dot V,m(Q\cup Z)}\,|_{(U\smallsetminus Q)^I_{\overline{\mathbb{F}}_q}} \to (U\smallsetminus Q)^I_{\overline{\mathbb{F}}_q}$$

for the structure morphism, and recall that $\varprojlim_{m\geq 1} \mathrm{Sht}^I_{G,\dot V,m(Q\cup Z)}\,|_{(U\smallsetminus Q)^I_{\overline{\mathbb{F}}_q}}$ has a natural $\underline{G(F_{Q\cup Z})}$-action over $(U\smallsetminus Q)^I_{\overline{\mathbb{F}}_q}$ [54, 3.9]. Because the square

$$\begin{array}{ccc}
\Big[\varprojlim_{m\geq 1} \mathrm{Sht}^I_{G,\dot V,mZ}\,|_{(U\smallsetminus Q)^I_{\overline{\mathbb{F}}_q}}\Big]\Big/\underline{G(F_Z)} & \xrightarrow{\pi_{\infty Z}} & */\underline{G(\mathcal{O}_Q)}\times */\underline{G(F_Z)}\times (U\smallsetminus Q)^I_{\overline{\mathbb{F}}_q} \\
\downarrow & & \downarrow \\
\Big[\varprojlim_{m\geq 1} \mathrm{Sht}^I_{G,\dot V,m(Q\cup Z)}\,|_{(U\smallsetminus Q)^I_{\overline{\mathbb{F}}_q}}\Big]\Big/\underline{G(F_{Q\cup Z})} & \xrightarrow{\pi_{\infty(Q\cup Z)}} & */\underline{G(F_Q)}\times */\underline{G(F_Z)}\times (U\smallsetminus Q)^I_{\overline{\mathbb{F}}_q}
\end{array}$$

is cartesian, applying Lemma 5.2 and taking the colimit over Q endow

$$\pi_{\infty Z,!}\mathcal{S}_{\dot V} = \varinjlim_{m\geq 1} \pi_{mZ,!}\mathcal{S}_{\dot V} \in D(*/\underline{G(F_Z)},\Lambda)\otimes\Big[\big(\operatorname{Ind} D_{\mathrm{lis}}(U_{\overline{\mathbb{F}}_q},\Lambda)\big)^{(\mathrm{Frob}_U)^*_{\overline{\mathbb{F}}_q}}\Big]^{\otimes I}$$

with the structure of a module for the $\mathbb{E}_1$-algebra $\bigotimes_u \mathrm{End}_{G(F_u)}(\text{c-Ind}_{G(\mathcal{O}_u)}^{G(F_u)}\Lambda)$.

5.9. Our formula for $\pi_{\infty Z,!}\mathcal{S}_{\dot V}$ will involve the following *Hecke operators* of Fargues–Scholze. For all closed points v of C, recall from 1.9 the relative Fargues–Fontaine curve X_{S,F_v}. Write $\mathrm{Div}^1_v(S)$ for the set of degree 1 effective Cartier divisors on X_{S,F_v} as in [26, Definition II.1.19], so that Div^1_v is a small v-sheaf [26, p. 52]. For all objects V_v in $\mathrm{Rep}_\Lambda({}^LG_v)^I$, recall that we have a natural functor [65, p. 25]

$$T_{V_v} : D(\mathrm{Bun}_{G,F_v},\Lambda)\to D(\mathrm{Bun}_{G,F_v},\Lambda)\otimes D(\mathrm{Div}^1_v,\Lambda)^{\otimes I}.$$

For all z in Z, let $\dot V_z$ be an object in $\mathrm{Rep}_\Lambda({}^LG)^{I_z}$, and take $\dot V$ to be the object $\boxtimes_{z\in Z}\dot V_z$ in $\mathrm{Rep}_\Lambda({}^LG)^I$. Write V_z for the restriction of $\dot V_z$ to $({}^LG_z)^{I_z}$, and write V for the restriction of $\dot V$ to $\prod_{z\in Z}({}^LG_z)^{I_z}$. Then $\bigotimes_{z\in Z} T_{V_z}$ yields a natural functor

$$T_V : \bigotimes_{z\in Z} D(\mathrm{Bun}_{G,F_z},\Lambda)\to\bigotimes_{z\in Z}\Big[D(\mathrm{Bun}_{G,F_z},\Lambda)\otimes D(\mathrm{Div}^1_z,\Lambda)^{\otimes I_z}\Big].$$

Recall from [65, Proposition 3.2] that $\bigotimes_{z\in Z} D(\mathrm{Bun}_{G,F_z},\Lambda) = D(\prod_{z\in Z}\mathrm{Bun}_{G,F_z},\Lambda)$.

5.10. We restrict ℓ-adic sheaves on $U_{\overline{\mathbb{F}}_q}$ to local Galois representations as follows. Write $\breve F_v$ for the completion of the maximal unramified extension of F_v. Fix an isomorphism $\overline{\mathbb{F}}_q^\times\cong\mathbb{Q}/\mathbb{Z}[\frac{1}{q}]$. For all z in Z, choose a uniformizer π of $\mathcal{O}_z$, and choose a compatible system $\{\pi^{1/n}\}_{p\nmid n}$ of n-th roots of π in $\overline{F}_z$. These choices induce

- group schemes ID_{F_z} and WD_{F_z} over Λ by B.10,
- a functor $\Psi_z : D(\mathrm{Spa}\,\mathbb{C}_z,\Lambda)\to D(*,\Lambda)$ by B.11,
- compatible identifications

$$D(\mathrm{Spd}\,\breve F_z,\Lambda)\cong D_{\mathrm{qcoh}}\big((\mathrm{Spec}\,\Lambda)/\mathrm{ID}_{F_z}\big)\ \text{ and }\ D(\mathrm{Div}^1_z,\Lambda)\cong D_{\mathrm{qcoh}}\big((\mathrm{Spec}\,\Lambda)/\mathrm{WD}_{F_z}\big)$$

such that the composition $D(\mathrm{Spd}\,\breve F_z,\Lambda)\longrightarrow D(\mathrm{Spa}\,\mathbb{C}_z,\Lambda)\xrightarrow{\Psi_z} D(*,\Lambda)$ corresponds to pullback along $\mathrm{Spec}\,\Lambda\to(\mathrm{Spec}\,\Lambda)/\mathrm{ID}_{F_z}$, by Lemma B.10.

Moreover, Grothendieck's ℓ-adic monodromy theorem implies that these choices also induce a functor $D_{\mathrm{cons}}(U_{\overline{\mathbb{F}}_q}, \Lambda) \to D_{\mathrm{perf}}((\operatorname{Spec}\Lambda)/\mathrm{ID}_{F_z})$, arising from pullback along $\operatorname{Spec}\breve{F}_z \to U_{\overline{\mathbb{F}}_q}$. Ind-extending and taking Frobenius-equivariant objects yields a functor $D(U_{\overline{\mathbb{F}}_q}, \Lambda)^{(\mathrm{Frob}_U)^*_{\overline{\mathbb{F}}_q}} \to D_{\mathrm{qcoh}}\big((\operatorname{Spec}\Lambda)/\mathrm{WD}_{F_z}\big)$.

5.11. Finally, we prove Theorem E. Recall from 2.4 the natural open embedding $i_1 : */\underline{G(F_z)} \hookrightarrow \mathrm{Bun}_{G,F_z}$, so that taking the product over all z in Z yields an open embedding $i_1 : */\underline{G(F_Z)} \hookrightarrow \prod_{z\in Z} \mathrm{Bun}_{G,F_z}$.

Theorem. *The image of $\pi_{\infty Z,!}\mathcal{S}_{\dot{V}}$ in*

$$D(*/\underline{G(F_Z)}, \Lambda) \otimes \bigotimes_{z\in Z} D_{\mathrm{qcoh}}\big((\operatorname{Spec}\Lambda)/\mathrm{WD}_{F_z}\big)^{\otimes I_z}$$

is naturally isomorphic to $i_1^*(T_V(\mathrm{loc}_{Z,!}\Lambda))$ as modules for $\bigotimes_u \operatorname{End}_{G(F_u)}(\text{c-Ind}_{G(\mathcal{O}_u)}^{G(F_u)}\Lambda)$.

Proof. After pulling back along $\operatorname{Spec}\Lambda \to (\operatorname{Spec}\Lambda)/\mathrm{WD}_{F_z}$, we claim that the images of $\pi_{\infty Z,!}\mathcal{S}_{\dot{V}}$ and $i_1^*(T_V(\mathrm{loc}_{Z,!}\Lambda))$ in $D(*/\underline{G(F_Z)}, \Lambda)$ are naturally isomorphic as modules for $\bigotimes_u \operatorname{End}_{G(F_u)}(\text{c-Ind}_{G(\mathcal{O}_u)}^{G(F_u)}\Lambda)$. To see this, first let us write

$$\mathrm{Sht} \coloneqq \Big[\varprojlim_{m\geq 1} \mathrm{Sht}^I_{G,\dot{V},mZ}\,|_{(U\smallsetminus Q)^I_{\overline{\mathbb{F}}_q}} \Big] \Big/ \underline{G(F_Z)}$$

for convenience.

Next, Theorem 5.6 enables us, after replacing V_z with its restriction to the diagonal, to assume that I_z is a singleton for all z in Z. Consider the diagram

$$\begin{array}{ccccc}
(\mathrm{Sht}\,|_{\prod_{z\in Z}\operatorname{Spa}\mathbb{C}_z})^{\Diamond} & \longrightarrow & \Big[\prod_{z\in Z}\mathrm{Gr}_{G,V_z}\,|_{\operatorname{Spa}\mathbb{C}_z}\Big]\Big/\underline{G(F_Z)} & \longrightarrow & */\underline{G(F_Z)} \times \prod_{z\in Z}\operatorname{Spa}\mathbb{C}_z \\
 & & \big\downarrow & & \big\downarrow{\scriptstyle i_1\times\mathrm{id}} \\
 & & \prod_{z\in Z}\mathrm{Hck}_{G,V_z}\,|_{\operatorname{Spa}\mathbb{C}_z} & \xrightarrow{h_2} & \prod_{z\in Z}\mathrm{Bun}_{G,F_z} \times \prod_{z\in Z}\operatorname{Spa}\mathbb{C}_z \\
 & & \big\downarrow{\scriptstyle h_1} & & \\
\big\downarrow & & & & \\
\mathrm{Bun}_{G,U} & \xrightarrow{\mathrm{loc}_Z} & \prod_{z\in Z}\mathrm{Bun}_{G,F_z}, & &
\end{array}$$

where $\mathrm{Hck}_{G,V_z}\,|_{\operatorname{Spa}\mathbb{C}_z}$ denotes the Hecke stack over $\operatorname{Spa}\mathbb{C}_z$ as in [26, p. 16], and h_1 and h_2 denote the morphisms as in [26, p. 317]. The left square is cartesian by 4.4 and Theorem 3.10, and the right square is cartesian by [26, p. 97]. Note that the composition in the top row equals the restriction of $\pi^{\Diamond}_{\infty Z}$ to $\prod_{z\in Z}\operatorname{Spa}\mathbb{C}_z$.

Proper base change and the above diagram imply that the image of $i_1^*(T_V(\mathrm{loc}_{Z,!}\Lambda))$ in $D(*/\underline{G(F_Z)}, \Lambda)$ has the following description. For all z in Z, write $\mathcal{S}_{V_z}$ for the object in $D_{\mathrm{mot}}(\mathrm{Hck}_{G,V_z}\,|_{\operatorname{Spa}\mathbb{C}_z})$ associated with V_z via motivic geometric Satake as in [65, Theorem 5.7], and write $\mathcal{S}_V$ for the pullback of $\boxtimes_{z\in Z}\mathcal{S}_{V_z}$ to $(\mathrm{Sht}\,|_{\prod_{z\in Z}\operatorname{Spa}\mathbb{C}_z})^{\Diamond}$. The discussion on [65, p. 25] shows that $\pi^{\Diamond}_{\infty Z,!}\mathcal{S}_V$ lies in the full subcategory

$$D_{\mathrm{mot}}(*/\underline{G(F_Z)}, \Lambda) \otimes_{D_{\mathrm{mot}}(*)} \bigotimes_{z\in Z} D_{\mathrm{mot}}(\operatorname{Spa}\mathbb{C}_z) \hookrightarrow D_{\mathrm{mot}}\big(*/\underline{G(F_Z)} \times \prod_{z\in Z}\operatorname{Spa}\mathbb{C}_z\big),$$

where $\bigotimes_{z\in Z}$ is over $D_{\mathrm{mot}}(*)$, and Proposition B.11 implies that applying

$$\bigotimes_{z\in Z}\Psi_z : D_{\mathrm{mot}}(*/\underline{G(F_Z)}) \otimes_{D_{\mathrm{mot}}(*)} \bigotimes_{z\in Z} D_{\mathrm{mot}}(\operatorname{Spa}\mathbb{C}_z) \to D_{\mathrm{mot}}(*/\underline{G(F_Z)},\Lambda)$$

yields an object whose image under

$$D_{\mathrm{mot}}(*/\underline{G(F_Z)}) \to D(*/\underline{G(F_Z)},\Lambda)$$

is precisely the image of $i_1^*(T_V(\mathrm{loc}_{Z,!}\Lambda))$ in $D(*/\underline{G(F_Z)},\Lambda)$.

We lift this to a Zariski-constructible version as follows. Note that $\mathcal{S}_V$ lies in

$$D_{\mathrm{mot}}^{U_{\mathbb{F}_q}^I\backslash\prod_{z\in Z}\operatorname{Spa}\mathbb{C}_z}\big((\mathrm{Sht}\,|_{\prod_{z\in Z}\operatorname{Spa}\mathbb{C}_z})^{\lozenge}\big) \subseteq D_{\mathrm{mot}}\big((\mathrm{Sht}\,|_{\prod_{z\in Z}\operatorname{Spa}\mathbb{C}_z})^{\lozenge}\big),$$

so 5.5 yields an object

$$(\triangle)\qquad \pi_{\infty Z,!}^{\lozenge}\mathcal{S}_V \in D_{\mathrm{mot}}(*/\underline{G(F_Z)}) \otimes_{D_{\mathrm{mot}}(*)} \operatorname{Ind} D_{\mathrm{mot}}^{U_{\mathbb{F}_q}^I\backslash\prod_{z\in Z}\operatorname{Spa}\mathbb{C}_z}\big(\prod_{z\in Z}\operatorname{Spa}\mathbb{C}_z\big)$$

whose image under Υ is identified with the previously mentioned

$$\pi_{\infty Z,!}^{\lozenge}\mathcal{S}_V \in D_{\mathrm{mot}}(*/\underline{G(F_Z)}) \otimes_{D_{\mathrm{mot}}(*)} D_{\mathrm{mot}}\big(\prod_{z\in Z}\operatorname{Spa}\mathbb{C}_z\big).$$

The Zariski-constructible version enables us to compare with ℓ-adic realizations. More precisely, the proof of [55, Lemma 6.4] identifies $r_{\ell,(\mathrm{Sht}\,|_{\prod_{z\in Z}\operatorname{Spa}\mathbb{C}_z})^{\lozenge}}(\mathcal{S}_V)$ with $\rho(\mathcal{S}_{\dot V})$, so 5.5 identifies the image of $(\triangle)$ under

$$\begin{aligned} r_{\ell,\prod_{z\in Z}\operatorname{Spa}\mathbb{C}_z} : D_{\mathrm{mot}}(*/\underline{G(F_Z)}) &\otimes_{D_{\mathrm{mot}}(*)} \operatorname{Ind} D_{\mathrm{mot}}^{U_{\mathbb{F}_q}^I\backslash\prod_{z\in Z}\operatorname{Spa}\mathbb{C}_z}\big(\prod_{z\in Z}\operatorname{Spa}\mathbb{C}_z\big) \\ &\to D(*/\underline{G(F_Z)},\Lambda)\otimes \operatorname{Ind} D_{\text{ét}}^{U_{\mathbb{F}_q}^I\backslash\prod_{z\in Z}\operatorname{Spa}\mathbb{C}_z}\big(\prod_{z\in Z}\operatorname{Spa}\mathbb{C}_z,\Lambda\big) \end{aligned}$$

with $\rho(\pi_{Z\infty,!}\mathcal{S}_{\dot V})$. Theorem 5.6 indicates that $\pi_{Z\infty,!}\mathcal{S}_{\dot V}$ lies in the full subcategory

$$D(*/\underline{G(F_Z)},\Lambda)\otimes D(U_{\overline{\mathbb{F}}_q},\Lambda)^{\otimes I} \hookrightarrow D(*/\underline{G(F_Z)},\Lambda)\otimes D(U_{\overline{\mathbb{F}}_q}^I,\Lambda).$$

Finally, consider the diagram

$$\begin{array}{ccc} \bigotimes_{z\in Z}\operatorname{Ind} D_{\mathrm{mot}}^{U_{\mathbb{F}_q}\backslash\operatorname{Spa}\mathbb{C}_z}(\operatorname{Spa}\mathbb{C}_z) & \xrightarrow{\otimes_{z\in Z}\Upsilon} & \bigotimes_{z\in Z} D_{\mathrm{mot}}(\operatorname{Spa}\mathbb{C}_z) \\ \downarrow{\scriptstyle \otimes_{z\in Z} r_{\ell,\operatorname{Spa}\mathbb{C}_z}} & & \\ \bigotimes_{z\in Z}\operatorname{Ind} D_{\text{ét}}^{U_{\mathbb{F}_q}\backslash\operatorname{Spa}\mathbb{C}_z}(\operatorname{Spa}\mathbb{C}_z,\Lambda) & \xleftarrow{\otimes_{z\in Z}\rho} & D(U_{\overline{\mathbb{F}}_q},\Lambda)^{\otimes I}, \end{array}$$

where the top arrow is an equivalence by Proposition B.9 and [64, Proposition 10.1], and the left arrow is identified with $\bigotimes_{z\in Z}\Psi_z$ by Proposition B.11. Then the claim follows from the observation that external tensor product induces a morphism from

this diagram to

$$\begin{array}{ccc} \operatorname{Ind} D_{\mathrm{mot}}^{U_{\overline{\mathbb{F}}_q}^I \setminus \prod_{z\in Z} \operatorname{Spa} \mathbb{C}_z}\big(\prod_{z\in Z} \operatorname{Spa} \mathbb{C}_z\big) & \xrightarrow{\Upsilon} & D_{\mathrm{mot}}\big(\prod_{z\in Z} \operatorname{Spa} \mathbb{C}_z\big) \\ \Big\downarrow{\scriptstyle r_{\ell,\prod_{z\in Z} \operatorname{Spa} \mathbb{C}_z}} & & \\ \operatorname{Ind} D_{\text{ét}}^{U_{\overline{\mathbb{F}}_q} \setminus \operatorname{Spa} \mathbb{C}_z}\big(\prod_{z\in Z} \operatorname{Spa} \mathbb{C}_z, \Lambda\big) & \xleftarrow{\rho} & D(U_{\overline{\mathbb{F}}_q}^I, \Lambda). \end{array}$$

By instead working over $\prod_{z\in Z} \operatorname{Spd} \breve{F}_z$ and keeping track of the partial Frobenii, we obtain the stated result. □

5.12. As an application of Theorem 5.11, we can give a quick proof that the excursion operators constructed by V. Lafforgue and Xue agree with those constructed by Fargues–Scholze. We proved this originally in [55, Theorem 6.13].

Recall from Example 2.3 the set $\ker^1(F, G)$. For any finite set I, object $\dot{V}$ in $\operatorname{Rep}_\Lambda({}^L G)^I$, morphism $x : \mathbf{1} \to \dot{V}|_{\Delta(\widehat{G})}$, morphism $\xi : \dot{V}|_{\Delta(\widehat{G})} \to \mathbf{1}$, and element $\gamma_\bullet$ in W_F, recall that work of V. Lafforgue and Xue defines a Λ-linear endomorphism $S_{I,\dot{V},x,\xi,\gamma_\bullet}$ of $C_c^\infty(\coprod_{\alpha\in\ker^1(F,G)} G_\alpha(F)\backslash G(\mathbb{A}), \Lambda)$ [72, Proposition 2.2.1].

Write $\mathfrak{z}(G(F_v), \Lambda)$ for the Bernstein center of $G(F_v)$ over Λ, and write V for the restriction of $\dot{V}$ to $({}^L G_v)^I$. When $\gamma_\bullet$ lies in W_{F_v}, recall that work of Fargues–Scholze defines an element $\mathfrak{z}_{I,V,x,\xi,\gamma_\bullet}$ of $\mathfrak{z}(G(F_v), \Lambda)$ [26, Theorem VIII.4.1].

Corollary. *The element $\mathfrak{z}_{I,V,x,\xi,\gamma_\bullet}$ acts on $C_c^\infty(\coprod_{\alpha\in\ker^1(F,G)} G_\alpha(F)\backslash G(\mathbb{A}), \Lambda)$ via $S_{I,\dot{V},x,\xi,\gamma_\bullet}$.*

Proof. After shrinking U, we can assume that v lies in Z. Take I_z to be empty for all z in $Z \smallsetminus v$, so that $I_v = I$. The counit $i_{1,!} i_1^* \to \mathrm{id}$ induces a commutative diagram

$$\begin{array}{ccccccc} i_1^* T_{\mathbf{1}} i_{1,!} i_1^* \mathrm{loc}_{Z,!}\Lambda & \xrightarrow{x} & i_1^* T_V i_{1,!} i_1^* \mathrm{loc}_{Z,!}\Lambda & \xrightarrow{\gamma_\bullet} & i_1^* T_V i_{1,!} i_1^* \mathrm{loc}_{Z,!}\Lambda & \xrightarrow{\xi} & i_1^* T_{\mathbf{1}} i_{1,!} i_1^* \mathrm{loc}_{Z,!}\Lambda \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ i_1^* T_{\mathbf{1}} \mathrm{loc}_{Z,!}\Lambda & \xrightarrow{x} & i_1^* T_V \mathrm{loc}_{Z,!}\Lambda & \xrightarrow{\gamma_\bullet} & i_1^* T_V \mathrm{loc}_{Z,!}\Lambda & \xrightarrow{\xi} & i_1^* T_{\mathbf{1}} \mathrm{loc}_{Z,!}\Lambda. \end{array}$$

Because $T_{\mathbf{1}}$ is naturally isomorphic to the identity, the left and right arrows are isomorphisms, and the corners are identified with $i_1^* \mathrm{loc}_{Z,!}\Lambda$. By using Lemma 2.6 and arguing as in Example 2.3, we see that $H^0(i_1^* \mathrm{loc}_{Z,!}\Lambda)$ is naturally isomorphic to $C_c^\infty(\coprod_{\alpha\in\ker^1(F,G)} G_\alpha(F)\backslash G(\mathbb{A})/G(\mathbb{O}_U), \Lambda)$. After applying $H^0(-)$ to the above diagram, the top arrow equals the action of $\mathfrak{z}_{I,V,x,\xi,\gamma_\bullet}$ by construction, and the bottom arrow equals $S_{I,\dot{V},x,\xi,\gamma_\bullet}$ by Theorem 5.11. This indicates that the restrictions of $\mathfrak{z}_{I,V,x,\xi,\gamma_\bullet}$ and $S_{I,\dot{V},x,\xi,\gamma_\bullet}$ to $C_c^\infty(\coprod_{\alpha\in\ker^1(F,G)} G_\alpha(F)\backslash G(\mathbb{A})/G(\mathbb{O}_U), \Lambda)$ coincide. Finally, taking $\bigcup_Z$ yields the desired result. □

5.13. *Remarks.*

1) The proof of Corollary 5.12 only relies on Theorem 5.11 after taking cohomology groups. In particular, one does not need to appeal to Theorem 5.6; see Remark 5.7.
2) The main ingredient for the representation-theoretic results in [55] is [55, Theorem 6.13]. Therefore one can use Corollary 5.12 to prove all the representation-theoretic results in [55]; see the proof of [55, Theorem 6.16].

