# How shtukas were invented

## (not "How to invent shtukas")

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# Motivation: elliptic modules over $\mathbb{C}_\infty$

Number theorists use the moduli space of elliptic curves to construct Galois representations. Elliptic curves are $\mathbb{Z}$-module objects, and over $\mathbb{C}$ they are given by $\mathbb{C}/\Lambda$ for discrete free $\mathbb{Z}$-submodules $\Lambda \subset \mathbb{C}$ of rank 2.

How can we imitate this over function fields? Let $X$ be a smooth proper geometrically connected curve over $\mathbb{F}_q$, write $F$ for its function field, let $\infty$ be a closed point of $X$, and write $\mathbb{C}_\infty$ for the completion of the algebraic closure of $F_\infty$. Write $A$ for $H^0(X \smallsetminus \infty, \mathcal{O}_X)$.

**Definition**

An *elliptic (or Drinfeld) module of rank n* over $\mathbb{C}_\infty$ is a discrete locally free $A$-submodule $\Lambda \subset \mathbb{C}_\infty$ of rank $n$. A morphism of elliptic modules $\alpha : \Lambda_1 \to \Lambda_2$ over $\mathbb{C}_\infty$ is an element $\alpha$ of $\mathbb{C}_\infty$ such that $\alpha\Lambda_1 \subseteq \Lambda_2$.

**Example (Carlitz)**

Take $X = \mathbb{P}^1_{\mathbb{F}_q}$, the usual $\infty$, and $\Lambda = A = \mathbb{F}_q[t] \hookrightarrow \mathbb{C}_\infty = (\overline{\mathbb{F}_q((\frac{1}{t}))})^\wedge$.

We ought to study $\mathbb{C}_\infty/\Lambda$. We have an analogue of the Weierstrass $\wp$ function:

$$\wp_\Lambda(z) := z \prod_{\lambda \neq 0 \in \Lambda} \left(1 - \frac{z}{\lambda}\right).$$

**Proposition (Drinfeld)**

This converges for all $z$ in $\mathbb{C}_\infty$, and the resulting map $\wp_\Lambda : \mathbb{C}_\infty \to \mathbb{C}_\infty$ induces an isomorphism $\mathbb{C}_\infty/\Lambda \xrightarrow{\sim} \mathbb{C}_\infty$ of topological $\mathbb{F}_q$-vector spaces.

However, $\wp_\Lambda$ doesn't preserve the $A$-module structure on both sides!

**Proposition (Drinfeld)**

Let $a$ be in $A$. Under $\wp_\Lambda$, the action of $a$ on $\mathbb{C}_\infty/\Lambda$ corresponds to a polynomial map $\varphi_a : \mathbb{C}_\infty \to \mathbb{C}_\infty$ over $\mathbb{C}_\infty$.

**Example (Carlitz)**

One can show that the Carlitz module (is isomorphic to an elliptic module that) satisfies $\varphi_t(z) = tz + z^q$.

# Elliptic modules in general

Since $\wp_\Lambda$ and multiplication by $a$ are $\mathbb{F}_q$-linear, $\varphi_a$ is as well. Therefore

$$\varphi_a = c_0 \operatorname{id} + c_1\sigma + \cdots + c_d\sigma^{\circ d},$$

where $\sigma = z^q$. Hence we get a ring homomorphism $\varphi : A \to \operatorname{End}_{\mathrm{Grp}}(\mathcal{O}_{\mathbb{C}_\infty})$. Because $\wp_\Lambda$ has leading term $z$, we see that $c_0 = a$ above.

**Definition**

Let $S$ be an $A$-scheme. An *elliptic module of rank* $n$ *over* $S$ is a line bundle $\mathcal{L}$ over $S$ along with a ring homomorphism $\varphi : A \to \operatorname{End}_{\mathrm{Grp}}(\mathcal{L})$ such that

- the derivative of $\varphi$ equals $A \to H^0(S, \mathcal{O}_S) \to \operatorname{End}_{\mathcal{O}_S}(\operatorname{Lie}\mathcal{L})$,
- Zariski-locally on $S$, every $\varphi_a$ is isomorphic to a map of the form $c_0 \operatorname{id} + c_1\sigma + \cdots + c_d\sigma^{\circ d}$, where $d = -n\deg(\infty)v_\infty(a)$.