6. Relation with Langlands duality

In this section, we discuss the role that $\mathrm{Bun}_{G,F}$ plays in the Langlands correspondence. First, we recall the moduli of Galois representations and their derived categories of ind-coherent sheaves, which are the main players on the spectral side. By restricting Galois representations to decomposition groups, we obtain a map res_Z analogous to loc_Z; we study the pushforward of the coherent dualizing complex ω along res_Z.

Next, we recall the spectral action of Fargues–Scholze, which enables us to state their categorical Langlands conjecture. We can then state Conjecture F, and we explain how Conjecture F implies conjectures of Arinkin–Gaitsgory–Kazhdan–Raskin–Rozenblyum–Varshavsky and Zhu. Finally, we prove Conjecture F when G is commutative; this is Theorem G.

6.1. We start by recalling the stacks of Galois representations that we will use. **For the rest of this paper, assume that $\ell \neq 2$.**[11] Write $\mathrm{LS}_{{}^L G,U}$ for the derived algebraic stack over Λ of continuous ${}^L G$-valued representations of W_U

- as in [74, (3.24)] when Z is nonempty, or
- as in [74, Remark 3.36] when Z is empty,

using our choice of $\sqrt{q}$ to replace ${}^c G$ with ${}^L G$ [74, Remark 3.1]. When Z is nonempty, write $\mathrm{LS}^{\square}_{{}^L G,U}$ for its framed version as in [74, (3.24)], which the proof of [74, Theorem 3.29] shows is a disjoint union of derived affine schemes almost of finite type over Λ.

Similarly, for all closed points v of C, write $\mathrm{LS}_{{}^L G,F_v}$ for the algebraic stack over Λ of continuous ${}^L G_v$-valued representations of W_{F_v} as in [74, (3.3)], and write $\mathrm{LS}^{\square}_{{}^L G,F_v}$ for its framed version as in [74, (3.3)]. Then [65, Corollary 4.5] implies that $\mathrm{LS}_{{}^L G,F_v}$ is the base change to Λ of the algebraic stack from [65, Definition 4.3], and [26, Theorem VIII.1.3] shows that $\mathrm{LS}^{\square}_{{}^L G,F_v}$ is a disjoint union of affine schemes of finite type over Λ.

For all closed points u of U, write $\mathrm{LS}_{{}^L G,\mathcal{O}_u} \subseteq \mathrm{LS}_{{}^L G,F_u}$ for the closed substack of unramified ${}^L G_v$-valued representations of W_{F_v} as in [74, p. 82]. Write $\mathrm{LS}^{\square}_{{}^L G,\mathcal{O}_u}$ for its framed version, which is naturally isomorphic to $\widehat{G}$.

6.2. We will consider ind-coherent sheaves on these stacks. Namely, recall the Λ-linear ind-coherent 6-functor formalism $\mathrm{IndCoh}(-)$ from [63, Lecture VIII.7] on

$$\{\text{derived qcqs schemes almost of finite type over } \Lambda\},$$

where we use the notation of Gaitsgory–Rozenblyum [31, p. 273] for the associated functors instead of the notation of [57, Definition A.5.6]. Then [42, Theorem 3.4.11] extends this to a 6-functor formalism on

$$\{\text{derived algebraic stacks over } \Lambda\}.$$

Let D be one of $\{U \neq C, F_v\}$. Then 6.1 implies that the value of IndCoh on $\mathrm{LS}_{{}^L G,D}$ is naturally equivalent to $\mathrm{Ind}\, D^{\mathrm{qc}}_{\mathrm{coh}}(\mathrm{LS}_{{}^L G,D})$.

[11] This arises from the fact that de Jong's conjecture [18, Conjecture 1.1], which is used to prove basic properties of $\mathrm{LS}_{{}^L G,U}$ [74, Theorem 3.29], is only known when $\ell \neq 2$ [28].

6.3. When restricted to bounded below objects, there is no difference between ind-perfect and ind-coherent objects on $\mathrm{LS}_{{}^L G,D}$. More precisely, applying Ind to the inclusion $D^{\mathrm{qc}}_{\mathrm{perf}}(\mathrm{LS}_{{}^L G,D}) \subseteq D^{\mathrm{qc}}_{\mathrm{coh}}(\mathrm{LS}_{{}^L G,D})$ yields a fully faithful functor

$$\Xi : \mathrm{Ind}\, D^{\mathrm{qc}}_{\mathrm{perf}}(\mathrm{LS}_{{}^L G,D}) \hookrightarrow \mathrm{Ind}\, D^{\mathrm{qc}}_{\mathrm{coh}}(\mathrm{LS}_{{}^L G,D}).$$

Write $\Psi : \mathrm{Ind}\, D^{\mathrm{qc}}_{\mathrm{coh}}(\mathrm{LS}_{{}^L G,D}) \to \mathrm{Ind}\, D^{\mathrm{qc}}_{\mathrm{perf}}(\mathrm{LS}_{{}^L G,D})$ for its right adjoint. Then arguing as in the proof of [29, Proposition 1.2.4] shows that Ψ restricts to an equivalence

$$\bigcup_{n\in\mathbb{Z}} \big[\mathrm{Ind}\, D^{\mathrm{qc}}_{\mathrm{coh}}(\mathrm{LS}_{{}^L G,D})\big]^{\geq n} \xrightarrow{\sim} \bigcup_{n\in\mathbb{Z}} \big[\mathrm{Ind}\, D^{\mathrm{qc}}_{\mathrm{perf}}(\mathrm{LS}_{{}^L G,D})\big]^{\geq n},$$

which we use to identify $D^{\mathrm{qc}}_{\mathrm{coh}}(\mathrm{LS}_{{}^L G,D})$ with its image under Ψ.

6.4. Restriction induces the following morphisms between our stacks of Galois representations. For all closed points u of U and z in Z, write

$$\mathrm{res}_u : \mathrm{LS}_{{}^L G,U} \to \mathrm{LS}_{{}^L G,\mathcal{O}_u} \text{ and } \mathrm{res}_z : \mathrm{LS}_{{}^L G,U} \to \mathrm{LS}_{{}^L G,F_z}$$

for the resulting morphisms over Λ. Write $\mathrm{res}_Z : \mathrm{LS}_{{}^L G,U} \to \prod_{z\in Z} \mathrm{LS}_{{}^L G,F_z}$ for the morphism $(\mathrm{res}_z)_{z\in Z}$, where all products are over Λ. By 6.1, the value of IndCoh on $\prod_{z\in Z} \mathrm{LS}_{{}^L G,F_z}$ is naturally equivalent to

$$\bigotimes_{z\in Z} \mathrm{Ind}\, D^{\mathrm{qc}}_{\mathrm{coh}}(\mathrm{LS}_{{}^L G,F_z}).$$

6.5. For places in U, we have the following (derived) *spectral* Hecke algebra action on $\mathrm{res}_{Z,*}(\omega_{\mathrm{LS}_{{}^L G,U}})$. Let Q be a finite nonempty set of closed points of U, and write

$$\mathrm{res}_Q : \mathrm{LS}_{{}^L G,U} \to \textstyle\prod_{u\in Q} \mathrm{LS}_{{}^L G,\mathcal{O}_u}$$

for the morphism $(\mathrm{res}_u)_{u\in U}$. Then [74, Lemma 3.34] implies that the square

$$\begin{array}{ccc} \mathrm{LS}_{{}^L G,U} & \xrightarrow{(\mathrm{res}_Q,\mathrm{res}_Z)} & \big(\prod_{u\in Q} \mathrm{LS}_{{}^L G,\mathcal{O}_u}\big) \times \big(\prod_{z\in Z} \mathrm{LS}_{{}^L G,F_z}\big) \\ \downarrow & & \downarrow \\ \mathrm{LS}_{{}^L G,U\smallsetminus Q} & \longrightarrow & \big(\prod_{u\in Q} \mathrm{LS}_{{}^L G,F_u}\big) \times \big(\prod_{z\in Z} \mathrm{LS}_{{}^L G,F_z}\big) \end{array}$$

is derived cartesian. Moreover, 6.1 implies that bottom arrow is $*$-able, so the same holds for the top arrow. Therefore Lemma 5.2 endows $\mathrm{res}_{Z,*}(\omega_{\mathrm{LS}_{{}^L G,U}})$ in $\bigotimes_{z\in Z} \mathrm{Ind}\, D^{\mathrm{qc}}_{\mathrm{coh}}(\mathrm{LS}_{{}^L G,F_z})$ with the structure of a module for the $\mathbb{E}_1$-algebra

$$\mathrm{End}_{D_{\mathrm{coh}}(\prod_{u\in Q} \mathrm{LS}_{{}^L G,F_u})}(\mathscr{O}_{\prod_{u\in Q} \mathrm{LS}_{{}^L G,\mathcal{O}_u}}) = \bigotimes_{u\in Q} \mathrm{End}_{D_{\mathrm{coh}}(\mathrm{LS}_{{}^L G,F_u})}(\mathscr{O}_{\mathrm{LS}_{{}^L G,\mathcal{O}_u}})$$

over Λ. Taking the colimit over Q shows that $\mathrm{res}_{Z,*}(\omega_{\mathrm{LS}_{{}^L G,U}})$ is naturally a module for the $\mathbb{E}_1$-algebra $\bigotimes_u \mathrm{End}_{D_{\mathrm{coh}}(\mathrm{LS}_{{}^L G,F_u})}(\mathrm{LS}_{{}^L G,\mathcal{O}_u})$, where u runs over closed points of U.

6.6. Write $D^{\mathrm{qc}}_{\mathrm{coh}}(\mathrm{LS}_{{}^L G,F_v})_{\mathrm{Nilp}} \subseteq D^{\mathrm{qc}}_{\mathrm{coh}}(\mathrm{LS}_{{}^L G,F_v})$ for the full subcategory of objects with nilpotent singular support as in [26, VIII.2.2.2]. It is $\operatorname{Ind} D^{\mathrm{qc}}_{\mathrm{coh}}(\mathrm{LS}_{{}^L G,F_v})_{\mathrm{Nilp}}$ that appears in the categorical local Langlands conjecture, so we need to check that $\mathrm{res}_{Z,*}(\mathrm{LS}_{{}^L G,U})$ lies in this subcategory.

Proposition. *The object* $\mathrm{res}_{Z,*}(\omega_{\mathrm{LS}_{{}^L G,U}})$ *in* $\bigotimes_{z\in Z} \operatorname{Ind} D^{\mathrm{qc}}_{\mathrm{coh}}(\mathrm{LS}_{{}^L G,F_z})$ *lies in*

$$\bigotimes_{z\in Z} \operatorname{Ind} D^{\mathrm{qc}}_{\mathrm{coh}}(\mathrm{LS}_{{}^L G,F_z})_{\mathrm{Nilp}}.$$

Proof. When Z is empty, this condition is also empty. When Z is nonempty, [74, Theorem 3.29] implies that $\mathrm{LS}^{\square}_{{}^L G,U}$ is quasismooth over Λ, so [7, Corollary 2.2.8][12] shows that its dualizing complex $\omega_{\mathrm{LS}^{\square}_{{}^L G,U}}$ is a shift of a line bundle. By smooth descent, the same holds for $\omega_{\mathrm{LS}_{{}^L G,U}}$, so the result follows from 6.3. □

6.7. The local automorphic and spectral sides are related by the following *spectral action* of Fargues–Scholze. Write Z_G for the center of G over F, and **for the rest of this paper, assume that $\pi_0((Z_G)_{\overline{F}})$ is invertible in Λ.** Then [65, Theorem 6.1] and 6.1 show that there is a natural Λ-linear action of $D_{\mathrm{perf}}(\mathrm{LS}_{{}^L G,F_v})$ on $D(\mathrm{Bun}_{G,F_v},\Lambda)^{\omega}$. Write V_v for the vector bundle on $\mathrm{LS}_{{}^L G,F_v}$ associated with the object V_v in $\mathrm{Rep}_{\Lambda}({}^L G)^I$, so that by construction V_v in $D_{\mathrm{perf}}(\mathrm{LS}_{{}^L G,F_v})$ acts on $D(\mathrm{Bun}_{G,F_v},\Lambda)^{\omega}$ via T_{V_v}.

By restricting to $D^{\mathrm{qc}}_{\mathrm{perf}}(\mathrm{LS}_{{}^L G,F_v})$ and applying Ind, the above naturally extends to a Λ-linear colimit-preserving action of $\operatorname{Ind} D^{\mathrm{qc}}_{\mathrm{perf}}(\mathrm{LS}_{{}^L G,F_v})$ on $D(\mathrm{Bun}_{G,F_v},\Lambda)$.

6.8. We now use the spectral action to state the categorical local Langlands conjecture of Fargues–Scholze. **For the rest of this paper, assume that G is quasisplit over F.** Then we can choose B to be a Borel subgroup of G over F and T to be a maximal subtorus of B over F. Write N for the unipotent radical of B, and let $\psi : N(\mathbb{A}) \to \Lambda^{\times}$ be a continuous homomorphism trivial on $N(F)$ such that, for all closed points v of C, its restriction ψ_v to $N(F_v)$ is generic.

Consider the colimit-preserving functor $a_{\psi_v} : \operatorname{Ind} D^{\mathrm{qc}}_{\mathrm{perf}}(\mathrm{LS}_{{}^L G,F_v}) \to D(\mathrm{Bun}_{G,F_v},\Lambda)$ given by acting as in 6.7 on $i_{1,!}\text{c-Ind}^{G(F_v)}_{N(F_v)}\psi_v$. Denote its right adjoint by

$$c_{\psi_v} : D(\mathrm{Bun}_{G,F_v},\Lambda) \to \operatorname{Ind} D^{\mathrm{qc}}_{\mathrm{perf}}(\mathrm{LS}_{{}^L G,F_v}).$$

Conjecture ([26, Conjecture X.3.5])**.** *The functor c_{ψ_v} restricts to an equivalence*

$$D(\mathrm{Bun}_{G,F_v},\Lambda)^{\omega} \xrightarrow{\sim} D^{\mathrm{qc}}_{\mathrm{coh}}(\mathrm{LS}_{{}^L G,F_v})_{\mathrm{Nilp}}.$$

Consequently, applying Ind *yields an equivalence*

$$\mathbb{L}_{\psi_v} : D(\mathrm{Bun}_{G,F_v},\Lambda) \xrightarrow{\sim} \operatorname{Ind} D^{\mathrm{qc}}_{\mathrm{coh}}(\mathrm{LS}_{{}^L G,F_v})_{\mathrm{Nilp}}.$$

Conjecture 6.8 is known when G is a torus [76, Theorem 6.4.1].

[12] While [7] works over an algebraically closed field of characteristic zero, the proof of [7, Corollary 2.2.8] does not use this assumption.

6.9. To compare the actions of the (derived) automorphic and spectral Hecke algebras for places in U, we consider the following conjecture of Zhu.

Conjecture ([74, Conjecture 4.11])**.** *Let u be a closed point of U.*

1) Assume Conjecture 6.8 for u. Then $\mathbb{L}_{\psi_u}(i_{1,!}\text{c-Ind}_{G(\mathcal{O}_u)}^{G(F_u)}\Lambda) \cong \mathscr{O}_{\mathrm{LS}_{L_{G,\mathcal{O}_u}}}$.
2) There is a natural isomorphism of $\mathbb{E}_1$-algebras over Λ

$$\mathrm{End}_{G(F_u)}(\text{c-Ind}_{G(\mathcal{O}_u)}^{G(F_u)}\Lambda) \cong \mathrm{End}_{D_{\mathrm{coh}}(\mathrm{LS}_{L_{G,F_u}})}(\mathscr{O}_{\mathrm{LS}_{L_{G,\mathcal{O}_u}}}).$$

Note that Conjecture 6.9.1) implies Conjecture 6.9.2). The proof of [76, Theorem 6.4.1] shows that Conjecture 6.9.1) is known when G is a torus; see [74, Proposition 4.13] for an explicit description of the resulting isomorphism in Conjecture 6.9.2).

Conjecture 6.9.2) is known when Λ equals L [75, Theorem 5.3 (2)].

6.10. Finally, we arrive at Conjecture F.

Conjecture. *Assume Conjecture 6.8 for all z in Z, which yields an equivalence*

$$\bigotimes_{z\in Z}\mathbb{L}_{\psi_z} : \bigotimes_{z\in Z} D(\mathrm{Bun}_{G,F_z},\Lambda) \xrightarrow{\sim} \bigotimes_{z\in Z}\mathrm{Ind}\, D_{\mathrm{coh}}^{\mathrm{qc}}(\mathrm{LS}_{L_{G,F_z}})_{\mathrm{Nilp}}.$$

Assume Conjecture 6.9.2) for all closed points u in U, which yields an isomorphism

$$\bigotimes_u \mathrm{End}_{G(F_u)}(\text{c-Ind}_{G(\mathcal{O}_u)}^{G(F_u)}\Lambda) \cong \bigotimes_u \mathrm{End}_{D_{\mathrm{coh}}(\mathrm{LS}_{L_{G,F_u}})}(\mathscr{O}_{\mathrm{LS}_{L_{G,\mathcal{O}_u}}}).$$

Then the $\bigotimes_u \mathrm{End}_{G(F_u)}(\text{c-Ind}_{G(\mathcal{O}_u)}^{G(F_u)}\Lambda)$-module

$$\mathrm{loc}_{Z,!}\Lambda \in \bigotimes_{z\in Z} D(\mathrm{Bun}_{G,F_z},\Lambda)$$

corresponds under $\bigotimes_{z\in Z}\mathbb{L}_{\psi_z}$ to the $\bigotimes_u \mathrm{End}_{D_{\mathrm{coh}}(\mathrm{LS}_{L_{G,F_u}})}(\mathscr{O}_{\mathrm{LS}_{L_{G,\mathcal{O}_u}}})$-module

$$\mathrm{res}_{Z,*}(\omega_{\mathrm{LS}_{L_{G,U}}}) \in \bigotimes_{z\in Z}\mathrm{Ind}\, D_{\mathrm{coh}}^{\mathrm{qc}}(\mathrm{LS}_{L_{G,F_z}})_{\mathrm{Nilp}}.$$

6.11. *Example.* Assume that Z is empty. Then Example 2.3 identifies $\mathrm{Bun}_{G,C}$ with the constant stack over $\overline{\mathbb{F}}_q$ associated with the groupoid ($\heartsuit$), so

$$\mathrm{loc}_{\varnothing,!}\Lambda = \bigoplus_{\alpha\in\ker^1(F,G)} \Gamma_c(\underline{G_\alpha(F)\backslash G(\mathbb{A})/G(\mathbb{O})},\Lambda),$$

where we remind the reader that $\Gamma_c(\underline{G_\alpha(F)\backslash G(\mathbb{A})/G(\mathbb{O})},\Lambda)$ can have cohomology in nonzero degrees when $\Lambda \neq L$ [72, Remark 0.0.3]. On the other hand, by definition

$$\mathrm{res}_{\varnothing,*}(\omega_{\mathrm{LS}_{L_{G,C}}}) = \Gamma(\mathrm{LS}_{L_{G,C}},\omega_{\mathrm{LS}_{L_{G,C}}}).$$

Altogether, we see that Conjecture 6.10 specializes to [8, Conjecture 24.8.6] when G is split, $\Lambda = L$, and Z is empty.

6.12. More generally, Conjecture 6.10 implies the following formula for the cohomology of stacks of global shtukas for *extended pure inner twists* of G, which is a conjecture of Zhu [74, Conjecture 4.49]. Let b be an element in $G(U_{\overline{\mathbb{F}}_q})$ whose image in $B(F,G)$ is basic, and recall from 2.9 the parahoric group scheme G_b over U. Using fpqc descent, extend G_b to a parahoric group scheme over C. Since G_b is an inner twist of G over F, we have compatible identifications ${}^LG = {}^LG_b$ and ${}^LG_v = {}^LG_{b,v}$. Write

$$\pi^{G_b}_{\infty Z} : \varprojlim_{m\geq 1} \mathrm{Sht}^I_{G_b,\dot V,mZ}\,|_{(U\smallsetminus Q)^I_{\overline{\mathbb{F}}_q}} \to (U\smallsetminus Q)^I_{\overline{\mathbb{F}}_q}$$

for the structure morphism.