**Theorem (Drinfeld)**

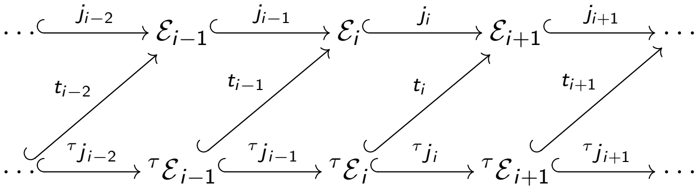
*When $S = \operatorname{Spec}\mathbb{C}_\infty$, this agrees with our previous definition.*

# Elliptic sheaves

On a proper curve over $\mathbb{C}$, Krichever related certain rings of differential operators to certain bundles. By analogizing $\frac{\mathrm{d}}{\mathrm{dx}}$ with $\sigma$ and considering the ring $\varphi(A)$, Drinfeld developed a similar description of elliptic modules:

### Definition

Let $S$ be an $A$-scheme. An *elliptic sheaf of rank $n$* over $S$ is a commutative diagram of locally free $\mathcal{O}_{X\times S}$-modules of rank $n$



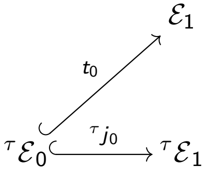

such that the $\mathcal{E}_i \to \mathcal{E}_{i+n\deg(\infty)}$ are isomorphic to the canonical map $\mathcal{E}_i \to \mathcal{E}_i(\infty)$, the coker $t_i$ are line bundles over the graph of $S \to X \smallsetminus \infty$, and $\deg(\mathcal{E}_0|_{X\times\bar{s}}) = n(g-1)+1$ for all geometric points $\bar{s}$ of $S$.

Write $\mathrm{pr}_2$ for projection. One can use Riemann–Roch to show that $\mathcal{L} := \mathrm{pr}_{2,*}(\mathcal{E}_0)$ is a line bundle over $S$. One can show that $P := \varinjlim_i \mathrm{pr}_{2,*}(\mathcal{E}_i|_{(X\setminus\infty)\times S})) \cong \mathcal{L}\{\sigma\}$ as an $\mathcal{O}_S\{\sigma\}$-module, where the $\sigma$-action is given by $t_i$. As the $A$-action commutes with the $\mathcal{O}_S\{\sigma\}$-action, we get a ring homomorphism $\varphi : A \to \mathrm{End}_{\mathcal{O}_S\{\sigma\}}(P) \to \mathrm{End}_{\mathrm{Grp}}(\mathcal{L})$.

**Theorem (Drinfeld)**

*This construction yields an equivalence from elliptic sheaves of rank $n$ over $S$ to elliptic modules of rank $n$ over $S$.*

Note that repeatedly taking pushforwards and then shifting back down (using $\mathcal{E}_{i+n\deg(\infty)} = \mathcal{E}_i(\infty)$) shows that the subdiagram




determines the entire elliptic sheaf.

# Drinfeld shtukas

The moduli space of elliptic curves only associates Galois representations to automorphic forms for $\mathrm{GL}_2/\mathbb{Q}$ satisfying a certain condition at $\mathbb{Q}_\infty = \mathbb{R}$ (namely, to modular forms). The moduli space of elliptic sheaves has a similar restriction at $\infty$, and this corresponds to restricting $j_i$ to be modifications at $\infty$. But we can simply ignore this restriction:

### Definition

Let $S$ be an $\mathbb{F}_q$-scheme. A *(left) Drinfeld shtuka of rank $n$* over $S$ is a pair of morphisms $x, y : S \to X$ along with a diagram of locally free $\mathcal{O}_{X\times S}$-modules of rank $n$

$$\mathcal{E} \xleftarrow{t} \mathcal{E}' \xrightarrow{j} {}^\tau\mathcal{E}$$

such that $\operatorname{coker} t$ is a line bundle over the graph of $x$, and $\operatorname{coker} j$ is a line bundle over the graph of $y$.

Note that the moduli space of Drinfeld shtukas maps to $X^2$.

# Shtukas in general

We generalize this as follows. Let $I$ be a finite set, and let $I_1, \ldots, I_k$ be an ordered partition of $I$. Let $G/\mathbb{F}_q$ be a split connected reductive group, and choose a split maximal subtorus $T$ and Borel subgroup $B \supseteq T$. Let $\underline{\omega} = (\omega_i)_{i \in I}$ be an $I$-tuple of dominant coweights of $G$.

**Definition**

Let $S$ be an $\mathbb{F}_q$-scheme. Write $\mathrm{Sht}_{G,I,\underline{\omega}}^{(I_1,\ldots,I_k)}$ for the prestack whose $S$-points parametrize

- an $I$-tuple $(x_i)_{i \in I}$ of morphisms $S \to X$,
- $G$-bundles $\mathcal{G}_0, \ldots, \mathcal{G}_k$ over $X \times S$,
- isomorphisms $\phi_j : \mathcal{G}_{j-1}|_{X \times S \smallsetminus \bigcup_{i \in I_j} \Gamma_{x_i}} \xrightarrow{\sim} \mathcal{G}_j|_{X \times S \smallsetminus \bigcup_{i \in I_j} \Gamma_{x_i}}$ whose relative position at $\Gamma_{x_i}$ is bounded by $\sum_{x_h = x_i} \omega_h$ for all $i$ in $I_j$,
- an isomorphism $\theta : \mathcal{G}_k \xrightarrow{\sim} {}^\tau\mathcal{G}_0$.