Recall the open embedding $i_b : *\big/\underline{G_b(F_z)} \hookrightarrow \mathrm{Bun}_{G,F_z}$ [26, Theorem III.4.5], so that taking the product over all z in Z yields an open embedding

$$i_b : *\big/\underline{G_b(F_Z)} \hookrightarrow \textstyle\prod_{z\in Z} \mathrm{Bun}_{G,F_z}\,.$$

For all z in Z, let K_z be a compact open subgroup of $G_b(F_z)$. Write K for $\prod_{z\in Z} K_z$, and consider the structure morphism

$$\pi^{G_b}_K : \Big[\varprojlim_{m\geq 1} \mathrm{Sht}^I_{G_b,\dot V,mZ}\,|_{(U\smallsetminus Q)^I_{\overline{\mathbb{F}}_q}}\Big]\Big/\underline{K} \to (U\smallsetminus Q)^I_{\overline{\mathbb{F}}_q}\,.$$

Write $\dot V$ for the vector bundle on $\mathrm{LS}_{{}^LG,U}$ associated with the object $\dot V$ in $\mathrm{Rep}_\Lambda({}^LG)^I$, and write $\mathbb{D}(-)$ for the Grothendieck–Serre dual.

Proposition. *Assume Conjecture 6.10. Then the image of* $\pi^{G_b}_{K,!}\mathcal{S}_{\dot V}$ *in*

$$\bigotimes_{z\in Z} D_{\mathrm{qcoh}}\big((\mathrm{Spec}\,\Lambda)/\mathrm{WD}_{F_z}\big)^{\otimes I_z}$$

is isomorphic as modules for $\bigotimes_u \mathrm{End}_{G(F_u)}(\text{c-Ind}^{G(F_u)}_{G(\mathcal{O}_u)}\Lambda)$ *to*

$$\Gamma\big(\mathrm{LS}_{{}^LG,U}, \dot V \otimes \mathrm{res}^!_Z(\boxtimes_{z\in Z}\mathbb{D}(\mathbb{L}_{\psi_z}(i_{b,!}\,\text{c-Ind}^{G_b(F_z)}_{K_z})))\big).$$

Proof. Because b is basic, Proposition 2.9 and Theorem 5.11 identify

$$\begin{aligned}\pi^{G_b}_{K,!}\mathcal{S}_{\dot V} = (\pi^{G_b}_{\infty Z,!}\mathcal{S}_{\dot V})^K &= \mathrm{Hom}_{G_b(F_Z)}\big(\text{c-Ind}^{G_b(F_Z)}_K \Lambda, \pi^{G_b}_{\infty Z,!}\mathcal{S}_{\dot V}\big)\\ &= \mathrm{Hom}_{G_b(F_Z)}\big(\text{c-Ind}^{G_b(F_Z)}_K \Lambda, i_b^*(T_V(\mathrm{loc}_{Z,!}\Lambda))\big)\\ &= \mathrm{Hom}\big(i_{b,!}\,\text{c-Ind}^{G_b(F_Z)}_K \Lambda, T_V(\mathrm{loc}_{Z,!}\Lambda)\big).\end{aligned}$$

By Conjecture 6.10 and the projection formula, this is isomorphic to

$$\begin{aligned}&\mathrm{Hom}\big(\boxtimes_{z\in Z}\mathbb{L}_{\psi_z}(i_{b,!}\,\text{c-Ind}^{G_b(F_z)}_{K_z}\Lambda), V\otimes \mathrm{res}_{Z,*}(\omega_{\mathrm{LS}_{{}^LG,U}})\big)\\ =\ &\mathrm{Hom}\big(\boxtimes_{z\in Z}\mathbb{L}_{\psi_z}(i_{b,!}\,\text{c-Ind}^{G_b(F_z)}_{K_z}\Lambda), \mathrm{res}_{Z,*}(\dot V\otimes\omega_{\mathrm{LS}_{{}^LG,U}})\big).\end{aligned}$$

Finally, applying Grothendieck–Serre duality yields

$$\begin{aligned}&\mathrm{Hom}\big(\mathrm{res}_{Z,!}(\mathbb{D}(\dot V\otimes\omega_{\mathrm{LS}_{{}^LG,U}})), \boxtimes_{z\in Z}\mathbb{D}(\mathbb{L}_{\psi_z}(i_{b,!}\,\text{c-Ind}^{G_b(F_z)}_{K_z}))\big)\\ =\ &\mathrm{Hom}\big(\mathrm{res}_{Z,!}(\dot V^\vee), \boxtimes_{z\in Z}\mathbb{D}(\mathbb{L}_{\psi_z}(i_{b,!}\,\text{c-Ind}^{G_b(F_z)}_{K_z}))\big)\\ =\ &\Gamma\big(\mathrm{LS}_{{}^LG,U}, \dot V\otimes \mathrm{res}^!_Z(\boxtimes_{z\in Z}\mathbb{D}(\mathbb{L}_{\psi_z}(i_{b,!}\,\text{c-Ind}^{G_b(F_z)}_{K_z})))\big).\end{aligned}$$

□

6.13. *Remark.* When $b=1$, it is expected that $\mathbb{L}_{\psi_z}(i_{1,!}\,\text{c-Ind}^{G(F_z)}_{K_z}\Lambda)$ is isomorphic to its Grothendieck–Serre dual [74, Remark 4.27].

6.14. In this rest of this section, our goal is to prove Conjecture 6.10 when $G = T$ is a torus. We start by recalling work of Langlands [53] on his conjectures for tori. Write $W_{\widetilde{F}/F} \coloneqq W_F/\overline{[W_{\widetilde{F}}, W_{\widetilde{F}}]}$ for the relative Weil group of $\widetilde{F}/F$, so that global class field theory gives a natural short exact sequence of locally profinite groups

$$1 \longrightarrow \widetilde{F}^\times\backslash\mathbb{A}_{\widetilde{F}}^\times \longrightarrow W_{\widetilde{F}/F} \longrightarrow \mathrm{Gal}(\widetilde{F}/F) \longrightarrow 1.$$

Write $\mathbb{T}$ for the locally profinite group $\big[T(\widetilde{F})\backslash T(\mathbb{A}_{\widetilde{F}})\big]^{\mathrm{Gal}(\widetilde{F}/F)}$. Then the long exact sequence for group cohomology yields a natural short exact sequence

$$1 \longrightarrow T(F)\backslash T(\mathbb{A}) \longrightarrow \mathbb{T} \longrightarrow \ker^1(F, T) \longrightarrow 1$$

of locally profinite groups. Recall that the transfer map induces an isomorphism

$$H_1(W_{\widetilde{F}/F}, X_*(T)) \xrightarrow{\sim} H_1(\widetilde{F}^\times\backslash\mathbb{A}_{\widetilde{F}}^\times, X_*(T))^{\mathrm{Gal}(\widetilde{F}/F)} = \mathbb{T}$$

[53, p. 233], and for all $\mathrm{Gal}(\widetilde{F}/F)$-stable open subgroups P of $\widetilde{F}^\times\backslash\mathbb{A}_{\widetilde{F}}^\times$, write

$$\Theta(P) \coloneqq \ker\big[H_1(W_{\widetilde{F}/F}, X_*(T)) \to H_1(W_{\widetilde{F}/F}/P, X_*(T))\big].$$

Lemma. *Under the identification $H_1(W_{\widetilde{F}/F}, X_*(T)) \xrightarrow{\sim} \mathbb{T}$, the subgroup $\Theta(P)$ corresponds to the image of $P \otimes_{\mathbb{Z}} X_*(T) \subseteq T(\widetilde{F})\backslash T(\mathbb{A}_{\widetilde{F}})$ under the norm map.*

Proof. The five term exact sequence for group homology identifies $\Theta(P)$ with the image of $H_1(P, X_*(T)) = P \otimes_{\mathbb{Z}} X_*(T)$ under the corestriction map. Postcomposing the corestriction map with the transfer map yields the norm map, as desired. □

6.15. Assume that Z is nonempty. We will stratify $\mathrm{LS}_{{}^L T,U}$ by bounding ramification as follows. Let K be a $\mathrm{Gal}(\widetilde{F}/F)$-stable compact open subgroup of $\widetilde{F}_{\widetilde{Z}}^\times$, and write $\mathrm{LS}^\square_{{}^L T,K}$ for the presheaf on {affine schemes over Λ} whose $\mathrm{Spec}\, A$-points consist of 1-cocycles $W_{\widetilde{F}/F}/\mathbb{O}_{\widetilde{U}}^\times K \to \widehat{T}(A)$. Since $W_{\widetilde{F}/F}/\mathbb{O}_{\widetilde{U}}^\times K$ is finitely generated, $\mathrm{LS}^\square_{{}^L T,K}$ is an affine scheme of finite type over Λ. Write $\mathrm{LS}_{{}^L T,K}$ for the algebraic stack $\mathrm{LS}^\square_{{}^L T,K}/\widehat{T}$ over Λ.

For any finitely generated abelian group M, write $\underline{\mathrm{Hom}}(M, \mathbb{G}_m)$ for its Cartier dual over Λ. Note that $\underline{\mathrm{Hom}}(M, \mathbb{G}_m)$ is a flat affine scheme of finite type over Λ.

Proposition. *The algebraic stack $\mathrm{LS}_{{}^L T,K}$ is naturally a gerbe over*

$$\underline{\mathrm{Hom}}\big(\mathbb{T}/\mathbb{O}_U^\times \mathrm{Nm}_{\widetilde{F}/F}(K \otimes_{\mathbb{Z}} X_*(T)), \mathbb{G}_m\big)$$

banded by $\widehat{T}^{\mathrm{Gal}(\widetilde{F}/F)}$.

Proof. Write H^{pre} for the presheaf on {affine schemes over Λ} whose $\mathrm{Spec}\, A$-points consist of the set-theoretic quotient

$$\mathrm{LS}^\square_{{}^L T,K}(A)/\widehat{T}(A) = H^1\big(W_{\widetilde{F}/F}/\mathbb{O}_{\widetilde{U}}^\times K, \widehat{T}(A)\big),$$

and write H for its fpqc-sheafification. Using the presentation $\mathrm{LS}_{{}^L T,K} = \mathrm{LS}^\square_{{}^L T,K}/\widehat{T}$, we see the inertia stack of $\mathrm{LS}_{{}^L T,K}$ is naturally isomorphic to $\widehat{T}^{\mathrm{Gal}(\widetilde{F}/F)} \times \mathrm{LS}_{{}^L T,K}$ over $\mathrm{LS}_{{}^L T,K}$. Therefore [69, Tag 06QJ] shows that $\mathrm{LS}_{{}^L T,K}$ is naturally a gerbe over H banded by $\widehat{T}^{\mathrm{Gal}(\widetilde{F}/F)}$.

Since $X^*(T) \otimes_{\mathbb{Z}} A^\times = \widehat{T}(A)$, the natural map

$$H^1\big(W_{\widetilde{F}/F}/\mathbb{O}_{\widetilde{U}}^\times K, X^*(T) \otimes_{\mathbb{Z}} A^\times\big) \to \mathrm{Hom}\left(H_1\big(W_{\widetilde{F},F}/\mathbb{O}_{\widetilde{U}}^\times K, X_*(T)\big), A^\times\right)$$

yields a morphism $\iota^{\mathrm{pre}} : H^{\mathrm{pre}} \to \underline{\mathrm{Hom}}(H_1(W_{\widetilde{F},F}/\mathbb{O}_{\widetilde{U}}^\times K, X_*(T)), \mathbb{G}_m)$. When $A^\times$ is divisible, the above map is an isomorphism, so [76, Lemma 4.1.1] implies that ι^{pre} fpqc-sheafifies to an isomorphism $\iota : H \xrightarrow{\sim} \underline{\mathrm{Hom}}(H_1(W_{\widetilde{F},F}/\mathbb{O}_{\widetilde{U}}^\times K, X_*(T)), \mathbb{G}_m)$. Finally, Lemma 6.14 shows that the transfer map yields an isomorphism

$$H_1\big(W_{\widetilde{F},F}/\mathbb{O}_{\widetilde{U}}^\times K, X_*(T)\big) \xrightarrow{\sim} \mathbb{T}/\mathbb{O}_U^\times \mathrm{Nm}_{\widetilde{F}/F}(K \otimes_{\mathbb{Z}} X_*(T)). \qquad \square$$

6.16. Assume that Z is nonempty. Using work of Langlands, we can prove the following explicit description of $\mathrm{LS}_{{}^L T,U}$.

Theorem. *The derived algebraic stack $\mathrm{LS}_{{}^L T,U}$ is classical and equals*

$$\bigcup_K \mathrm{LS}_{{}^L T,K},$$

where K runs over $\mathrm{Gal}(\widetilde{F}/F)$-stable pro-$p$ open subgroups of $\widetilde{F}_{\widetilde{Z}}^\times$ and the transition morphisms are clopen embeddings.

Proof. Recall that the $\mathrm{Spec}\, A$-points of $\mathrm{LS}^{\square,\mathrm{cl}}_{{}^L T,U}$ consist of 1-cocycles $\varphi : W_U \to \widehat{T}(A)$ that are continuous with respect to the ind-ℓ-adic topology on A as in [74, Remark 2.54]. Because $\varphi|_{W_{\widetilde{U}}}$ is a homomorphism and $\widehat{T}(A)$ is abelian, φ factors through $W_U \to W_{\widetilde{F}/F}/\mathbb{O}_{\widetilde{U}}^\times$. Since $\widetilde{F}^\times \backslash \mathbb{A}_{\widetilde{F}}^\times / \mathbb{O}_{\widetilde{U}}^\times$ is locally pro-p, there exists a $\mathrm{Gal}(\widetilde{F}/F)$-stable pro-$p$ open subgroup K of $\widetilde{F}_{\widetilde{Z}}^\times$ such that φ factors through $W_{\widetilde{F}/F}/\mathbb{O}_{\widetilde{U}}^\times K$. This shows that

$$\mathrm{LS}^{\square,\mathrm{cl}}_{{}^L T,U} = \bigcup_K \mathrm{LS}^{\square}_{{}^L T,K},$$

where K runs over $\mathrm{Gal}(\widetilde{F}/F)$-stable pro-$p$ open subgroups of $\widetilde{F}_{\widetilde{Z}}^\times$, and the proof of [26, Theorem VIII.1.3] implies that the transition morphisms are clopen embeddings. Quotienting by $\widehat{T}$ yields an analogous description

$$\mathrm{LS}^{\mathrm{cl}}_{{}^L T,U} = \bigcup_K \mathrm{LS}_{{}^L T,K}\,.$$

Because the finitely generated abelian group $\mathbb{T}/\mathbb{O}_U^\times \mathrm{Nm}_{\widetilde{F}/F}(K \otimes_{\mathbb{Z}} X_*(T))$ has rank $\dim \widehat{T}^{\mathrm{Gal}(\widetilde{F}/F)}$, its Cartier dual $\underline{\mathrm{Hom}}(\mathbb{T}/\mathbb{O}_U^\times \mathrm{Nm}_{\widetilde{F}/F}(K \otimes_{\mathbb{Z}} X_*(T)), \mathbb{G}_m)$ has relative dimension $\dim \widehat{T}^{\mathrm{Gal}(\widetilde{F}/F)}$ over Λ. Hence Proposition 6.15 indicates that $\mathrm{LS}_{{}^L T,K}$ has relative dimension 0 over Λ. Finally, $\mathrm{LS}_{{}^L T,U}$ also has relative virtual dimension 0 over Λ, so [36, Proposition B.0.1] implies that $\mathrm{LS}_{{}^L T,U}$ is classical. $\square$

6.17. Finally, we prove Theorem G.

Theorem. *Conjecture 6.10 holds when $G = T$ is a torus.*

Proof. Using Proposition 6.12, we can assume that Z is nonempty. Then Theorem 6.16 and Proposition 6.15 indicate that $\mathrm{LS}_{{}^L T,U}$ is naturally a gerbe over

$$\underline{\mathrm{Hom}}(\mathbb{T}/\mathbb{O}_U^\times, \mathbb{G}_m) \coloneqq \bigcup_K \underline{\mathrm{Hom}}\big(\mathbb{T}/\mathbb{O}_U^\times \mathrm{Nm}_{\widetilde{F}/F}(K \otimes_{\mathbb{Z}} X_*(T)), \mathbb{G}_m\big)$$

banded by $\widehat{T}^{\mathrm{Gal}(\widetilde{F}/F)}$, where K runs over $\mathrm{Gal}(\widetilde{F}/F)$-stable pro-$p$ open subgroups of $\widetilde{F}^{\times}_{\widetilde{Z}}$. This implies that $\omega_{\mathrm{LS}_{L_{T,U}}}$ is isomorphic to $\mathscr{O}_{\mathrm{LS}_{L_{T,U}}}$. Using [76, Proposition 4.3.2], this also implies that we have a decomposition

$$D^{\mathrm{qc}}_{\mathrm{perf}}(\mathrm{LS}_{L_{T,U}}) = \bigoplus_{\chi \in X_*(T)_{W_F}} D^{\mathrm{qc}}_{\mathrm{perf}}(\mathrm{LS}_{L_{T,U}})_{\chi}$$

such that $D^{\mathrm{qc}}_{\mathrm{perf}}(\mathrm{LS}_{L_{T,U}})_{\mathbf{1}}$ is naturally equivalent to $D^{\mathrm{qc}}_{\mathrm{perf}}(\underline{\mathrm{Hom}}(\mathbb{T}/\mathbb{O}_U^{\times}, \mathbb{G}_m))$, the latter of which [76, Lemma 4.2.1] identifies with $D(*/\underline{\mathbb{T}/\mathbb{O}_U^{\times}}, \Lambda)^{\omega}$. For all χ in $X_*(T)_{W_F}$, tensoring with the line bundle associated with χ also yields an equivalence $D^{\mathrm{qc}}_{\mathrm{perf}}(\mathrm{LS}_{L_{T,U}})_{\mathbf{1}} \xrightarrow{\sim} D^{\mathrm{qc}}_{\mathrm{perf}}(\mathrm{LS}_{L_{T,U}})_{\chi}$.

For all z in Z, [76, Lemma 6.2.3] and [76, Lemma 4.2.1] indicate that we have an analogous decomposition

$$D^{\mathrm{qc}}_{\mathrm{perf}}(\mathrm{LS}_{L_{T,F_z}}) = \bigoplus_{\chi_z \in X_*(T)_{W_{F_z}}} D^{\mathrm{qc}}_{\mathrm{perf}}(\mathrm{LS}_{L_{T,F_z}})_{\chi_z},$$

where $D^{\mathrm{qc}}_{\mathrm{perf}}(\mathrm{LS}_{L_{T,F_z}})_{\chi_z}$ is naturally equivalent to $D(*/\underline{T(F_z)}, \Lambda)^{\omega}$ for all χ_z in $X_*(T)_{W_{F_z}}$. This induces a decomposition

$$(\clubsuit) \qquad \bigotimes_{z \in Z} \mathrm{Ind}\, D^{\mathrm{qc}}_{\mathrm{perf}}(\mathrm{LS}_{L_{T,F_z}}) = \prod_{\chi_\bullet \in \bigoplus_{z\in Z} X_*(T)_{W_{F_z}}} D(*/\underline{T(F_Z)}, \Lambda).$$

Under these identifications, $\mathrm{res}_{Z,*}(\omega_{L_{T,U}})$ corresponds to $C_c^{\infty}(\mathbb{T}/\mathbb{O}_U^{\times}, \Lambda)$ in the factors where $\chi_\bullet$ lies in $\ker(\bigoplus_{z\in Z} X_*(T)_{W_{F_z}} \to X_*(T)_{W_F})$ and 0 in the other factors.

Since T is a torus, every element in the Kottwitz set is basic. Therefore $\mathrm{Bun}_{T,U}$ equals $\mathrm{Bun}^{\mathrm{ss}}_{T,U}$, so Theorem 4.11 and Lemma 2.6 naturally identify $\mathrm{Bun}_{T,U}$ with

$$\coprod_b \underline{T(F)} \backslash \underline{T(\mathbb{A})} / \underline{T(\mathbb{O}_U) T(F_Z)},$$

where b runs over elements in $B(F,T)$ satisfying $\mathrm{loc}_u(b) = 1$ for all closed points u of U. Hence, for all $b_\bullet$ in $\prod_{z\in Z} B(F_z, T)$, the corresponding factor of $\mathrm{loc}_{Z,!}\Lambda$ in

$$(\spadesuit) \qquad \bigotimes_{z \in Z} D(\mathrm{Bun}_{T,F_z}, \Lambda) = \prod_{b_\bullet \in \prod_{z\in Z} B(F_z,T)} D(*/\underline{T(F_Z)}, \Lambda)$$

equals $\bigoplus_{b'} C_c^{\infty}(T(F)\backslash T(\mathbb{A})/T(\mathbb{O}_U), \Lambda)$, where b' runs over elements in $B(F,T)$ satisfying $\mathrm{loc}_z(b') = b_z$ for all z in Z and $\mathrm{loc}_u(b') = 1$ for all closed points u of U. By [50, Proposition 15.6], the set of such b' is nonempty if and only if $(\kappa(b_z))_z$ lies in $\ker(\bigoplus_{z\in Z} X_*(T)_{W_{F_z}} \to X_*(T)_{W_F})$, and when this occurs, [50, p. 83] shows that this set is a $\ker^1(F,T)$-torsor. Using 6.14, this implies that the images of $\mathrm{loc}_{Z,!}\Lambda$ in $(\spadesuit)$ and $\mathrm{res}_{Z,*}(\omega_{L_{T,U}})$ in $(\clubsuit)$ are isomorphic. Finally, [76, Remark 6.4.6] indicates that $\bigotimes_{z\in Z} \mathbb{L}_{\psi_z}$ is given precisely by composing the equivalences $(\spadesuit)$ and $(\clubsuit)$. □

Appendix A. GAGA over sousperfectoid spaces

Our goal in this section is to prove Theorem H. Actually, we prove a generalization of Theorem H over any complete Tate Huber pair, though stating it requires analytic geometry à la Clausen–Scholze [66]; see Theorem A.18. While we were finalizing this paper, a generalization of Theorem A.18 was proved independently by Wang [71, Theorem 4.4.4] along very similar lines.

The proof of Theorem A.18 closely follows Clausen–Scholze's proof of GAGA for complex analytic spaces [14, Theorem 13.10]. Namely, we start by using analytic geometry à la Clausen–Scholze to define a flexible notion of analytification that "only analytifies some functions"; said differently, it "remembers some algebraicity." By combining this with an argument using *discrete* Huber pairs, this already yields a GAGA theorem for *all* solid quasi-coherent sheaves. Next, we prove an analogue of Grauert's coherence theorem. Using this, we prove that the previous GAGA theorem restricts to an equivalence on (pseudo)coherent sheaves.

We conclude by applying Theorem H to prove that the analytification of algebraic stacks of bundles agrees with analytic stacks of bundles, answering a question of Heuer–Xu [41, Remark 8.1.2]. This was also proved independently by Wang [71].

Notation. For any (animated) condensed ring $\mathcal{A}$, write $D_{\mathrm{cond}}(\mathcal{A})$ for its derived category of condensed $\mathcal{A}$-modules, and recall that there is a natural fully faithful functor $D(\mathcal{A}(*)) \hookrightarrow D_{\mathrm{cond}}(\mathcal{A})$.

A.1. We start with some notation on *analytic rings*. Write AnRing for the ∞-category of (normalized animated) analytic rings as in [66, Definition 12.11], and for any analytic ring $(\mathcal{A}, \mathcal{M})$, write $D(\mathcal{A}, \mathcal{M}) \subseteq D_{\mathrm{cond}}(\mathcal{A})$ for the full subcategory of complete objects as in [66, Remark 12.5]. Recall that $D(\mathcal{A}, \mathcal{M}) \subseteq D_{\mathrm{cond}}(\mathcal{A})$ determines the analytic ring structure $(\mathcal{A}, \mathcal{M})$ on the underlying condensed ring $\mathcal{A}$.

Recall that $(\mathcal{A}, \mathcal{M}) \mapsto D(\mathcal{A}, \mathcal{M})$ yields a functor $D(-) : \mathrm{AnRing} \to \mathrm{Sym}$.

Lemma. *The functor $D(-) : \mathrm{AnRing} \to \mathrm{Sym}$ preserves pushouts.*

Proof. This is a special case of [22, Proposition 4.1.14]. □

A.2. Next, let us recall the theory of modules over analytic rings. For any $(\mathcal{A}, \mathcal{M})$ in AnRing, write $\mathrm{AnSpec}(\mathcal{A}, \mathcal{M})$ for the corresponding object of $\mathrm{AnRing}^{\mathrm{op}}$. Endow $\mathrm{AnRing}^{\mathrm{op}}$ with the topology generated by finite families of jointly conservative steady localizations, as in [66, p. 95]. Consider the following classes of morphisms $\mathrm{AnSpec}(\mathcal{B}, \mathcal{N}) \to \mathrm{AnSpec}(\mathcal{A}, \mathcal{M})$ in $\mathrm{AnRing}^{\mathrm{op}}$:

(I) the functor $D(\mathcal{A}, \mathcal{M}) \to D(\mathcal{B}, \mathcal{N})$ is an open embedding as in [14, p. 61],
(P) $(\mathcal{B}, \mathcal{N})$ is induced from $(\mathcal{A}, \mathcal{M})$ along $\mathcal{A} \to \mathcal{B}$ as in [66, Proposition 12.8].

Write E for the class of morphisms in $\mathrm{AnRing}^{\mathrm{op}}$ of the form $p \circ i$ for p in P and i in I. Then the proof of [60, Lemma 3.2.5][13] shows that

- I and P form a suitable decomposition of E as in [57, Definition A.5.9],
- the functor $D(-) : \mathrm{AnRing} \to \mathrm{Sym}$ satisfies the conditions in [42, Proposition A.5.10] with respect to I and P,

so [42, Proposition A.5.10] endows $\mathrm{AnSpec}(\mathcal{A}, \mathcal{M}) \mapsto D(\mathcal{A}, \mathcal{M})$ with the structure of a 6-functor formalism on $\mathrm{AnRing}^{\mathrm{op}}$ with E as its class of !-able morphisms. By [66, Proposition 12.18], this 6-functor formalism is sheafy as in [42, Definition 3.4.1].

Write {analytic stacks} for the ∞-category of sheaves of anima on $\mathrm{AnRing}^{\mathrm{op}}$. Then [42, Theorem 3.4.11] extends the above 6-functor formalism on $\mathrm{AnRing}^{\mathrm{op}}$ to a sheafy 6-functor formalism on {analytic stacks}. Note that the ∞-category of analytic spaces as in [66, Definition 13.5] is a full subcategory of {analytic stacks}.

[13] While [60, Lemma 3.2.5] restricts to the full subcategory of $\mathrm{AnRing}^{\mathrm{op}}$ consisting of solid affinoids as in [60, Definition 2.6.6], this hypothesis is not used in the proof of [60, Lemma 3.2.5].

A.3. There is a notion of *nuclearity* [14, Definition 8.5 (1)] for modules over analytic rings, and it enjoys the following property with respect to induced analytic ring structures. Let $(\mathcal{A},\mathcal{M})$ be an analytic ring, let $\mathcal{B}$ be an algebra in $D(\mathcal{A},\mathcal{M})$, and write $(\mathcal{B},\mathcal{N})$ for the analytic ring induced from $(\mathcal{A},\mathcal{M})$ along $\mathcal{A}\to\mathcal{B}$ as in [66, Proposition 12.8]. Then Lemma A.1 implies that $D(\mathcal{B},\mathcal{N})$ is naturally equivalent to $\mathrm{Mod}_{\mathcal{B}}(D(\mathcal{A},\mathcal{M}))$, so the image of $-\otimes_{(\mathcal{B},\mathcal{N})}-$ in $D(\mathcal{A},\mathcal{M})$ is given by $-\otimes_{\mathcal{B}}-$.

Lemma. *Assume that $\mathcal{B}$ in $D(\mathcal{A},\mathcal{M})$ is nuclear. Then an object C of $D(\mathcal{B},\mathcal{N})$ is nuclear if and only if its image in $D(\mathcal{A},\mathcal{M})$ is nuclear.*

Proof. Recall that C in $D(\mathcal{B},\mathcal{N})$ is nuclear if and only if, for all extremally disconnected S, the natural morphism

$$(\underline{\mathrm{Hom}}_{\mathcal{B}}(\mathcal{N}[S],\mathcal{B})\otimes_{\mathcal{B}} C)(*)\to\underline{\mathrm{Hom}}_{\mathcal{B}}(\mathcal{N}[S],C)(*)$$

is an isomorphism. Adjunction shows that $\underline{\mathrm{Hom}}_{\mathcal{B}}(\mathcal{N}[S],-)=\underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{M}[S],-)$, and because $\mathcal{B}$ in $D(\mathcal{A},\mathcal{M})$ is nuclear, [1, Proposition 5.35] shows that $\underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{M}[S],\mathcal{B})$ is naturally isomorphic to $\underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{M}[S],\mathcal{A})\otimes_{(\mathcal{A},\mathcal{M})}\mathcal{B}$. Therefore the above becomes

$$(\underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{M}[S],\mathcal{A})\otimes_{(\mathcal{A},\mathcal{M})} C)(*)\to\underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{M}[S],C)(*),$$

and this is an isomorphism for all extremally disconnected S if and only if the image of C in $D(\mathcal{A},\mathcal{M})$ is nuclear. □

A.4. Consequently, we can check nuclearity on a cover in the following situation. Let $(\mathcal{A},\mathcal{M})$ be an analytic ring, let $\mathcal{B}$ be a nuclear algebra in $D(\mathcal{A},\mathcal{M})$, and write $(\mathcal{B},\mathcal{N})$ for the analytic ring induced from $(\mathcal{A},\mathcal{M})$ along $\mathcal{A}\to\mathcal{B}$ as in [66, Proposition 12.8].

Let $\{\mathcal{B}_j\}_{j=1}^r$ be a finite family of nuclear algebras in $D(\mathcal{B},\mathcal{N})$, and write $(\mathcal{B}_j,\mathcal{N}_j)$ for the analytic ring induced from $(\mathcal{B},\mathcal{N})$ along $\mathcal{B}\to\mathcal{B}_j$ as in [66, Proposition 12.8] for all $1\le j\le r$. Assume that $\left\{(\mathcal{B},\mathcal{N})\to(\mathcal{B}_j,\mathcal{N}_j)\right\}_{j=1}^r$ is a family of jointly conservative steady localizations.

Proposition. *An object C of $D(\mathcal{B},\mathcal{N})$ is nuclear if and only if $C\otimes_{\mathcal{B}}\mathcal{B}_j$ in $D(\mathcal{B}_j,\mathcal{N}_j)$ is nuclear for all $1\le j\le r$.*

Proof. Since the functor $D(\mathcal{B},\mathcal{N})\to D(\mathcal{B}_j,\mathcal{N}_j)$ is symmetric monoidal, it preserves trace-class morphisms as in [14, Definition 8.1]. It also preserves colimits, so it preserves basic nuclear objects as in [14, Definition 8.5 (2)] and hence nuclear objects by [14, Theorem 8.6 (2)].

Conversely, assume that $C\otimes_{\mathcal{B}}\mathcal{B}_j$ in $D(\mathcal{B}_j,\mathcal{N}_j)$ is nuclear for all $1\le j\le r$. Then Lemma A.3 indicates that the image of $C\otimes_{\mathcal{B}}\mathcal{B}_j$ in $D(\mathcal{B},\mathcal{N})$ is nuclear. Descent shows that C is naturally isomorphic to a finite limit with terms of the form $C\otimes_{\mathcal{B}}\bigotimes_{j\in J}\mathcal{B}_j$ for nonempty subsets J of $\{1,\dots,r\}$, and because the $C\otimes_{\mathcal{B}}\bigotimes_{j\in J}\mathcal{B}_j$ are nuclear, [14, Theorem 8.6 (1)] implies that C is also nuclear. □

A.5. Recall the notion of *pseudocompact* objects [14, Definition 9.9]. In general, we can check pseudocompactness on covers with bounded Tor-dimension; for us, the following situation will suffice. Let $(\mathcal{A},\mathcal{M})$ be an analytic ring. Let $\left\{(\mathcal{A},\mathcal{M})\to(\mathcal{A}_j,\mathcal{M}_j)\right\}_{j=1}^r$ be a finite family of jointly conservative steady localizations such that, for all $1\le j\le r$, the functor $D(\mathcal{A},\mathcal{M})\to D(\mathcal{A}_j,\mathcal{M}_j)$ is the open embedding as in [14, p. 61] corresponding to an idempotent algebra $\mathcal{I}_j$ in $D^{\le 0}(\mathcal{A},\mathcal{M})$.

Proposition. *An object C of $D(\mathcal{A},\mathcal{M})$ is pseudocompact if and only if its image in $D(\mathcal{A}_j,\mathcal{M}_j)$ is pseudocompact for all $1\leq j\leq r$.*

Proof. Because $D(\mathcal{A}_j,\mathcal{M}_j)\to D(\mathcal{A},\mathcal{M})$ is left t-exact and preserves direct sums, if C is pseudocompact, then its image in $D(\mathcal{A}_j,\mathcal{M}_j)$ is pseudocompact.

Conversely, assume that the image of C in $D(\mathcal{A}_j,\mathcal{M}_j)$ is pseudocompact for all $1\leq j\leq r$. Since the composition $D(\mathcal{A},\mathcal{M})\to D(\mathcal{A}_j,\mathcal{M}_j)\to D(\mathcal{A},\mathcal{M})$ is given by $\underline{\mathrm{Hom}}_{\mathcal{A}}(\mathrm{fib}(\mathcal{A}\to\mathcal{I}_j),-)$ and $\mathcal{I}_j$ lies in $D^{\leq 0}(\mathcal{A},\mathcal{M})$, we see that $D(\mathcal{A},\mathcal{M})\to D(\mathcal{A}_j,\mathcal{M}_j)$ sends $D^{\geq 0}(\mathcal{A},\mathcal{M})$ to $D^{\geq -1}(\mathcal{A}_j,\mathcal{M}_j)$. Descent shows that $\mathrm{Hom}_{\mathcal{A}}(C,-)$ is naturally isomorphic to a finite limit with terms of the form

$$\mathrm{Hom}_{\bigotimes_{j\in J}(\mathcal{A}_j,\mathcal{M}_j)}(C\otimes_{(\mathcal{A},\mathcal{M})}\bigotimes_{j\in J}(\mathcal{A}_j,\mathcal{M}_j),-\otimes_{(\mathcal{A},\mathcal{M})}\bigotimes_{j\in J}(\mathcal{A}_j,\mathcal{M}_j))$$

for nonempty subsets J of $\{1,\dots,r\}$. The above shows that $-\otimes_{(\mathcal{A},\mathcal{M})}\bigotimes_{j\in J}(\mathcal{A}_j,\mathcal{M}_j)$ sends $D^{\geq 0}(\mathcal{A},\mathcal{M})$ to $D^{\geq -(\#J)}\big(\bigotimes_{j\in J}(\mathcal{A}_j,\mathcal{M}_j)\big)$. Because $C\otimes_{(\mathcal{A},\mathcal{M})}\bigotimes_{j\in J}(\mathcal{A}_j,\mathcal{M}_j)$ is pseudocompact, this implies that C is also pseudocompact. □

A.6. Next, let us recall the analytic rings associated with complete Huber pairs. For any complete Huber pair (A,A^+), write $(A,A^+)_\blacksquare$ for the associated analytic ring as in [66, Proposition 13.16], write $(A,A^+)_\blacksquare[-]$ for its functor of measures, and write $D_\blacksquare(A,A^+)$ for $D((A,A^+)_\blacksquare)$. When A^+ equals A°, we omit it from our notation. By [1, Proposition 3.29], we see that $D_\blacksquare(A,A^+)$ is naturally equivalent to $\mathrm{Mod}_A(D_\blacksquare(A^+))$.

Lemma. *We have the following pushout squares in* AnRing*:*

$$\begin{array}{ccc}(\mathbb{Z}[X,Y],\mathbb{Z})_\blacksquare & \longrightarrow & (\mathbb{Z}[X,Y],\mathbb{Z}[X])_\blacksquare\\ \downarrow & & \downarrow\\ (\mathbb{Z}[X,Y],\mathbb{Z}[Y])_\blacksquare & \longrightarrow & \mathbb{Z}[X,Y]_\blacksquare,\end{array}$$

$$\begin{array}{ccc}(\mathbb{Z}[XY],\mathbb{Z})_\blacksquare & \longrightarrow & \mathbb{Z}[XY]_\blacksquare\\ \downarrow & & \downarrow\\ (\mathbb{Z}[X,Y],\mathbb{Z})_\blacksquare & \longrightarrow & (\mathbb{Z}[X,Y],\mathbb{Z}[XY])_\blacksquare,\end{array}\qquad\begin{array}{ccc}(\mathbb{Z}[X+Y],\mathbb{Z})_\blacksquare & \longrightarrow & \mathbb{Z}[X+Y]_\blacksquare\\ \downarrow & & \downarrow\\ (\mathbb{Z}[X,Y],\mathbb{Z})_\blacksquare & \longrightarrow & (\mathbb{Z}[X,Y],\mathbb{Z}[X+Y])_\blacksquare.\end{array}$$

Proof. This follows from combining [66, Proposition 12.12] with the characterization of $(-,-)_\blacksquare$ from [1, Proposition 3.32]. □

A.7. The following analytic rings correspond to the complements of the closed[14] unit disk centered at the origin and at infinity. Recall that $\mathbb{Z}((T^{-1}))$ is an idempotent algebra in $D_\blacksquare(\mathbb{Z}[T],\mathbb{Z})$ [67, Observation 8.7], and the corresponding open embedding as in [14, p. 61] is $D_\blacksquare(\mathbb{Z}[T],\mathbb{Z})\to D_\blacksquare(\mathbb{Z}[T])$ [67, Observation 8.10]. Also, recall that the complex

$$\mathbb{Z}[\![U]\!]\otimes_{\mathbb{Z}}\mathbb{Z}[T]\xrightarrow{UT-1}\mathbb{Z}[\![U]\!]\otimes_{\mathbb{Z}}\mathbb{Z}[T]$$

is a projective resolution of $\mathbb{Z}((T^{-1}))$ in $D_\blacksquare(\mathbb{Z}[T],\mathbb{Z})$ [67, Observation 8.6].

Recall that $\mathbb{Z}[\![U]\!]$ is naturally isomorphic to $\mathbb{Z}_\blacksquare[\mathbb{N}\cup\{\infty\}]/\mathbb{Z}[\{\infty\}]$, where multiplication by U is induced by the endomorphism $n\mapsto n+1$ of $\mathbb{N}$ [1, Lemma 3.11].

[14] "Closed" in the sense of being defined by a non-strict inequality; our closed unit disks are actually open in the affine line, so their complements in the affine line are actually closed.

Lemma. *$\mathbb{Z}[\![T]\!]$ is also an idempotent algebra in $D_{\blacksquare}(\mathbb{Z}[T], \mathbb{Z})$, and we have*

$$\mathbb{Z}((T^{-1})) \otimes_{(\mathbb{Z}[T],\mathbb{Z})_{\blacksquare}} \mathbb{Z}[\![T]\!] = 0.$$

Proof. By [67, Example 6.4], we obtain

$$\mathbb{Z}[\![T]\!] \otimes_{(\mathbb{Z}[T]_{\blacksquare},\mathbb{Z})} \mathbb{Z}[\![T]\!] = (\mathbb{Z}[\![T_1]\!] \otimes_{\mathbb{Z}_{\blacksquare}} \mathbb{Z}[\![T_2]\!])/(T_1 - T_2) = \mathbb{Z}[\![T_1, T_2]\!]/(T_1 - T_2) = \mathbb{Z}[\![T]\!].$$

Next, using the above projective resolution of $\mathbb{Z}((T^{-1}))$, we see that

$$\mathbb{Z}((T^{-1})) \otimes_{(\mathbb{Z}[T],\mathbb{Z})_{\blacksquare}} \mathbb{Z}[\![T]\!]$$

is the solidification of the complex

$$\mathbb{Z}[\![U]\!] \otimes_{\mathbb{Z}} \mathbb{Z}[\![T]\!] \xrightarrow{UT-1} \mathbb{Z}[\![U]\!] \otimes_{\mathbb{Z}} \mathbb{Z}[\![T]\!].$$

After solidification, [67, Example 6.4] indicates that both terms become $\mathbb{Z}[\![U, T]\!]$. Because $UT - 1$ is invertible in $\mathbb{Z}[\![U, T]\!]$, the resulting complex is indeed exact. □

A.8. Let (R, R^+) be a complete Huber pair.

Lemma. *We have a pushout square in* AnRing

$$\begin{array}{ccc} \mathbb{Z}_{\blacksquare} & \longrightarrow & \mathbb{Z}[T]_{\blacksquare} \\ \downarrow & & \downarrow \\ (R, R^+)_{\blacksquare} & \longrightarrow & (R\langle T\rangle, R^+\langle T\rangle)_{\blacksquare}. \end{array}$$

Proof. Write $(\mathcal{A}, \mathcal{M})$ for the pushout. Since [1, Lemma 3.8] indicates that $\mathbb{Z}_{\blacksquare} \to \mathbb{Z}[T]_{\blacksquare}$ is steady, we see that

$$\begin{aligned} \mathcal{A} &= R \otimes_{\mathbb{Z}_{\blacksquare}} \mathbb{Z}[T]_{\blacksquare} && \text{by [66, p. 84],} \\ &= R\langle T\rangle && \text{by [1, Proposition 3.14].} \end{aligned}$$

Next, [66, Proposition 12.12] indicates that $D(\mathcal{A}, \mathcal{M}) \subseteq D_{\mathrm{cond}}(R\langle T\rangle)$ is the full subcategory of objects whose images in $D_{\mathrm{cond}}(\mathbb{Z}[T])$ and $D_{\mathrm{cond}}(R)$ lie in $D_{\blacksquare}(\mathbb{Z}[T])$ and $D_{\blacksquare}(R, R^+)$, respectively. Because $R^+\langle T\rangle$ is topologically generated by $\mathbb{Z}[T]$ and R^+, [1, Proposition 3.32] implies that $D(\mathcal{A}, \mathcal{M})$ equals $D_{\blacksquare}(R\langle T\rangle, R^+\langle T\rangle)$. □

A.9. One can use analytic rings to define the following version of analytification, which remembers all algebraic functions. For any R-algebra A, write A^{alg} for the pushout in AnRing

$$\begin{array}{ccc} (R_{\mathrm{disc}}, R^+_{\mathrm{disc}})_{\blacksquare} & \longrightarrow & (A_{\mathrm{disc}}, (R^+)^{\sim}_{\mathrm{disc}})_{\blacksquare} \\ \downarrow & & \downarrow \\ (R, R^+)_{\blacksquare} & \longrightarrow & A^{\mathrm{alg}}. \end{array}$$

Since A_{disc} is nuclear in $D(R_{\mathrm{disc}}, R^+_{\mathrm{disc}})$, [66, Proposition 13.14] shows that the top arrow is steady. Therefore underlying condensed ring of A^{alg} is A, by [66, p. 84].

For any f in A, write $A_{|f|\leq 1}$ and $A_{|f|\geq 1}$ for the following pushouts in AnRing:

$$\begin{array}{ccc} (\mathbb{Z}[T], \mathbb{Z})_{\blacksquare} & \longrightarrow & \mathbb{Z}[T]_{\blacksquare} \\ \downarrow{\scriptstyle f} & & \downarrow \\ A^{\mathrm{alg}} & \longrightarrow & A_{|f|\leq 1}, \end{array} \qquad \begin{array}{ccc} (\mathbb{Z}[T], \mathbb{Z})_{\blacksquare} & \longrightarrow & (\mathbb{Z}[T^{\pm 1}], \mathbb{Z}[T^{-1}])_{\blacksquare} \\ \downarrow{\scriptstyle f} & & \downarrow \\ A^{\mathrm{alg}} & \longrightarrow & A_{|f|\geq 1}. \end{array}$$

Because the top arrows are steady localizations, the same holds for the bottom arrows.

A.10. The following analytic ring corresponds to the closure of the closed unit disk centered at the origin. For the rest of this section, assume that R is Tate.

Proposition. *The object $R\langle T\rangle$ in $D_{\blacksquare}(R, R^+)$ is nuclear. Moreover, the analytic ring induced from $(R, R^+)_{\blacksquare}$ along $R \to R\langle T\rangle$ as in* [66, Proposition 12.8] *is naturally isomorphic to $(R\langle T\rangle, R^+ + T \cdot R^{\circ\circ}\langle T\rangle)_{\blacksquare}$, and the induced morphism $R[T]^{\mathrm{alg}} \to (R\langle T\rangle, R^+ + T \cdot R^{\circ\circ}\langle T\rangle)_{\blacksquare}$ is a steady localization.*

Proof. Since R is Tate, the first statement follows from [2, Lemma 4.4]. For the second statement, write $(\mathcal{A}, \mathcal{M})$ for the analytic ring induced from $(R, R^+)_{\blacksquare}$ along $R \to R\langle T\rangle$ as in [66, Proposition 12.8]. Because $R\langle T\rangle$ lies in $D_{\blacksquare}(R, R^+)$, we see that $\mathcal{A} = R\langle T\rangle$. Next, [66, Proposition 12.12] indicates that $D(\mathcal{A}, \mathcal{M}) \subseteq D_{\mathrm{cond}}(R\langle T\rangle)$ is the full subcategory of objects whose image in $D_{\mathrm{cond}}(R)$ lies in $D_{\blacksquare}(R, R^+)$. Since $R^+ + T \cdot R^{\circ\circ}\langle T\rangle$ is topologically generated by R^+, [1, Proposition 3.32] implies that $D(\mathcal{A}, \mathcal{M})$ equals $D_{\blacksquare}(R\langle T\rangle, R^+ + T \cdot R^{\circ\circ}\langle T\rangle)$, as desired.

For the third statement, [66, Proposition 13.14] and the first statement imply

$$(R, R^+)_{\blacksquare} \to (R\langle T\rangle, R^+ + T \cdot R^{\circ\circ}\langle T\rangle)_{\blacksquare}$$

is steady, so [66, Proposition 12.15] shows that

$$R[T]^{\mathrm{alg}} \to (R\langle T\rangle, R^+ + T \cdot R^{\circ\circ}\langle T\rangle)_{\blacksquare}$$

is also steady. Finally, Lemma A.8 implies that the composition

$$R[T]^{\mathrm{alg}} \to (R\langle T\rangle, R^+ + T \cdot R^{\circ\circ}\langle T\rangle)_{\blacksquare} \to (R\langle T\rangle, R^+\langle T\rangle)_{\blacksquare}$$

is a localization. Because the right arrow is a localization, the same holds for the left arrow. □

A.11. For any R-algebra A and f in A, write $\overline{A}_{|f|\leq 1}$ for the pushout in AnRing

$$\begin{array}{ccc} R[T]^{\mathrm{alg}} & \longrightarrow & (R\langle T\rangle, R^+ + T \cdot R^{\circ\circ}\langle T\rangle)_{\blacksquare} \\ \downarrow{\scriptstyle f} & & \downarrow \\ A^{\mathrm{alg}} & \longrightarrow & \overline{A}_{|f|\leq 1}. \end{array}$$

Extending the construction in §A.9, the following analytic stacks yield a version of analytification that "only analytifies f". Choose a pseudouniformizer ϖ of R, form the colimits in {analytic stacks}

$$\mathrm{an}(A, f) \coloneqq \varinjlim_{n\geq 0} \mathrm{AnSpec}\, A_{|\varpi^n f|\leq 1} \text{ and } D^{\mathrm{an}}(f) \coloneqq \varinjlim_{n\geq 0} \mathrm{AnSpec}\, A_{|\varpi^{-n} f|\geq 1},$$

and note that they do not depend on the choice of ϖ. Moreover, the factorization

$$(R\langle \varpi T\rangle, R^+\langle \varpi T\rangle)_{\blacksquare} \to (R\langle T\rangle, R^+ + T \cdot R^{\circ\circ}\langle T\rangle)_{\blacksquare} \to (R\langle T\rangle, R^+\langle T\rangle)_{\blacksquare}$$

indicates that $\mathrm{an}(A, f)$ is naturally isomorphic to $\varinjlim_{n\geq 0} \mathrm{AnSpec}\, \overline{A}_{|\varpi^n f|\leq 1}$.

A.12. Our $\mathrm{an}(A, f)$ and $D^{\mathrm{an}}(f)$ enjoy the following basic properties.

Proposition.

1) For any r in R, we have $\mathrm{an}(R, r) = \mathrm{AnSpec}\, R^{\mathrm{alg}}$.

2) Write $\mathrm{an}(R[X,Y], X) \cap \mathrm{an}(R[X,Y], Y)$ *for the fiber product in* {*analytic stacks*}

$$\begin{array}{ccc} \mathrm{an}(R[X,Y], X) \cap \mathrm{an}(R[X,Y], Y) & \longrightarrow & \mathrm{an}(R[X,Y], X) \\ \downarrow & & \downarrow \\ \mathrm{an}(R[X,Y], Y) & \longrightarrow & \mathrm{AnSpec}\, R[X,Y]^{\mathrm{alg}}. \end{array}$$

Then $\mathrm{an}(R[X,Y], X) \cap \mathrm{an}(R[X,Y], Y) \to \mathrm{AnSpec}\, R[X,Y]^{\mathrm{alg}}$ *factors through* $\mathrm{an}(R[X,Y], XY) \to \mathrm{AnSpec}\, R[X,Y]^{\mathrm{alg}}$ *and* $\mathrm{an}(R[X,Y], X+Y) \to \mathrm{AnSpec}\, R[X,Y]^{\mathrm{alg}}$.

3) The composition $\mathrm{an}(R[T^{\pm 1}], T^{-1}) \to \mathrm{AnSpec}\, R[T^{\pm 1}]^{\mathrm{alg}} \to \mathrm{AnSpec}\, R[T]^{\mathrm{alg}}$ *is naturally isomorphic to* $D^{\mathrm{an}}(T) \to \mathrm{AnSpec}\, R[T]^{\mathrm{alg}}$ *over* $\mathrm{AnSpec}\, R[T]^{\mathrm{alg}}$.

4) $\{\mathrm{an}(R[T], T), D^{\mathrm{an}}(T)\}$ *forms a cohomologically étale (as in* [63, Definition 6.12]*) steady localization cover of* $\mathrm{AnSpec}\, R[T]^{\mathrm{alg}}$.

5) Write $D^{\mathrm{an}}(X) \cup D^{\mathrm{an}}(Y)$ *for the pushout in* {*analytic stacks*}

$$\begin{array}{ccc} D^{\mathrm{an}}(X) \cap D^{\mathrm{an}}(Y) & \longrightarrow & D^{\mathrm{an}}(X) \\ \downarrow & & \downarrow \\ D^{\mathrm{an}}(Y) & \longrightarrow & D^{\mathrm{an}}(X) \cup D^{\mathrm{an}}(Y). \end{array}$$

Then $D^{\mathrm{an}}(X+Y) \to \mathrm{AnSpec}\, R[X,Y]^{\mathrm{alg}}$ *factors through*

$$D^{\mathrm{an}}(X) \cup D^{\mathrm{an}}(Y) \to \mathrm{AnSpec}\, R[X,Y]^{\mathrm{alg}}.$$

Proof. For part 1), it suffices to show that $R_{|r| \leq 1} = R^{\mathrm{alg}}$ when r lies in $R^{\circ\circ}$. By A.7, the functor $D(R[T]^{\mathrm{alg}}) \to D(R[T]_{|T| \leq 1})$ is the open embedding as in [14, p. 61] corresponding to the idempotent algebra $\mathbb{Z}((T^{-1})) \otimes_{\mathbb{Z}_\blacksquare} (R, R^+)_\blacksquare$ in $D(R[T]^{\mathrm{alg}})$. The descriptions of $\mathbb{Z}((T))$ and $\mathbb{Z}[\![U]\!]$ from A.7 show that $\mathbb{Z}((T^{-1})) \otimes_{\mathbb{Z}_\blacksquare} (R, R^+)_\blacksquare$ is given by the complex

$$\frac{(R, R^+)_\blacksquare[\mathbb{N} \cup \{\infty\}]}{R[\{\infty\}]} \otimes_R R[T] \xrightarrow{UT-1} \frac{(R, R^+)_\blacksquare[\mathbb{N} \cup \{\infty\}]}{R[\{\infty\}]} \otimes_R R[T].$$

Since r lies in $R^{\circ\circ}$, multiplication by $Ur - 1$ is invertible on $(R, R^+)_\blacksquare[\mathbb{N} \cup \{\infty\}]/R[\{\infty\}]$. Therefore, after base changing the above complex along the R-algebra morphism $R[T] \mapsto R$ that sends $T \mapsto r$, the resulting complex is indeed exact.

For part 2), it suffices to show that

$$R[X,Y]^{\mathrm{alg}} \to R[X,Y]_{|X| \leq 1} \otimes_{R[X,Y]^{\mathrm{alg}}} R[X,Y]_{|Y| \leq 1}$$

factors through $R[X,Y]^{\mathrm{alg}} \to R[X,Y]_{|XY| \leq 1}$ and $R[X,Y]^{\mathrm{alg}} \to R[X,Y]_{|X+Y| \leq 1}$. This follows from applying $- \otimes_{\mathbb{Z}_\blacksquare} (R, R^+)_\blacksquare$ to Lemma A.6 and the morphisms

$$(\mathbb{Z}[X,Y], \mathbb{Z}[XY])_\blacksquare \to \mathbb{Z}[X,Y]_\blacksquare \text{ and } (\mathbb{Z}[X,Y], \mathbb{Z}[X+Y])_\blacksquare \to \mathbb{Z}[X,Y]_\blacksquare.$$

For part 3), note that T is invertible in $(\mathbb{Z}[T^{\pm 1}], \mathbb{Z}[T^{-1}])_\blacksquare$, and $T \mapsto T^{-1}$ induces an isomorphism $(\mathbb{Z}[T^{\pm 1}], \mathbb{Z}[T])_\blacksquare \xrightarrow{\sim} (\mathbb{Z}[T^{\pm 1}], \mathbb{Z}[T^{-1}])_\blacksquare$. Applying $- \otimes_{\mathbb{Z}_\blacksquare} (R, R^+)_\blacksquare$ to this shows that $R[T]^{\mathrm{alg}} \to R[T]_{|T| \geq 1}$ factors through $R[T]^{\mathrm{alg}} \to R[T^{\pm 1}]^{\mathrm{alg}}$ and

that the analytic rings $R[T]_{|T|\geq 1}$ and $R[T^{\pm 1}]_{|T^{-1}|\leq 1}$ are naturally isomorphic over $R[T^{\pm 1}]^{\mathrm{alg}}$. This implies the desired result.

For part 4), [60, Lemma 3.2.9] indicates that it suffices to show that

$$\left\{R[T]_{|T|\leq 1}, R[T]_{|T|\geq 1}\right\}$$

forms a family of jointly conservative steady localizations of $R[T]^{\mathrm{alg}}$ such that, after applying $D(-)$, the resulting functors are open embeddings as in [14, p. 61]. This follows from applying $-\otimes_{\mathbb{Z}_\blacksquare}(R, R^+)_\blacksquare$ to Lemma A.7.

For part 5), Lemma A.7 shows that $\mathbb{Z}[\![T]\!]$ is an idempotent algebra in $D_\blacksquare(\mathbb{Z}[T], \mathbb{Z})$, and A.7 implies that $D_\blacksquare(\mathbb{Z}[T], \mathbb{Z}) \to D_\blacksquare(\mathbb{Z}[T^{\pm 1}, T^{-1}])$ is the open embedding as in [14, p. 61] corresponding to $\mathbb{Z}[\![T]\!]$. Hence applying $-\otimes_{\mathbb{Z}_\blacksquare}(R, R^+)_\blacksquare$ to the morphism $\mathbb{Z}[\![X+Y]\!] \to \mathbb{Z}[\![X, Y]\!]$ of idempotent algebras in $D_\blacksquare(\mathbb{Z}[X, Y], \mathbb{Z})$ yields the desired result. □

A.13. Using the above formalism and an argument with discrete Huber pairs, one can prove the following GAGA theorem for *all* (solid) quasi-coherent sheaves as in Clausen–Scholze [14]. Let A be a finitely generated R-algebra, and let $A^+ \subseteq A$ be the integral closure of a finitely generated R-subalgebra. If A^+ equals the integral closure of the R-subalgebra generated by $f_1, \ldots, f_m$, write $\mathrm{an}(A, A^+)$ for the intersection $\bigcap_{i=1}^m \mathrm{an}(A, f_i)$ in {analytic stacks} over $\mathrm{AnSpec}\, A^{\mathrm{alg}}$. By Proposition A.12.1), Proposition A.12.2), and proof of [14, Lemma 6.11], this does not depend on the choice of $f_1, \ldots, f_m$.

Theorem. *The assignment* $\mathrm{Spa}(A_{\mathrm{disc}}, A^+_{\mathrm{disc}}) \mapsto \mathrm{an}(A, A^+)$ *sends rational covers to steady localization covers. Consequently, the assignments* $\mathrm{Spec}\, A \mapsto \mathrm{an}(A, R^\sim)$ *and* $\mathrm{Spec}\, A \mapsto \mathrm{an}(A, A)$ *naturally glue to functors*

$$(-)^{\mathrm{alg}} \text{ and } (-)^{\mathrm{an}} : \{\text{schemes locally of finite type over } R\} \to \{\text{analytic stacks}\},$$

respectively, along with a natural transformation $(-)^{\mathrm{an}} \to (-)^{\mathrm{alg}}$. *Moreover,* $(-)^{\mathrm{an}}$ *sends Zariski covers to cohomologically étale steady localization covers. Finally, for any proper scheme* X *over* R, *the morphism* $X^{\mathrm{an}} \to X^{\mathrm{alg}}$ *is an isomorphism.*

Proof. The first statement follows from the description of rational covers of

$$\mathrm{Spa}(A_{\mathrm{disc}}, A^+_{\mathrm{disc}})$$

from [14, Lemma 6.12] and Proposition A.12. The second statement follows from the first statement and the fact that the assignments $\mathrm{Spec}\, A \mapsto \mathrm{Spa}(A_{\mathrm{disc}}, R^\sim_{\mathrm{disc}})$ and $\mathrm{Spec}\, A \mapsto \mathrm{Spa}(A_{\mathrm{disc}}, A_{\mathrm{disc}})$ send rational covers to rational covers. The third statement follows from Proposition A.12.4). Finally, the fourth statement follows from the valuative criterion for properness. □

Remark. By [66, Proposition 13.6], the functors $(-)^{\mathrm{alg}}$ and $(-)^{\mathrm{an}}$ are valued in the full subcategory of analytic spaces as in [66, Definition 13.5].

A.14. We will later need the notion of *pseudocoherent* sheaves on schemes, as well as its relation with finite presentation. For any scheme X, write $D_{\mathrm{pc}}(X)$ for the full subcategory of $D_{\mathrm{qc}}(X)$ consisting of pseudocoherent objects as in [69, Tag 08CB]. When $X = \mathrm{Spec}\, A$ is affine, recall from [69, Tag 08E7] that this is equivalent to the full subcategory $D_{\mathrm{pc}}(A) \subseteq D(A)$ of pseudocoherent objects as in [69, Tag 064Q].

Lemma. *Let A be a finitely generated R-algebra. Then A is a finitely presented R-algebra if and only if, for all R-algebra surjections $R[T_1, \dots, T_m] \twoheadrightarrow A$, the object A lies in $D_{\mathrm{pc}}(R[T_1, \dots, T_m])$.*

Proof. Let $R[T_1, \dots, T_m] \twoheadrightarrow A$ be an R-algebra surjection with A lying in

$$D_{\mathrm{pc}}(R[T_1, \dots, T_m]).$$

Then [69, Tag 064T] implies that A is a finitely presented R-algebra.

Conversely, assume that A is a finitely presented R-algebra, and let

$$\alpha : R[T_1, \dots, T_m] \twoheadrightarrow A$$

be an R-algebra surjection. By [69, Tag 00R2], $\ker \alpha$ is finitely generated. Therefore noetherian approximation yields a noetherian ring R_0, a surjection

$$\alpha_0 : R_0[T_1, \dots, T_m] \twoheadrightarrow A_0$$

of R_0-algebras, and a ring homomorphism $R_0 \to R$ such that the base change of (A_0, α_0) to R is isomorphic to (A, α). Because $R_0[T_1, \dots, T_m]$ is noetherian, A_0 lies in $D_{\mathrm{pc}}(R_0[T_1, \dots, T_m])$. Hence A lies in $D_{\mathrm{pc}}(R[T_1, \dots, T_m])$. □

A.15. By construction, the analytifications from Theorem A.13 are covered by analytic rings of the following form. For any finitely generated R-algebra A with generators $\underline{f} = (f_1, \dots, f_m)$, write $A_{|\underline{f}|\leq 1}$ and $\overline{A}_{|\underline{f}|\leq 1}$ for these pushouts in AnRing:

$$\begin{array}{ccc} (\mathbb{Z}[T_1, \dots, T_m], \mathbb{Z})_{\blacksquare} & \longrightarrow & \mathbb{Z}[T_1, \dots, T_m]_{\blacksquare} \\ \downarrow{\scriptstyle \underline{f}} & & \downarrow \\ A^{\mathrm{alg}} & \longrightarrow & A_{|\underline{f}|\leq 1}, \end{array}$$

$$\begin{array}{ccc} R[T_1, \dots, T_m]^{\mathrm{alg}} & \longrightarrow & (R\langle T_1, \dots, T_m\rangle, R^+ + (T_1, \dots, T_m) \cdot R^{\circ\circ}\langle T_1, \dots, T_m\rangle)_{\blacksquare} \\ \downarrow{\scriptstyle \underline{f}} & & \downarrow \\ A^{\mathrm{alg}} & \longrightarrow & \overline{A}_{|\underline{f}|\leq 1}. \end{array}$$

For any scheme X locally of finite type over R, Theorem A.13 shows that X^{an} is glued from objects of the form $\operatorname{AnSpec} A_{|\underline{f}|\leq 1}$ along covers as in A.5. Therefore Proposition A.5 indicates that the full subcategory of pseudocompact objects in $D(A_{|\underline{f}|\leq 1})$ naturally glues to a full subcategory of $D(X^{\mathrm{an}})$. We say that an object of $D(X^{\mathrm{an}})$ is *pseudocompact* if it lies in this full subcategory.

Theorem A.13 and A.11 show that X^{an} is also glued from objects of the form $\operatorname{AnSpec} \overline{A}_{|\underline{f}|\leq 1}$ along covers as in A.4. Hence Proposition A.4 indicates that the full subcategory of nuclear objects in $D(\overline{A}_{|f|\leq 1})$ naturally glues to a full subcategory of $D(X^{\mathrm{an}})$. We say that an object of $D(X^{\mathrm{an}})$ is *nuclear* if it lies in this full subcategory.

Finally, write $D_{\mathrm{pc}}(X^{\mathrm{an}})$ for the full subcategory of pseudocompact nuclear objects in $D(X^{\mathrm{an}})$. Recall from [1, Theorem 5.50][15] that

$$D_{\mathrm{pc}}((\operatorname{Spec} R)^{\mathrm{an}}) \subseteq D((\operatorname{Spec} R)^{\mathrm{an}}) = D_{\blacksquare}(R, R^+)$$

[15]While [1, Section 5] assumes that (R, R^+) is sheafy, this hypothesis is not used in the proof of [1, Theorem 5.50].

equals the image of $D_{\mathrm{pc}}(R) \subseteq D(R) \hookrightarrow D_{\blacksquare}(R, R^+)$.

A.16. For the analytifications from Theorem A.13, !-pullback of quasi-coherent sheaves commutes with arbitrary direct sums of uniformly bounded below objects. More precisely, for any scheme X locally of finite type over R, write

$$\pi : X^{\mathrm{an}} \to (\operatorname{Spec} R)^{\mathrm{an}} = \operatorname{AnSpec}(R, R^+)_{\blacksquare}$$

for $(-)^{\mathrm{an}}$ applied to the structure morphism $X \to \operatorname{Spec} R$.

Proposition. *Assume that X is of finite presentation over R, and let d be an integer. Then there exists an integer e such that, for all families $(M_\alpha)_\alpha$ of objects of $D^{\geq d}_{\blacksquare}(R, R^+)$, the family $(\pi^! M_\alpha)_\alpha$ lies in $D^{\geq e}(X^{\mathrm{an}})$ and the natural morphism*

$$\bigoplus_\alpha \pi^! M_\alpha \to \pi^! \big(\bigoplus_\alpha M_\alpha \big)$$

is an isomorphism.

Proof. Theorem A.13 indicates that $(-)^{\mathrm{an}}$ sends Zariski covers to cohomologically étale steady localization covers, so by replacing X with an open cover, we can assume that $X = \operatorname{Spec} A$ for a finitely presented R-algebra A.

Let $\underline{f} = (f_1, \dots, f_m)$ be generators of A. Then X^{an} is isomorphic to

$$\varinjlim_{n \geq 0} \operatorname{AnSpec} A_{|\varpi^n \underline{f}| \leq 1},$$

where the transition morphisms are cohomologically étale, so it suffices to replace X^{an} with $\operatorname{AnSpec} A_{|\underline{f}| \leq 1}$. Write $\underline{T} = (T_1, \dots, T_m)$ for the standard generators of $R[T_1, \dots, T_m]$, and note we have a pushout square in AnRing

$$\begin{array}{ccc} (R[T_1, \dots, T_m]_{\mathrm{disc}}, R^+_{\mathrm{disc}})_{\blacksquare} & \longrightarrow & R[T_1, \dots, T_m]_{|\underline{T}| \leq 1} \\ \downarrow{\scriptstyle \underline{f}} & & \downarrow{\scriptstyle \underline{f}} \\ (A_{\mathrm{disc}}, (R^+)^{\sim}_{\mathrm{disc}})_{\blacksquare} & \longrightarrow & A_{|\underline{f}| \leq 1}. \end{array}$$

Now [1, Proposition 3.22] and the characterization of $(-,-)_{\blacksquare}$ from [1, Proposition 3.32] imply that the left arrow is induced as in [66, Proposition 12.8], so the same holds for the right arrow. Therefore the functor

$$(\operatorname{AnSpec} \underline{f})^! : D(R[T_1, \dots, T_m]_{|\underline{T}| \leq 1}) \to D(A_{|\underline{f}| \leq 1})$$

is given by $\underline{\operatorname{Hom}}_{R\langle T_1, \dots, T_m \rangle}(R\langle T_1, \dots, T_m \rangle \otimes_{R[T_1, \dots, T_m]} A, -)$, where we identify

$$R[T_1, \dots, T_m]_{|\underline{T}| \leq 1} = (R\langle T_1, \dots, T_m \rangle, R^+\langle T_1, \dots, T_m \rangle)_{\blacksquare}$$

using Lemma A.8. Lemma A.14 shows that A lies in $D_{\mathrm{pc}}(R[T_1, \dots, T_m])$, so

$$R\langle T_1, \dots, T_m \rangle \otimes_{R[T_1, \dots, T_m]} A \tag{$\Finv$}$$

lies in $D_{\mathrm{pc}}(R\langle T_1, \dots, T_m \rangle)$. By [1, Theorem 5.50], ($\Finv$) yields a pseudocompact object of $D(R[T_1, \dots, T_m]_{|T| \leq 1})$ as in [14, Definition 9.9], which implies that $(\operatorname{AnSpec} \underline{f})^!$ commutes with direct sums of objects of $D^{\geq n}(R[T_1, \dots, T_m]_{|T| \leq 1})$ for all integers n. Because ($\Finv$) also lies in $D^{\leq 0}(R[T_1, \dots, T_m]_{|T| \leq 1})$, the functor $(\operatorname{AnSpec} \underline{f})^!$ is also left t-exact.

Hence we can assume that $A = R[T_1, \dots, T_m]$. Consider the morphisms

$$\operatorname{AnSpec} A_{|\underline{T}|\leq 1} \xrightarrow{j} \operatorname{AnSpec} A^{\mathrm{alg}} \xrightarrow{p} \operatorname{AnSpec}(R, R^+)_{\blacksquare},$$

and note that the restriction of π to $\operatorname{AnSpec} A_{|\underline{T}|\leq 1}$ is $p \circ j$. We claim that $j^! p^!$ is naturally isomorphic to $j^* p^*[m]$. First, [60, Lemma 3.2.9] and A.7 show that j is cohomologically étale, so $j^* = j^!$. Since p is induced, for all objects M of $D_{\blacksquare}(R, R^+)$ we have $j^! p^! M = j^! \underline{\mathrm{Hom}}_R(A, M)$. Next, note that $\underline{\mathrm{Hom}}_R(A, M)$ is naturally isomorphic to $M(\!(T_1^{-1}, \dots, T_m^{-1})\!)/M[T_1, \dots, T_m]$, where $M(\!(T_1^{-1}, \dots, T_m^{-1})\!)$ is a module for the idempotent algebra $\mathbb{Z}(\!(T_j^{-1})\!) \otimes_{\mathbb{Z}_{\blacksquare}} (R, R^+)_{\blacksquare}$ in $D(R[T_j]^{\mathrm{alg}})$ for all $1 \leq j \leq m$. By A.7, the intersection of the open embeddings as in [14, p. 61] corresponding to these idempotent algebras is $j^!$, so applying $j^!$ to the tensor product of the exact triangles

$$M[T_j] \longrightarrow M(\!(T_j^{-1})\!) \longrightarrow M(\!(T_j^{-1})\!)/M[T_j] \xrightarrow{+1}$$

for all $1 \leq j \leq m$ implies that

$$\begin{aligned} j^! \underline{\mathrm{Hom}}_R(A, M) &= j^! M(\!(T_1^{-1}, \dots, T_m^{-1})\!)/M[T_1, \dots, T_m] = j^! M[T_1, \dots, T_m][m] \\ &= j^! p^* M[m] = j^* p^* M[m]. \end{aligned}$$

Finally, the desired result follows immediately from the claim. □

A.17. As a consequence, we prove the following finitude result for proper pushforwards, which is the analogue of Grauert's coherence theorem.

Corollary. *For any proper scheme X of finite presentation over R, the functor*

$$\pi_* : D(X^{\mathrm{an}}) \to D_{\blacksquare}(R, R^+)$$

sends $D_{\mathrm{pc}}(X^{\mathrm{an}})$ to $D_{\mathrm{pc}}(R)$.

Proof. Theorem A.13 indicates that $X^{\mathrm{an}} \xrightarrow{\sim} X^{\mathrm{alg}}$. Because X^{alg} is covered by finitely many objects of the form $\operatorname{AnSpec}(\mathcal{A}, \mathcal{M})$, it is quasicompact, so the same holds for X^{an}. Therefore A.15 shows that X^{an} is covered by finitely many objects of the form $\operatorname{AnSpec} \overline{A}_{|\underline{f}|\leq 1}$. Since X is separated over R, the intersection of these objects over X^{an} are also of this form, so Lemma A.3 and [14, Theorem 8.6 (1)] imply that π_* sends nuclear objects of $D(X^{\mathrm{an}})$ to nuclear objects of $D_{\blacksquare}(R, R^+)$ as in [14, Definition 8.5 (1)].

We claim that π_* is naturally isomorphic to $\pi_!$. To see this, note that Proposition A.10 indicates that $(R, R^+)_{\blacksquare} \to \overline{A}_{|\underline{f}|\leq 1}$ is induced as in [66, Proposition 12.8]. Hence [42, Lemma 4.5.5] implies that $\operatorname{AnSpec} \overline{A}_{|\underline{f}|\leq 1} \to \operatorname{AnSpec}(R, R^+)_{\blacksquare}$ is cohomologically prim as in as in [42, Definition 4.5.1.(b)] with trivial codualizing complex, and applying [42, Remark 4.4.11.(iii)] to the cover of X^{an} by such objects shows that the same holds for π. This yields the claim.

Let C be a pseudocompact object of $D(X^{\mathrm{an}})$, and let d be an integer. Then Proposition A.16 shows there exists an integer e such that, for all families $(M_\alpha)_\alpha$ of objects of $D_{\blacksquare}^{\geq d}(R, R^+)$, the family $(\pi^! M_\alpha)_\alpha$ lies in $D^{\geq e}(X^{\mathrm{an}})$ and satisfies

$$\bigoplus_\alpha \pi^! M_\alpha = \pi^! \big(\bigoplus_\alpha M_\alpha \big).$$

Because C is pseudocompact, the claim indicates that

$$\begin{aligned}\mathrm{Hom}(\pi_* C, \bigoplus_\alpha M_\alpha) &= \mathrm{Hom}(C, \pi^! \bigoplus_\alpha M_\alpha) \\ &= \mathrm{Hom}(C, \bigoplus_\alpha \pi^! M_\alpha) = \bigoplus_\alpha \mathrm{Hom}(C, \pi^! M_\alpha) = \bigoplus_\alpha \mathrm{Hom}(\pi_* C, M_\alpha).\end{aligned}$$

Therefore $\pi_* C$ is pseudocompact, as desired. □

A.18. Using Corollary A.17, we prove that Theorem A.13 restricts to an equivalence between subcategories of pseudocoherent objects as follows. For any scheme X locally of finite type over R, Theorem A.13 shows that X^{alg} is glued from objects of the form $\mathrm{AnSpec}\, A^{\mathrm{alg}}$ along covers induced from Zariski covers. Hence A.14 indicates that the full subcategory $D_{\mathrm{pc}}(A) \subseteq D(A) \hookrightarrow D(A^{\mathrm{alg}})$ naturally glues to a full subcategory of $D(X^{\mathrm{alg}})$ that is naturally equivalent to $D_{\mathrm{pc}}(X)$.

Consider the functor $D(X^{\mathrm{alg}}) \to D(X^{\mathrm{an}})$. When $X = \mathrm{Spec}\, A$ is affine, [1, Theorem 5.50] and [14, Theorem 8.6 (1)] imply that this sends $D_{\mathrm{pc}}(A)$ to $D_{\mathrm{pc}}((\mathrm{Spec}\, A)^{\mathrm{an}})$, so in general this sends $D_{\mathrm{pc}}(X)$ to $D_{\mathrm{pc}}(X^{\mathrm{an}})$.

Theorem. *For any proper scheme X of finite presentation over R, the resulting functor $D_{\mathrm{pc}}(X) \to D_{\mathrm{pc}}(X^{\mathrm{an}})$ is an equivalence.*

Proof. Full faithfulness follows from Theorem A.13. For essential surjectivity, let C be an object of $D_{\mathrm{pc}}(X^{\mathrm{an}})$. By full faithfulness, it suffices to prove that, for all affine open subschemes $j : \mathrm{Spec}\, A \hookrightarrow X$, the image of C under the composition

$$D_{\mathrm{pc}}(X^{\mathrm{an}}) \subseteq D(X^{\mathrm{an}}) \xleftarrow{\sim} D(X^{\mathrm{alg}}) \xrightarrow{j^{\mathrm{alg},*}} D(A^{\mathrm{alg}})$$

lies in the image of $D_{\mathrm{pc}}(A) \subseteq D(A) \hookrightarrow D(A^{\mathrm{alg}})$. Since $j^{\mathrm{alg},*}$ preserves pseudocompact objects, it suffices to prove that $j^{\mathrm{alg},*}C$ lies in the image of $D(A) \hookrightarrow D(A^{\mathrm{alg}})$. Noetherian approximation yields a finitely generated ideal sheaf $\mathscr{Z}$ of X whose vanishing locus equals the closed subset $X \smallsetminus \mathrm{Spec}\, A \subseteq X$, so after replacing X with its blowup along $\mathscr{Z}$, we can assume that $\mathrm{Spec}\, A = X \smallsetminus D$ for an effective Cartier divisor D on X. Then $j^{\mathrm{alg}}_* j^{\mathrm{alg},*} C = \varinjlim_{n \geq 0} C \otimes \mathscr{O}_X(nD)$ and hence

$$\pi_* j^{\mathrm{alg}}_* j^{\mathrm{alg},*} C = \varinjlim_{n \geq 0} \pi_* \big(C \otimes \mathscr{O}_X(nD)\big).$$

Note that $\pi_* j^{\mathrm{alg}}_* j^{\mathrm{alg},*} C$ is the image of $j^{\mathrm{alg},*}C$ in $D_{\blacksquare}(R, R^+)$. Because $C \otimes \mathscr{O}_X(nD)$ lies in $D_{\mathrm{pc}}(X^{\mathrm{an}})$, Corollary A.17 shows that $\pi_*\big(C \otimes \mathscr{O}_X(nD)\big)$ lies in $D_{\mathrm{pc}}(R)$, so the image of $j^{\mathrm{alg},*}C$ in $D_{\blacksquare}(R, R^+)$ lies in the image of $D(R) \hookrightarrow D_{\blacksquare}(R, R^+)$. Therefore $j^{\mathrm{alg},*}C$ lies in the image of $D(A) \hookrightarrow D(A^{\mathrm{alg}})$, as desired. □

A.19. Finally, we prove Theorem H. Recall that the assignment

$$\mathrm{Spa}(A, A^+) \mapsto \mathrm{AnSpec}(A, A^+)_{\blacksquare}$$

from Tate affinoid adic spaces to {analytic stacks} sends rational covers to steady localization covers [60, Proposition 2.3.2], so it naturally glues to a functor

$$(-)_{\blacksquare} : \{\text{analytic adic spaces}\} \to \{\text{analytic stacks}\}.$$

For the rest of this section, assume that R is sousperfectoid, and write S for $\mathrm{Spa}(R, R^+)$. Then for any smooth scheme X over R, the adic space X^{an}_S as in Definition 1.1 exists by [26, Proposition IV.4.9].

Theorem. *For any smooth proper scheme X over R, the functor*

$$\{\textit{vector bundles on } X\} \to \{\textit{vector bundles on } X_S^{\mathrm{an}}\}$$

is an equivalence of categories.

Proof. By covering X^{an} by small enough steady subspaces of the form $\mathrm{AnSpec}\, A_{|\underline{f}|\leq 1}$, we see that X^{an} is isomorphic to $(X_S^{\mathrm{an}})_{\blacksquare}$. Then Theorem A.18 yields a natural equivalence $D_{\mathrm{pc}}(X) \xrightarrow{\sim} D_{\mathrm{pc}}((X_S^{\mathrm{an}})_{\blacksquare})$, hence a natural equivalence between the full subcategories consisting of perfect objects C as in [14, Proposition 9.2] such that $- \otimes C$ is t-exact.

For $D_{\mathrm{pc}}(X)$, [69, Tag 0658] shows that this subcategory is naturally equivalent to $\{\text{vector bundles on } X\}$. For $D_{\mathrm{pc}}((X_S^{\mathrm{an}})_{\blacksquare})$, applying [1, Theorem 5.50] and [69, Tag 0658] to the above cover of $X^{\mathrm{an}} \cong (X_S^{\mathrm{an}})_{\blacksquare}$ by objects of the form $\mathrm{AnSpec}\, A_{|f|\leq 1}$ shows that this subcategory is naturally equivalent to $\{\text{vector bundles on } X_S^{\mathrm{an}}\}$. □

A.20. As an application of Theorem A.19, we conclude by proving that the analytification of algebraic stacks of bundles agrees with analytic stacks of bundles. This answers a question of Heuer–Xu [41, Remark 8.1.2].

Let K be an algebraically closed nonarchimedean field over $\mathbb{Q}_p$, and write Perfd_K for the category of affinoid perfectoid spaces over $\mathrm{Spa}\, K$. Endow Perfd_K with the v-topology. Recall from [41, Definition 8.4.4] the diamondification functor

$$(-)^{\diamondsuit} : \{\text{algebraic stacks over } K\} \to \{\text{small v-stacks on } \mathrm{Perfd}_K\}.$$

Let X be a smooth proper scheme over K, and let G be a linear algebraic group over K. Write $\mathcal{B}_{G,X}$ and $\mathcal{H}_{G,X}$ for the algebraic stacks over K of G-bundles and G-Higgs bundles on X, respectively. Write $\mathrm{Bun}_{G,X}$ and $\mathrm{Hig}_{G,X}$ for the small v-stacks on Perfd_K as in [40, Definition 7.12.(1)] and [40, Definition 7.12.(2)], respectively.

Theorem. *The natural morphisms of v-stacks on* Perfd_K

$$\mathcal{B}_{G,X}^{\diamondsuit} \to \mathrm{Bun}_{G,X} \quad \textit{and} \quad \mathcal{H}_{G,X}^{\diamondsuit} \to \mathrm{Hig}_{G,X}$$

are isomorphisms.

Proof. This follows immediately from Theorem A.19. □

Appendix B. ℓ-adic realizations of Berkovich motives

In §5 and §6, we use Berkovich (i.e. overconvergent) motivic sheaves as in [64, 65] because they enjoy good formal properties. For example, there are !-pushforwards, and when evaluated on condensed anima, they can be described in terms of classical sheaves on topological spaces. We start this section by recalling this material.

Difficulties arise when we need to compare Berkovich motives for (analytifications of) moduli stacks of global shtukas $\mathrm{Sht}_{G,V}^{I}$ with their classical ℓ-adic sheaves. Ideally, we would start from a motivic intersection cohomology sheaf on $\mathrm{Sht}_{G,V}^{I}$, analytify it, and then pass to ℓ-adic realizations. However, such motivic intersection cohomology sheaves are only known when G is split [59], and one would still need to compare Berkovich motives with classical motives. To avoid this, we develop some tools that keep track of the relationship between ℓ-adic sheaves on varieties, ℓ-adic sheaves on their analytifications, and motivic sheaves on said analytifications.

We conclude this section by recalling the description of motivic sheaves on $\breve{F}_v$ and Div_v^1 and describing their relationship with motivic nearby cycles.

Notation. In this section, we work over $\overline{\mathbb{F}}_q$.

B.1. We start by recalling the theory of *Berkovich motives* [64]. Endow the category $\{\text{perfectoid rings over } \overline{\mathbb{F}}_q\}^{\mathrm{op}}$ with the arc-topology as in [64, Definition 3.1], which is subcanonical by [64, Theorem 3.14]. Write $\{\text{arc-sheaves over } \overline{\mathbb{F}}_q\}$ for the ∞-category of sheaves of anima on $\{\text{perfectoid rings over } \overline{\mathbb{F}}_q\}^{\mathrm{op}}$, and recall from [64, p. 54] the $\mathbb{Z}[\frac{1}{q}]$-linear overconvergent motivic 6-functor formalism $D_{\mathrm{mot}}(-)$ on $\{\text{arc-sheaves over } \overline{\mathbb{F}}_q\}$.

B.2. When evaluating on stacks arising from topological spaces (i.e. *condensed anima*), Berkovich motives enjoy the following description. Recall from [42, Construction 3.5.16] the $\mathbb{Z}[\frac{1}{q}]$-linear 6-functor formalism $D(-,\mathbb{Z}[\frac{1}{q}])$ on {condensed anima}, and write

$$\underline{(-)} : \{\text{condensed anima}\} \to \{\text{arc-sheaves over } \overline{\mathbb{F}}_q\}$$

for the left Kan extension of the assignment $X \mapsto \underline{X}$ from profinite sets. Then the proof of [65, Proposition 2.1] implies we have a morphism of 3-functor formalisms

$$D(-,\mathbb{Z}[\tfrac{1}{q}]) \to D_{\mathrm{mot}}(\underline{(-)})$$

on {condensed anima} with respect to !-able morphisms for $D(-,\mathbb{Z}[\frac{1}{q}])$.

Proposition. *This induces an isomorphism of* 6*-functor formalisms*

$$D(-,\mathbb{Z}[\tfrac{1}{q}]) \otimes_{D(\mathbb{Z}[\frac{1}{q}])} D_{\mathrm{mot}}(*) \xrightarrow{\sim} D_{\mathrm{mot}}(\underline{(-)})$$

on {*condensed anima*} *with respect to* !*-able morphisms for* $D(-,\mathbb{Z}[\frac{1}{q}])$.

Proof. When evaluating on finite sets, this is immediate. For any cofiltered limit $X = \varprojlim_i X_i$ of finite sets, we have $\underline{X} = \varprojlim_i \underline{X_i}$, so [64, Lemma 10.4] shows that

$$\begin{aligned} D(X,\mathbb{Z}[\tfrac{1}{q}]) \otimes_{D(\mathbb{Z}[\frac{1}{q}])} D_{\mathrm{mot}}(*) &= \big[\varinjlim_i D(X_i,\mathbb{Z}[\tfrac{1}{q}])\big] \otimes_{D(\mathbb{Z}[\frac{1}{q}])} D_{\mathrm{mot}}(*) \\ &= \varinjlim_i \big[D(X_i,\mathbb{Z}[\tfrac{1}{q}]) \otimes_{D(\mathbb{Z}[\frac{1}{q}])} D_{\mathrm{mot}}(*)\big] \\ &= \varinjlim_i D_{\mathrm{mot}}(\underline{X_i}) = D_{\mathrm{mot}}(\underline{X}). \end{aligned}$$

Finally, the result follows from the uniqueness in [65, Theorem 3.4.11]. □

B.3. Berkovich motives are related to étale sheaves as follows. Write

$$a'^* : \{\text{small v-stacks over } \overline{\mathbb{F}}_q\} \to \{\text{arc-sheaves over } \overline{\mathbb{F}}_q\}$$

for the left Kan extension of the assignment $\mathrm{Spa}(R,R^+) \mapsto \mathrm{Hom}_{\overline{\mathbb{F}}_q}(R,-)$ [65, p. 3]. For all positive integers m and any small v-stack X over $\overline{\mathbb{F}}_q$, [64, Proposition 12.3] yields a natural fully faithful symmetric monoidal functor

$$\mathrm{Mod}_{\mathbb{Z}/\ell^m}(D_{\mathrm{mot}}(a'^*X)) \hookrightarrow D_{\text{ét}}(X,\mathbb{Z}/\ell^m)$$

that is compatible with pullback and whose essential image consists of overconvergent objects as in [26, Proposition IV.2.4].

Using this, one can define ℓ-adic realization functors as follows. Write $\widehat{D}_{\text{ét}}(X,\mathbb{Z}_\ell)$ for $\varprojlim_{m\geq 1} D_{\text{ét}}(X,\mathbb{Z}/\ell^m)$, so that taking $\varprojlim_{m\geq 1}$ of the above induces a functor

$$\widehat{r}_{\ell,X} : D_{\mathrm{mot}}(a'^*X) \to \varprojlim_{m\geq 1} \mathrm{Mod}_{\mathbb{Z}/\ell^m}(D_{\mathrm{mot}}(a'^*X)) \hookrightarrow \widehat{D}_{\text{ét}}(X,\mathbb{Z}_\ell).$$

Write $\widehat{D}(\mathbb{Z}_\ell)$ for $\varprojlim_{m\geq 1} D(\mathbb{Z}/\ell^m)$. Then the natural functor $D_{\mathrm{perf}}(\mathbb{Z}_\ell)\to\widehat{D}(\mathbb{Z}_\ell)$ is fully faithful [10, Lemma 4.2], and [26, p. 168] identifies $\widehat{D}_{\text{ét}}(*,\mathbb{Z}_\ell)$ with $\widehat{D}(\mathbb{Z}_\ell)$. Under this identification, [64, Theorem 11.1] shows that $\widehat{r}_{\ell,*}$ restricts to a functor $D_{\mathrm{mot}}(*)^\omega\to D_{\mathrm{perf}}(\mathbb{Z}_\ell)$, so [64, Theorem 11.1] indicates that applying Ind yields a functor $r_{\ell,*}: D_{\mathrm{mot}}(*)\to D(\mathbb{Z}_\ell)$.

B.4. For the rest of this section, we will use the following base change of Berkovich motives. For any small v-stack X over $\overline{\mathbb{F}}_q$, write $D_{\mathrm{mot}}(X)$ for $D_{\mathrm{mot}}(a'^*X)$.

Let Λ be a $\mathbb{Z}_\ell$-algebra, and consider the composition

$$D_{\mathrm{mot}}(*)\xrightarrow{r_{\ell,*}}(\mathbb{Z}_\ell)\longrightarrow D(\Lambda).$$

Write $D(X,\Lambda)$ for the symmetric monoidal ∞-category $D_{\mathrm{mot}}(X)\otimes_{D_{\mathrm{mot}}(*)}D(\Lambda)$, so that $D(-,\Lambda)$ is a Λ-linear 6-functor formalism on $\{$small v-stacks over $\overline{\mathbb{F}}_q\}$.

As a consequence of Proposition B.2, evaluating $D(-,\Lambda)$ on classifying stacks yields categories of smooth representations. More precisely, let G be a locally pro-p group, let K be a compact open subgroup of G, and write $j: */\underline{K}\to */\underline{G}$ for the associated morphism of small v-stacks.

Corollary. *The symmetric monoidal ∞-category $D(*/\underline{G},\Lambda)$ is naturally equivalent to the derived category of smooth representations of G over Λ. Under this identification, $j_!: D(*/\underline{K},\Lambda)\to D(*/\underline{G},\Lambda)$ corresponds to* c-Ind_K^G.

Proof. View G as a condensed set, and consider the condensed anima $*/G$. Then the arc-sheaf $*/G$ is naturally isomorphic to $a'^*(*/\underline{G})$, so the result follows from Proposition B.2, [42, Proposition 5.3.10], and [42, Proposition 5.4.4]. □

B.5. To facilitate comparisons with classical étale ℓ-adic sheaf theories for varieties over $\overline{\mathbb{F}}_q$ in §5, we will consider the following notion of *Zariski-constructible* motives (or étale sheaves) relative to a base. Let Z be a separated finite type scheme over $\overline{\mathbb{F}}_q$, and let C be a small v-stack over $Z^\diamondsuit$.

Definition. For any separated finite type scheme X over Z,

a) Write $D_{\mathrm{mot}}^{Z\backslash C}(X_C^\diamondsuit)\subseteq D_{\mathrm{mot}}(X_C^\diamondsuit)$ for the full subcategory generated under cones and retracts by objects of the form $f_{C,!}^\diamondsuit\mathbb{Z}[\frac{1}{q}](n)$, where $f: Y\to X$ runs over separated finite type morphisms and n runs over integers.
b) For all positive integers m, write $D_{\text{ét}}^{Z\backslash C}(X_C^\diamondsuit,\mathbb{Z}/\ell^m)\subseteq D_{\text{ét}}(X_C^\diamondsuit,\mathbb{Z}/\ell^m)$ for the full subcategory generated under cones and retracts by objects of the form $f_{C,!}^\diamondsuit(\mathbb{Z}/\ell^m)$, where $f: Y\to X$ runs over separated finite type morphisms.
c) Write $D_{\text{ét}}^{Z\backslash C}(X_C^\diamondsuit,\mathbb{Z}_\ell)\subseteq\widehat{D}_{\text{ét}}(X_C^\diamondsuit,\mathbb{Z}_\ell)$ for the full subcategory

$$\varprojlim_{m\geq 1} D_{\text{ét}}^{Z\backslash C}(X_C^\diamondsuit,\mathbb{Z}/\ell^m).$$

B.6. One immediately has the following relationship between étale sheaves on varieties over $\overline{\mathbb{F}}_q$ and étale sheaves on their analytifications. Note that $D_{\mathrm{mot}}((-)_C^\diamondsuit)$, $\widehat{D}_{\text{ét}}((-)_C^\diamondsuit,\mathbb{Z}_\ell)$ and $D(-,\mathbb{Z}_\ell)$ are 6-functor formalisms on

$$\{\text{separated finite type schemes over } Z\}.$$

Now [62, Proposition 27.1], [62, Proposition 27.2], and [62, Proposition 27.4] yield a fully faithful morphism of 3-functor formalisms

$$c^*: D(-,\mathbb{Z}_\ell)\hookrightarrow\widehat{D}_{\text{ét}}((-)^\diamondsuit,\mathbb{Z}_\ell)$$

on {separated finite type schemes over Z}, and proper base change shows that pull-back induces a morphism of 3-functor formalisms

$$|_C : \widehat{D}_{\text{ét}}((-)^{\Diamond}, \mathbb{Z}_\ell) \to \widehat{D}_{\text{ét}}((-)^{\Diamond}_C, \mathbb{Z}_\ell)$$

on {separated finite type schemes over Z}.

B.7. One also has the following relationship between Zariski-constructible motives and étale sheaves.

Proposition.

1) $D_{\text{mot}}((-)^{\Diamond}_C)$ restricts to a 3-functor formalism $D^{Z\setminus C}_{\text{mot}}((-)^{\Diamond}_C)$.
2) $\widehat{D}_{\text{ét}}((-)^{\Diamond}_C, \mathbb{Z}_\ell)$ restricts to a 3-functor formalism $D^{Z\setminus C}_{\text{ét}}((-)^{\Diamond}_C, \mathbb{Z}_\ell)$.
3) The composition $|_C \circ c^*$ sends $D_{\text{cons}}(-, \mathbb{Z}_\ell)$ to $D^{Z\setminus C}_{\text{ét}}((-)^{\Diamond}_C, \mathbb{Z}_\ell)$.
4) The functor $\widehat{r}_{\ell,(-)^{\Diamond}_C}$ sends $D^{Z\setminus C}_{\text{mot}}((-)^{\Diamond}_C)$ to $D^{Z\setminus C}_{\text{ét}}((-)^{\Diamond}_C, \mathbb{Z}_\ell)$ and induces a morphism of 3-functor formalisms on {separated finite type schemes over Z}.

Proof. Part 1) and part 2) follow immediately from proper base change. For part 3), recall that $D_{\text{cons}}(X, \mathbb{Z}/\ell^m)$ is generated under cones and retracts by objects of the form $f_!(\mathbb{Z}/\ell^m)$, where $f : Y \to X$ runs over separated finite type morphisms. Therefore B.6 and proper base change show that $|_C \circ c^*$ sends $D_{\text{cons}}(X, \mathbb{Z}/\ell^m)$ to $D^{Z\setminus C}_{\text{ét}}((-)^{\Diamond}_C, \mathbb{Z}/\ell^m)$. Finally, taking $\varprojlim_{m\geq 1}$ yields the desired result.

For the first statement in part 4), let $f : Y \to X$ be a separated finite type morphism, and let n be an integer. By Nagata compactification, there exists a proper morphism $\overline{f} : \overline{Y} \to X$ and an open embedding $j : Y \to \overline{Y}$ such that $f = \overline{f} \circ j$. Then $j^{\Diamond}_C$ is a partially proper open embedding, so the image of $j^{\Diamond}_{C,!}\mathbb{Z}[\frac{1}{q}](n)$ in $D_{\text{ét}}(\overline{Y}^{\Diamond}_C, \mathbb{Z}/\ell^m)$ is isomorphic to $j^{\Diamond}_{C,!}(\mathbb{Z}/\ell^m)$. Because $f^{\Diamond}_C$ is proper and of finite cohomological dimension as in [64, Definition 4.17], we see from [64, Proposition 12.3] that the image of $f^{\Diamond}_{C,!}\mathbb{Z}[\frac{1}{q}](n) = \overline{f}^{\Diamond}_{C,*}j^{\Diamond}_{C,!}\mathbb{Z}[\frac{1}{q}](n)$ in $D_{\text{ét}}(X^{\Diamond}_C, \mathbb{Z}/\ell^m)$ is isomorphic to $\overline{f}^{\Diamond}_{C,*}j^{\Diamond}_{C,!}(\mathbb{Z}/\ell^m) = f^{\Diamond}_{C,!}(\mathbb{Z}/\ell^m)$. Hence taking $\varprojlim_{m\geq 1}$ yields the desired statement. Finally, the second statement in part 4) follows from B.3 and the above argument. □

B.8. Finally, we Ind-extend the preceding discussion. Proposition B.7.2) implies that Ind-extending yields a morphism

$$\Upsilon : \operatorname{Ind} D^{Z\setminus C}_{\text{mot}}((-)^{\Diamond}_C) \to D_{\text{mot}}((-)^{\Diamond}_C)$$

of 3-functor formalisms on {separated finite type schemes over Z}, and Proposition B.7.3) indicates that applying Ind to $|_C \circ c^* : D_{\text{cons}}(-, \mathbb{Z}_\ell) \to D^{Z\setminus C}_{\text{ét}}((-)^{\Diamond}_C, \mathbb{Z}_\ell)$ yields a morphism of 3-functor formalisms

$$\rho : D(-, \mathbb{Z}_\ell) \to \operatorname{Ind} D^{Z\setminus C}_{\text{ét}}((-)^{\Diamond}_C, \mathbb{Z}_\ell)$$

on {separated finite type schemes over Z}. Finally, Proposition B.7.4) shows that $\widehat{r}_{\ell,(-)^{\Diamond}_C}$ induces a morphism of 3-functor formalisms

$$D^{Z\setminus C}_{\text{mot}}((-)^{\Diamond}_C) \to D^{Z\setminus C}_{\text{ét}}((-)^{\Diamond}_C, \mathbb{Z}_\ell)$$

on {separated finite type schemes over Z}, so applying Ind yields a morphism

$$r_{\ell,(-)^{\Diamond}_C} : \operatorname{Ind} D^{Z\setminus C}_{\text{mot}}((-)^{\Diamond}_C) \to \operatorname{Ind} D^{Z\setminus C}_{\text{ét}}((-)^{\Diamond}_C, \mathbb{Z}_\ell)$$

of 3-functor formalisms on {separated finite type schemes over Z}.

B.9. Write F for $\mathbb{F}_q(\!(\pi)\!)$, fix a separable closure $\overline{F}$ of F, and write C for the completion of $\overline{F}$. Then there is a simple description of Zariski-constructible étale sheaves on $\operatorname{Spa} C$ (relative to any Z). More precisely, $\widehat{D}_{\text{ét}}(\operatorname{Spa} C, \mathbb{Z}_\ell)$ is naturally isomorphic to $\widehat{D}(\mathbb{Z}_\ell)$, and under this identification, B.6 implies that

$$D_{\text{ét}}^{Z\setminus \operatorname{Spa} C}(\operatorname{Spa} C, \mathbb{Z}_\ell) = D_{\text{perf}}(\mathbb{Z}_\ell).$$

We prove an analogous description of Zariski-constructible motives on $\operatorname{Spa} C$:

Proposition. *We have* $D_{\text{mot}}^{Z\setminus \operatorname{Spa} C}(\operatorname{Spa} C) = D_{\text{mot}}(\operatorname{Spa} C)^\omega$.

Proposition B.9 and [64, Proposition 10.1] imply that

$$\Upsilon : \operatorname{Ind} D_{\text{mot}}^{Z\setminus \operatorname{Spa} C}(\operatorname{Spa} C) \to D_{\text{mot}}(\operatorname{Spa} C)$$

is an equivalence.

Proof. By considering $f : Z \times X \to Z$ for smooth projective schemes X over $\overline{\mathbb{F}}_q$, [64, Proposition 10.1] shows that $D_{\text{mot}}(\operatorname{Spa} C)^\omega$ lies in $D_{\text{mot}}^{Z\setminus \operatorname{Spa} C}(\operatorname{Spa} C)$. For the reverse inclusion, it suffices to prove that, for any separated finite type morphism $f : Y \to \operatorname{Spec} C$, the object $f_!^{\text{an},\lozenge}\mathbb{Z}[\frac{1}{q}]$ in $D_{\text{mot}}(\operatorname{Spa} C)$ is compact. Since $(-)^{\text{an},\lozenge}$ factors through limit perfection, we can assume that Y is reduced.

We now induct on $\dim Y$, where the $\dim Y = 0$ case is immediate. By excision [64, Proposition 4.25] and the induction hypothesis, we can assume that Y is irreducible. Then [17, Theorem 4.1] yields a smooth projective morphism $\widetilde{f} : \widetilde{Y} \to \operatorname{Spec} C$, a dense open subspace $V \subseteq Y$, a dense open subspace $i : \widetilde{V} \hookrightarrow \widetilde{Y}$, and a finite étale morphism $j : \widetilde{V} \to V$ over C. By excision [64, Proposition 4.25] and the induction hypothesis, we can assume that $Y = V$. Write d for the product of all primes $\ell \neq p$ dividing $\deg j$. Because $(f_!^{\text{an},\lozenge}\mathbb{Z}[\frac{1}{q}]) \otimes_{\mathbb{Z}[\frac{1}{q}]} \mathbb{Z}/d^m = f_!^{\text{an},\lozenge}(\mathbb{Z}/d^m)$, spreading out implies that the image of $f_!^{\text{an},\lozenge}\mathbb{Z}[\frac{1}{q}]$ in

$$\varprojlim_{m\geq 1} \operatorname{Mod}_{\mathbb{Z}/d^m}(D_{\text{mot}}(\operatorname{Spa} C)) = \prod_{\ell \mid d} \widehat{D}(\mathbb{Z}_\ell)$$

is compact. Therefore it suffices to prove that the image $f_!^{\text{an},\lozenge}\mathbb{Z}[\frac{1}{dq}]$ of $f_!^{\text{an},\lozenge}\mathbb{Z}[\frac{1}{q}]$ in $\operatorname{Mod}_{\mathbb{Z}[\frac{1}{dq}]}(D_{\text{mot}}(\operatorname{Spa} C))$ is compact.

The proof of [64, Proposition 10.1] shows that $\widetilde{f}_!^{\text{an},\lozenge}\mathbb{Z}[\frac{1}{dq}]$ is compact in

$$\operatorname{Mod}_{\mathbb{Z}[\frac{1}{dq}]}(D_{\text{mot}}(\operatorname{Spa} C)),$$

so excision [64, Proposition 4.25] and the induction hypothesis imply that the same holds for $(\widetilde{f} \circ i)_!^{\text{an},\lozenge}\mathbb{Z}[\frac{1}{dq}]$. Finally, the trace map exhibits $f_!^{\text{an},\lozenge}\mathbb{Z}[\frac{1}{dq}]$ as a retract of $(\widetilde{f} \circ i)_!^{\text{an},\lozenge}\mathbb{Z}[\frac{1}{dq}]$, which yields the desired result. □

B.10. Write $\breve{F}$ for $\overline{\mathbb{F}}_q(\!(\pi)\!)$, write $\phi : \operatorname{Spd} \breve{F} \to \operatorname{Spd} \breve{F}$ for the geometric q-Frobenius automorphism over $\overline{\mathbb{F}}_q$, and write Div^1 for the small v-sheaf $(\operatorname{Spd} \breve{F})/\phi^{\mathbb{Z}}$ over $\overline{\mathbb{F}}_q$.

Let us recall the following descriptions of motives on $\operatorname{Spd} \breve{F}$ and Div^1. Fix a group isomorphism $\overline{\mathbb{F}}_q^\times \cong \mathbb{Q}/\mathbb{Z}[\frac{1}{q}]$, and choose a compatible system $\{\pi^{1/n}\}_{p\nmid n}$ of

n-th roots of π in $\overline{F}$. Recall that these choices induce a short exact sequence of certain group schemes over $\mathbb{Z}[\frac{1}{q}]$ [65, p. 10]

$$1 \longrightarrow \mathrm{ID}_F \longrightarrow \mathrm{WD}_F \longrightarrow \underline{\mathbb{Z}} \longrightarrow 1.$$

Lemma. *These choices induce compatible equivalences*

$$D(\mathrm{Spd}\,\breve{F}, \Lambda) \cong D_{\mathrm{qcoh}}\big((\mathrm{Spec}\,\Lambda)/\mathrm{ID}_F\big) \text{ and } D(\mathrm{Div}^1, \Lambda) \cong D_{\mathrm{qcoh}}\big((\mathrm{Spec}\,\Lambda)/\mathrm{WD}_F\big).$$

Proof. Write $\mathrm{MG}_{\overline{\mathbb{F}}_q}$ for the algebraic stack over $\mathbb{Z}[\frac{1}{q}]$ from [64, p. 62], and recall that we have a natural functor $D_{\mathrm{qcoh}}(\mathrm{MG}_{\overline{\mathbb{F}}_q}) \to D_{\mathrm{mot}}(*)$ [64, p. 62]. Consider the morphism $\mathrm{Spec}\,\mathbb{Z}[\frac{1}{q}] \to \mathrm{MG}_{\overline{\mathbb{F}}_q}$ induced by the isomorphism $\overline{\mathbb{F}}_q^\times \cong \mathbb{Q}/\mathbb{Z}[\frac{1}{q}]$ as in [65, p. 10]; note that the isomorphism $\overline{\mathbb{F}}_q^\times \cong \mathbb{Q}/\mathbb{Z}[\frac{1}{q}]$ also induces a commutative square

$$\begin{array}{ccc} D_{\mathrm{qcoh}}(\mathrm{MG}_{\overline{\mathbb{F}}_q}) & \longrightarrow & D_{\mathrm{mot}}(*) \\ \downarrow & & \downarrow{\scriptstyle r_{\ell,*}} \\ D(\mathbb{Z}[\frac{1}{q}]) & \longrightarrow & D(\mathbb{Z}_\ell). \end{array}$$

Write $\mathrm{MG}_{\breve{F}}$ and $\mathrm{MG}_{\mathrm{Div}^1}$ for the algebraic stacks over $\mathrm{MG}_{\overline{\mathbb{F}}_q}$ from [65, Definition 4.1]. Then [65, Theorem 4.2] and its proof yield compatible equivalences

$$D_{\mathrm{qcoh}}(\mathrm{MG}_{\breve{F}}) \otimes_{D_{\mathrm{qcoh}}(\mathrm{MG}_{\overline{\mathbb{F}}_q})} D_{\mathrm{mot}}(*) \xrightarrow{\sim} D_{\mathrm{mot}}(\mathrm{Spd}\,\breve{F}),$$
$$D_{\mathrm{qcoh}}(\mathrm{MG}_{\mathrm{Div}^1}) \otimes_{D_{\mathrm{qcoh}}(\mathrm{MG}_{\overline{\mathbb{F}}_q})} D_{\mathrm{mot}}(*) \xrightarrow{\sim} D_{\mathrm{mot}}(\mathrm{Div}^1).$$

so the above commutative square induces compatible equivalences

$$D_{\mathrm{qcoh}}(\mathrm{MG}_{\breve{F}} \times_{\mathrm{MG}_{\overline{\mathbb{F}}_q}} \mathrm{Spec}\,\Lambda) \xrightarrow{\sim} D(\mathrm{Spd}\,\breve{F}, \Lambda),$$
$$D_{\mathrm{qcoh}}(\mathrm{MG}_{\mathrm{Div}^1} \times_{\mathrm{MG}_{\overline{\mathbb{F}}_q}} \mathrm{Spec}\,\Lambda) \xrightarrow{\sim} D(\mathrm{Div}^1, \Lambda).$$

Finally, [65, p. 10] compatibly identifies

$$\mathrm{MG}_{\breve{F}} \times_{\mathrm{MG}_{\overline{\mathbb{F}}_q}} \mathrm{Spec}\,\Lambda \cong (\mathrm{Spec}\,\Lambda)/\mathrm{ID}_F,$$
$$\mathrm{MG}_{\mathrm{Div}^1} \times_{\mathrm{MG}_{\overline{\mathbb{F}}_q}} \mathrm{Spec}\,\Lambda \cong (\mathrm{Spec}\,\Lambda)/\mathrm{WD}_F,$$

and we conclude by applying [31, Chapter 3, Proposition 3.5.3]. □

B.11. We conclude this section by describing how *motivic nearby cycles* behave after ℓ-adic realization. Write $f : \mathrm{Spa}\,C \to *$ for the structure morphism. Recall that f_* yields an equivalence $D_{\mathrm{mot}}(\mathrm{Spa}\,C) \xrightarrow{\sim} \mathrm{Mod}_{f_*\mathbb{Z}[\frac{1}{q}]}(D_{\mathrm{mot}}(*))$ and that our choice of $\{\pi^{1/n}\}_{p\nmid n}$ from B.10 induces a morphism $f_*\mathbb{Z}[\frac{1}{q}] \to \mathbb{Z}[\frac{1}{q}]$ of $\mathbb{E}_\infty$-algebras in $D_{\mathrm{mot}}(*)$ [64, p. 61]. Under this identification, write $\Psi : D_{\mathrm{mot}}(\mathrm{Spa}\,C) \to D_{\mathrm{mot}}(*)$ for the functor corresponding to $- \otimes_{f_*\mathbb{Z}[\frac{1}{q}]} \mathbb{Z}[\frac{1}{q}]$ [64, Definition 11.3].

Recall that pullback yields an equivalence [26, p. 168]

$$\widehat{D}_{\text{ét}}(*, \mathbb{Z}_\ell) \xrightarrow{\sim} \widehat{D}_{\text{ét}}(\mathrm{Spa}\,C, \mathbb{Z}_\ell).$$

Proposition. *The functor Ψ preserves compact objects. Moreover, we have a commutative square*

$$\begin{array}{ccc} D_{\mathrm{mot}}(\operatorname{Spa} C) & \xrightarrow{\widehat{r}_{\ell,\operatorname{Spa} C}} & \widehat{D}_{\text{ét}}(\operatorname{Spa} C, \mathbb{Z}_\ell) \\ \downarrow{\scriptstyle \Psi} & & \uparrow{\scriptstyle \wr} \\ D_{\mathrm{mot}}(*) & \xrightarrow{\widehat{r}_{\ell,*}} & \widehat{D}_{\text{ét}}(*, \mathbb{Z}_\ell). \end{array}$$

Proposition B.11 and B.3 imply that $\widehat{r}_{\ell,\operatorname{Spa} C}$ restricts to a functor

$$D_{\mathrm{mot}}(\operatorname{Spa} C)^\omega \to D_{\mathrm{perf}}(\mathbb{Z}_\ell),$$

so [64, Proposition 10.1] indicates that applying Ind yields a functor

$$r_{\ell,\operatorname{Spa} C} : D_{\mathrm{mot}}(\operatorname{Spa} C) \to D(\mathbb{Z}_\ell).$$

Proof. Because $f^* : D_{\mathrm{mot}}(*) \to D_{\mathrm{mot}}(\operatorname{Spa} C)$ is left adjoint to f_*, it corresponds to $- \otimes_{\mathbb{Z}[\frac{1}{q}]} f_* \mathbb{Z}[\frac{1}{q}]$. Therefore $\Psi \circ f^*$ is naturally isomorphic to the identity, so the first statement follows from [64, Proposition 10.1] and [64, Theorem 11.1].

For the second statement, the proof of [64, Proposition 11.4] implies that, for all positive integers m, applying $- \otimes_{\mathbb{Z}[\frac{1}{q}]} \mathbb{Z}/\ell^m$ to $f_* \mathbb{Z}[\frac{1}{q}] \to \mathbb{Z}[\frac{1}{q}]$ gives an isomorphism. This yields the desired result. □

Appendix C. Global rigid inner forms, by Peter Dillery

In the following, G always denotes a connected reductive group over F a global function field associated to the projective curve C with constant field $\mathbb{F}_q$. We will break from the notation of the rest of the paper by denoting F^s a separable closure of F inside a fixed algebraic closure $\overline{F}$. The set of all places of F will be denoted by V_F. We denote the Kottwitz gerbe over F by Kott_F, and the Kottwitz gerbe over each F_v by Kott_v. For a set of places Σ of F, we denote by $O_{F,\Sigma}$ the ring of Σ-integers of F.

C.1. Theorem 4.11 says that the $B(F,G)_{\mathrm{basic}}$-inner forms G_b of a quasi-split connected reductive group G appear naturally in the description of the semistable locus of $\mathrm{Bun}_{G,F}$. The goal of this appendix is to use results in the main body of the paper to prove fundamental geometric properties of the "extended" stack $\mathrm{Bun}^e_{G,F}$ introduced in [21], which has geometry analogous to that of $\mathrm{Bun}_{G,F}$ but whose semistable locus encodes global rigid inner forms of G, in the sense of [20], rather than just $B(F,G)_{\mathrm{basic}}$-inner forms. The advantage of using global rigid inner forms is that they enable a statement ([21, Conjecture 5.7]) of a multiplicity formula for discrete automorphic representations that applies to arbitrary connected reductive groups over F. It is reasonable to believe that a $\mathrm{Bun}^e_{G,F}$-adapted version of the machinery in the main part of this paper together with the conjectural $\mathrm{Kott}_v \times_{F_v} \mathrm{Kal}_v$-adapted refined Langlands correspondence from [21, Conjecture 5.1] at each v can be used to prove cases of this multiplicity formula. We emphasize that the construction presented here is inspired by the work of [23] in the local case.

C.2. We review the notion of local and global rigid inner forms. One can use class field theory to construct a global gerbe $\mathrm{Kal}_F \to \mathrm{Spec}(F)$ ([20, Proposition 3.21]) and for each place $v \in V_F$ a local gerbe $\mathrm{Kal}_v \to \mathrm{Spec}(F_v)$ ([19, Theorem 3.4]) which are banded by profinite multiplicative group schemes P and u_v, respectively. We refer the reader to [21, §2.1] for a more detailed summary of these two gerbes and their bands. We give these gerbes the étale topology inherited from $\mathrm{Spec}(F)$.

For $? \in \{F, v\}$, the basic cohomology $H^1(\mathrm{Kal}_?, G)_{\mathrm{basic}}$ is defined as all isomorphism classes of étale G-torsors on $\mathrm{Kal}_?$ whose restriction to the band of $\mathrm{Kal}_?$ is an F-rational homomorphism $P \to Z_G$ or $u_v \to Z_G$. A key property of these gerbes is that the natural map

$$H^1(\mathrm{Kal}_?, G)_{\mathrm{basic}} \to H^1(F, G_{\mathrm{ad}})$$

is always surjective. In other words, any inner form of G can be realized as coming from a G-torsor on one of these gerbes.

There is a variant of this gerbe defined in [23] and [21] which is better–suited to geometry:

Definition. Define $B_e(G)$ (resp. $B_e(G_{F_v})$) to be all isomorphism classes of étale G-torsors on the gerbe $\mathrm{Kott}_F \times_F \mathrm{Kal}_F$ (resp. $\mathrm{Kott}_v \times_{F_v} \mathrm{Kal}_v$) whose restriction to the band of Kal_F (resp. Kal_{F_v}) factors through Z_G. Define the subset $B_e(G)_{\mathrm{basic}}$ (resp. $B_e(G_{F_v})_{\mathrm{basic}}$) to be those classes whose restriction to the entire band of the gerbe $\mathrm{Kott}_F \times_F \mathrm{Kal}_F$ (resp. $\mathrm{Kott}_v \times_{F_v} \mathrm{Kal}_v$) factors through Z_G.

In the same way that there is a bijection $B(F, G)_{\mathrm{basic}} \xrightarrow{\sim} (\pi_1(G)[V_{F^s}]_0)_{\mathrm{Gal}(F^s/F)}$, one has a linear-algebraic description of $B_e(G)_{\mathrm{basic}}$ given by [21, Theorem 3.10] which recovers the $B(F, G)$-bijection and the $H^1(\mathrm{Kal}_F, G)_{\mathrm{basic}}$-bijection from [20, Theorem 4.11]. We omit the details here for sake of brevity, but remark that the proof relies on an explicit Tannakian description [21, Proposition 3.1] of the gerbe $\mathrm{Kott}_F \times_F \mathrm{Kal}_F$ generalizing the description of Kott_F using Drinfeld isoshtukas.

C.3. The key ingredient for defining the stack $\mathrm{Bun}^e_{G,F}$ is a generalization of Kal_F which is defined over a fixed affine open subset $U = C \setminus |\Sigma|$ of the curve C for a finite set of places Σ. This gerbe is denoted by $\mathrm{Kal}_{F,\Sigma} \to U$.

The gerbe $\mathrm{Kal}_{F,\Sigma}$ is built from finite level gerbes. Fix a set of lifts $\dot{\Sigma}$ of Σ in V_{F^s} and denote by F_Σ the maximal Σ-unramified extension of F. Fix also a cofinal system of pairs $\{(E_i, n_i)\}_{i \in \mathbb{N}}$ such that $n_i \in \mathbb{N}$, $F \subseteq E_i \subseteq F_\Sigma$ and if $j \geq i$ then $E_i \subseteq E_j$ and $n_i \mid n_j$. Define $P_{\dot{\Sigma},i}$ to be the $O_{F,\Sigma}$-group scheme Cartier dual to $(\frac{1}{n_i}\mathbb{Z}/\mathbb{Z})[\mathrm{Gal}(E_i/F) \times \Sigma_{F_i}]_{0,0}$ consisting of all elements whose $[(\sigma, w)]$-coefficient is zero unless $\sigma^{-1}(w) \in \dot{\Sigma}_{E_i}$. The subscript "$0, 0$" means we insist that the elements must be killed by both augmentation maps. There is ([21, Corollary 2.12]) a "level i" canonical class $\xi_i \in H^2_{\mathrm{fppf}}(O_{F,\Sigma}, P_{\dot{\Sigma},i})$, and these classes are compatible via a system of transition maps $\{P_{\dot{\Sigma},j} \to P_{\dot{\Sigma},i}\}$ defined loc. cit. such that $P_{\dot{\Sigma}} = \varprojlim P_{\dot{\Sigma},i}$.

To define the canonical class corresponding to $\mathrm{Kal}_{F,\Sigma}$, one takes the unique lift of $\varprojlim \xi_i$ in $H^2_{\mathrm{fppf}}(F, P_{\dot{\Sigma}})$ which, for each $v \in V_F$, localizes in some suitable sense to the class in $H^2_{\mathrm{fppf}}(F_v, u_v)$ corresponding to Kal_v—for the full details see [21, Theorem 2.16]. One can define a transition map $\mathrm{Kal}_{F,\Sigma'} \to \mathrm{Kal}_{F,\Sigma}$ for all $\Sigma \subseteq \Sigma'$ and pullback by these maps induces a bijection

$$\varinjlim_{\Sigma} H^1_{\text{ét}}(\mathrm{Kal}_{F,\Sigma}, G) \xrightarrow{\sim} H^1_{\text{ét}}(\mathrm{Kal}_F, G).$$

C.4. We now recall the definition of the stack $\mathrm{Bun}^e_{G,F}$ from [21, §4], which is inspired by the ideas of [23]. Each gerbe $\mathrm{Kal}_{F,\Sigma}$ defines ([21, Lemma 4.2]) an adic gerbe $\mathrm{Kal}^{\mathrm{ad}}_{F,\Sigma} \to \mathrm{Spa}(O_{F,\Sigma})$ (for the v-topology), and, for an affinoid perfectoid space S over $\mathbb{F}_q$, define $H^1_{\text{ét}}(\mathfrak{U}_S, G)_{\text{Kal-basic}}$ as all isomorphism classes of étale G-torsors $\mathscr{G}$ on $\mathfrak{U}_S := U^{\mathrm{an}}_S \times_{\mathrm{Spa}(O_{F,\Sigma})} \mathrm{Kal}^{\mathrm{ad}}_{F,\Sigma}$ whose restriction to the band is a morphism

$$(P^{\mathrm{ad}}_{\dot\Sigma})_S \xrightarrow{\lambda_{\mathscr{G}}} Z_{G,S}.$$

We denote all such torsors (not isomorphism classes) by $Z^1_{\text{ét}}(\mathfrak{U}_S, G)_{\mathrm{Kal}_F\text{-basic}}$.

The main definition is then ([21, Definition 4.6]):

Definition.

(1) Given U a dense affine open of C, define a functor $\mathrm{Bun}^e_{G,U}$ from affinoid perfectoid spaces over $\mathbb{F}_q$ to groupoids by

$$S \mapsto \{(\mathscr{G}, \phi) | \mathscr{G} \in Z^1_{\text{ét}}(\mathfrak{U}_S, G)_{\mathrm{Kal}_F\text{-basic}},\ \ \phi\colon \mathscr{G} \xrightarrow{\sim} (\mathrm{Frob}_S \times \mathrm{id})^*\mathscr{G}\};$$

(2) Define a functor $\mathrm{Bun}^e_{G,F}$ from affinoid perfectoid spaces over $\mathbb{F}_q$ to groupoids by

$$S \mapsto \varinjlim_U \mathrm{Bun}^e_{G,U}(S).$$

It is proved in [21, Lemma 4.7] that both of these functors are small v-stacks.

C.5. The assignment $(\mathscr{G}, \phi) \in \mathrm{Bun}^e_{G,U}(S) \mapsto \lambda_{\mathscr{G}}$ defines a map

$$\mathrm{Bun}^e_{G,U} \xrightarrow{\Lambda} \underline{\mathrm{Hom}}_U(P_{\dot\Sigma}, Z_G),$$

where $\underline{\mathrm{Hom}}_U(P_{\dot\Sigma}, Z_G)$ is the functor sending S to $\mathrm{Hom}_{U^{\mathrm{an}}_S}((P^{\mathrm{ad}}_{\dot\Sigma})_S, Z_{G,S})$. We now recall one more definition [21, Definition 4.11].

Observe that we have a canonical isomorphism for every geometric point $\bar s$ of a perfectoid space S over $\mathbb{F}_q$

$$\mathrm{Hom}_U(P_{\dot\Sigma}, Z_G) \xrightarrow{\sim} \mathrm{Hom}_{U^{\mathrm{an}}_{\bar s}}((P^{\mathrm{ad}}_{\dot\Sigma})_{\bar s}, Z_{G,\bar s})^{\mathrm{Frob}_{\bar s}}.$$

In view of this, given $\lambda \in \mathrm{Hom}(P^{\mathrm{ad}}_{\dot\Sigma}, Z_G)$, define $\mathrm{Bun}^{e,\lambda}_{G,U}$ as the substack characterized by

$$\mathrm{Bun}^{e,\lambda}_{G,U}(S) = \{x \in \mathrm{Bun}^e_{G,U}(S) | \lambda_{\bar x} = \lambda \in \mathrm{Hom}_{U^{\mathrm{an}}_{\bar s}}((P^{\mathrm{ad}}_{\dot\Sigma})_{\bar s}, Z_{G,\bar s})^{\mathrm{Frob}_{\bar s}}\},$$

where $\bar s$ runs over all geometric points of S and $\bar x$ denotes the image of x at $\bar s$.

A key result which allows us to use the results in the main body of paper in this new setting is [21, Corollary 4.13], which says that Λ induces a decomposition (as a v-stack)

$$\mathrm{Bun}^e_{G,U} \xrightarrow{\sim} \bigsqcup_{\lambda \in \mathrm{Hom}_F(P_{\dot\Sigma}, Z_G)} \mathrm{Bun}^{e,\lambda}_{G,U},$$

and analogously for $\mathrm{Bun}^e_{G,F}$.

There is an analogue [21, Definition 4.9] of the notions introduced in §2.9:

Definition.

(1) Define $\mathrm{Bun}^{e,1}_{G,F}$ to be the substack characterized by

$$\mathrm{Bun}^{e,1}_{G,F}(S) = \{x \in \mathrm{Bun}^e_{G,F}(S) | \bar x = \mathrm{triv} \in \mathrm{Bun}_{G,F}(\bar s)\},$$

where $\bar s$ runs over all geometric points of S and $\bar x$ denotes the image of x at $\bar s$.

(2) Given b an element of $B_e(G)_{\mathrm{basic}}$, define $\mathrm{Bun}^{e,b}_{G,F}$ as the substack characterized by $\mathrm{Bun}^{e,b}_{G,F}(S) = \{x \in \mathrm{Bun}^{e}_{G,F}(S) | \bar{x} = b \in \mathrm{Bun}_{G,F}(\bar{s})\}$, where $\bar{s}$ runs over all geometric points of S and $\bar{x}$ denotes the image of x at $\bar{s}$.

The identical twisting argument from the proof of Proposition 2.9 shows that:

Proposition.

(1) For $b \in B_e(G)_{\mathrm{basic}}$ pulled back from $\mathrm{Kott}_{F,\Sigma} \times_U \mathrm{Kal}_{F,\Sigma}$, there is a natural isomorphism $\mathrm{Bun}^{e}_{G,U} \xrightarrow{\sim} \mathrm{Bun}^{e}_{G_b,U}$ sending $\mathrm{Bun}^{e,b}_{G,U}$ to $\mathrm{Bun}^{e,1}_{G_b,U}$.

(2) For $b \in H^1(\mathrm{Kott}_{F,\Sigma} \times_U \mathrm{Kal}_{F,\Sigma})_{\mathrm{basic}}$ above a fixed $\lambda \in \mathrm{Hom}_U(P_{\dot{\Sigma}}, Z_G)$, we have an isomorphism $\mathrm{Bun}^{e,\lambda}_{G,U} \xrightarrow{\sim} \mathrm{Bun}_{G_b,U}$.

In the above $\mathrm{Kott}_{F,\Sigma}$ is a "U-level" version of Kott_F defined in [21, §2.3]. Combining this result with [21, Corollary 4.13] proves that the stack $\mathrm{Bun}^{e}_{G,U}$ is isomorphic to a disjoint union of the stacks $\mathrm{Bun}_{G',U}$ for varying inner forms G' of G (this is a global analogue of [23, Exemple 12.6]).

We therefore deduce from Proposition 4.5 that:

Corollary. *Each $\mathrm{Bun}^{e}_{G,U}$ is an Artin v-stack.*

C.6. We now prove some more refined results about the geometry of $\mathrm{Bun}^{e}_{G,F}$.

First, we deduce the following analogue of Theorem 2.10, which, in particular, identifies $\mathrm{Bun}^{e}_{G,U}(\mathrm{Spd}\,\overline{\mathbb{F}_q})$ with $B_e(G)$:

Theorem. *The perfect v-stack $\mathrm{Bun}^{e,\mathrm{red}}_{G,U}$ over $\mathbb{F}_q$ is the v-sheafification of*

$$\mathrm{Spec}\,B \mapsto \left\{ \begin{array}{c} \mathrm{Kal}_F\text{-basic } G\text{-torsors } \mathscr{G} \text{ on } U_B \times_U \mathrm{Kal}_{F,\Sigma} \text{ equipped with} \\ \text{an isomorphism } \phi : \mathscr{G} \xrightarrow{\sim} (\mathrm{Frob}_B \times \mathrm{id})^* \mathscr{G} \end{array} \right\}. \tag{1}$$

Moreover, when each connected component of $\mathrm{Spec}\,B$ is a valuation ring, no sheafification is needed.

Proof. [21, Corollary 4.13] implies that $\mathrm{Bun}^{e,\mathrm{red}}_{G,U} \xrightarrow{\sim} \bigsqcup_{\lambda \in \mathrm{Hom}_F(P_{\dot{\Sigma}}, Z_G)} \mathrm{Bun}^{e,\lambda,\mathrm{red}}_{G,U}$. We have seen in Proposition C.5 that picking $b \in H^1(\mathrm{Kott}_{F,\Sigma} \times \mathrm{Kal}_{F,\Sigma})_{\mathrm{basic}}$ above a fixed λ (we can do this by [21, Proposition 2.29]) gives an identification $\mathrm{Bun}^{e,\lambda}_{G,U} \xrightarrow{\sim} \mathrm{Bun}_{G_b,U}$. A similar (but easier) argument as in the proof of [21, Proposition 4.12] gives an analogous disjoint union decomposition of the functor (1) according to $\mathrm{Hom}_U(P_{\dot{\Sigma}}, Z_G)$, and twisting by b gives an isomorphism from the presheaf mapping $\mathrm{Spec}\,B$ to all $\mathscr{G}$ as in (1) such that $\lambda_{\mathscr{G}} = \lambda$ at all geometric points of B to the functor in $(*)$. We can then use Theorem 2.10 to deduce the result. □

C.7. This subsection is purely review from [21] and concerns the localization properties of the stack $\mathrm{Bun}^{e}_{G,F}$. First, we define for a place $v \in V_F$ the stack Bun^{e}_{G,F_v} to be the same as $\mathrm{Bun}^{e}_{G,U}$ but with F_v in place of $O_{F,\Sigma}$ and Kal_v in place of $\mathrm{Kal}_{F,\Sigma}$, as defined in [23, §12].

We recall from [21, §4.2] the functor $\mathrm{Bun}^{e}_{G,\mathbb{A}}$ defined by

$$S \mapsto \varinjlim_{\Sigma} \prod_{v \notin \Sigma} \mathrm{Bun}_{G,O_v}(S) \times \prod_{v \in \Sigma} \mathrm{Bun}^{e}_{G,F_v}(S),$$

where the limit is over all finite subsets of places Σ. This defines a small v-stack, and we have a morphism

$$\mathrm{Bun}^{e}_{G,F} \to \mathrm{Bun}^{e}_{G,\mathbb{A}}$$

induced by a localization map $\mathrm{Kal}_v \to \mathrm{Kal}_F$ at each place as discussed in [21, §2.1].

One then has [21, Proposition 4.16]:

Proposition. *There is a morphism*

$$\mathrm{Bun}^e_{G,U} \to \prod_{v\notin\Sigma} \mathrm{Bun}_{G,O_v} \times \prod_{v\in\Sigma} \mathrm{Bun}^{e,(\Sigma)}_{G,F_v} \subseteq \prod_{v\notin\Sigma} \mathrm{Bun}_{G,O_v} \times \prod_{v\in\Sigma} \mathrm{Bun}^e_{G,F_v}$$

which makes the square

$$\begin{array}{ccc} \mathrm{Bun}^e_{G,U} & \longrightarrow & \prod_{v\notin\Sigma} \mathrm{Bun}_{G,O_v} \times \prod_{v\in\Sigma} \mathrm{Bun}^{e,(\Sigma)}_{G,F_v} \\ \downarrow & & \downarrow \\ \mathrm{Bun}^e_{G,F} & \longrightarrow & \mathrm{Bun}^e_{G,\mathbb{A}} \end{array}$$

Cartesian.

In the above, $\mathrm{Bun}^{e,(\Sigma)}_{G,F_v}$ is a minor modification of the stack Bun^e_{G,F_v} adapted to the completion of F_Σ at a fixed place above v, see [21, §4.2] for the full details.

C.8. There is an analogue of the semistable locus of $\mathrm{Bun}_{G,F}$:

Definition. Set $\mathrm{Bun}^{e,\mathrm{ss}}_{G,F}$ to be the preimage of

$$\varinjlim \prod_{v\notin\Sigma} \mathrm{Bun}_{G,O_v} \times \prod_{v\in\Sigma} \mathrm{Bun}^{e,\mathrm{ss}}_{G,F_v}$$

in $\mathrm{Bun}^e_{F,G}$, where $\mathrm{Bun}^{e,\mathrm{ss}}_{G,F_v}$ is the local semistable locus as defined in [23]. This does not depend on the choice of localization maps, is a small v-stack, and is an open substack of $\mathrm{Bun}^{e,\mathrm{ss}}_{G,F}$.

One can explicitly describe the semistable locus of $\mathrm{Bun}^e_{G,F}$ using $B_e(G)_{\mathrm{basic}}$.

Theorem.

1. *The substack* $\mathrm{Bun}^{e,1}_{G,U} \subseteq \mathrm{Bun}^e_{G,U}$ *is open, and is isomorphic to* $*/\underline{G(O_{F,\Sigma})}$.
2. *For* $b \in B_e(G)_{\mathrm{basic}}$*, the substack* $\mathrm{Bun}^{e,b}_{G,F} \subseteq \mathrm{Bun}^e_{G,F}$ *is open and isomorphic to* $*/\underline{G_b(F)}$.
3. *We have an equality of stacks* $\mathrm{Bun}^{e,\mathrm{ss}}_{G,F} = \bigsqcup_{b\in B_e(G)_{\mathrm{basic}}} \mathrm{Bun}^{e,b}_{G,F}$.

Proof. For the first statement, we observe that the image of $\mathrm{Bun}^{e,1}_{G,U}$ via Λ is determined by its value at any geometric point, and is thus the trivial morphism 0_P. By Proposition C.5 we can identify the inclusion $\mathrm{Bun}^{e,1}_{G,U} \hookrightarrow \mathrm{Bun}^{e,0_P}_{G,U}$ with the inclusion $\mathrm{Bun}^1_{G,U} \hookrightarrow \mathrm{Bun}_{G,U}$, so that the desired result is true by combining [21, Proposition 4.12] with Theorem 2.8.

The second part is a consequence of the first statement and Proposition C.5. The disjointness part of the second statement can be deduced from the disjointness of the $\mathrm{Bun}^b_{G,F}$ after decomposing $\mathrm{Bun}^e_{G,F}$ using [21, Corollary 4.13]. Similarly, the verification that this union is exhaustive can be done using the decomposition (as stacks)

$$\mathrm{Bun}^{e,\mathrm{ss}}_{G,U} \xrightarrow{\sim} \bigsqcup_{\lambda\in\mathrm{Hom}_F(P_{\check{\Sigma}},Z_G)} \mathrm{Bun}^{e,\mathrm{ss},\lambda}_{G,U}$$

induced by [21, Corollary 4.13] and then observing that the identification $\mathrm{Bun}^{e,\lambda}_{G,U} \xrightarrow{\sim} \mathrm{Bun}^e_{G_b,U}$ for $b \in B_e(G)_{\mathrm{basic}}$ identifies $\mathrm{Bun}^{e,\mathrm{ss},\lambda}_{G,U}$ with $\mathrm{Bun}^{\mathrm{ss}}_{G_b,U}$. Now one can use Theorem 4.11. □

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Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, MA 02139
Email address: `sdlh@mit.edu`