Write $\pi$ for the canonical morphism $\mathrm{Sht}_{G,I,\underline{\omega}}^{(I_1,\ldots,I_k)} \to X^I$.

One can show that $\mathrm{Sht}_{G,I,\underline{\omega}}^{(I_1,\ldots,I_k)}$ is a Deligne–Mumford stack locally of finite type over $\mathbb{F}_q$. Because $\tau$ preserves connected components of $\mathrm{Bun}_G$, we see that $\mathrm{Sht}_{G,I,\underline{\omega}}^{(I_1,\ldots,I_k)}$ is empty when $\sum_{i\in I}\omega_i$ isn't in the coroot lattice, and one can show that the converse also holds. So assume this isn't the case.

Example (Drinfeld)

Take $I = \{1,2\}$ with the ordered partition $\{1\}\cup\{2\}$. Let $G = \mathrm{GL}_n$ with the standard $T$ and $B$, and set $\omega_1 = (0,\ldots,0,-1)$ and $\omega_2 = (1,0,\ldots,0)$. Then $\mathrm{Sht}_{G,I,\underline{\omega}}^{(I_1,I_2)}$ is the moduli space of Drinfeld shtukas.

Example ("No legs")

Suppose all the $\omega_i = 0$. Then all the $\phi_j$ extend to isomorphisms $\mathcal{G}_{j-1}\xrightarrow{\sim}\mathcal{G}_j$, so $\mathrm{Sht}_{G,I,\underline{\omega}}^{(I_1,\ldots,I_k)}$ only parametrizes $(x_i)_{i\in I}$ and an isomorphism $\mathcal{G}_0\xrightarrow{\sim}{}^{\tau}\mathcal{G}_0$. As intuition suggests, the latter data is parametrized by the discrete stack $\mathrm{Bun}_G(\mathbb{F}_q)$, so altogether $\mathrm{Sht}_{G,I,\underline{\omega}}^{(I_1,\ldots,I_k)} = X^I\times\mathrm{Bun}_G(\mathbb{F}_q)$.

Using a theorem of Harder, Lang's lemma, and fpqc descent, one can show $\mathrm{Bun}_G(\mathbb{F}_q)\cong G(F)\backslash G(\mathbb{A})/G(\mathbb{O})$ as groupoids. This is *Weil uniformization*.

## Example (Class field theory)

Let $G = \mathbb{G}_m$. Then the $\omega_i$ correspond to integers, and $\mathcal{G}_j$ is uniquely determined as $\mathcal{G}_0(\sum_{i \in I_1 \cup \cdots \cup I_j} \omega_i \Gamma_{x_i})$. So $\mathrm{Sht}_{G,I,\underline{\omega}}^{(I_1,\ldots,I_k)}$ only parametrizes $(x_i)_{i \in I}$ and an isomorphism $\mathcal{G}_0(\sum_{i \in I} \omega_i \Gamma_{x_i}) \xrightarrow{\sim} {}^{\tau}\mathcal{G}_0$. This yields a square

$$\begin{array}{ccc} \mathrm{Sht}_{G,I,\underline{\omega}}^{(I_1,\ldots,I_k)} & \longrightarrow & \mathrm{Pic} \\ \Big\downarrow \pi & & \Big\downarrow L \\ X^I & \xrightarrow{AJ_{\underline{\omega}}} & \mathrm{Pic}^0, \end{array}$$

where $L$ is the *Lang isogeny* $\mathcal{L} \mapsto {}^{\tau}\mathcal{L} \otimes \mathcal{L}^{-1}$, and $AJ_{\underline{\omega}}$ is the generalized *Abel–Jacobi map* $(x_i)_{i \in I} \mapsto \mathcal{O}(\sum_{i \in I} \omega_i \Gamma_{x_i})$. Because the square is Cartesian and $L$ is a $\mathrm{Pic}(\mathbb{F}_q)$-bundle, we see $\pi$ is a $\mathrm{Pic}(\mathbb{F}_q)$-bundle too.

Let $a$ in $\mathrm{Pic}(\mathbb{F}_q)$ have nonzero degree, and let $\chi : \mathrm{Pic}(\mathbb{F}_q)/a^{\mathbb{Z}} \to \overline{\mathbb{Q}}_\ell^\times$ be a character. We'll see that $\mathrm{Sht}_{G,I,\underline{\omega}}^{(I_1,\ldots,I_k)}/a^{\mathbb{Z}} \to X^I$ yields a homomorphism $\alpha : \pi_1^{\text{ét}}(X)^I \to \mathrm{Pic}(\mathbb{F}_q)/a^{\mathbb{Z}}$. It turns out that $\chi \circ \alpha = \prod_{i \in I} (\chi \circ \mathrm{Art})^{\omega_i}$, where $\mathrm{Art} : \pi_1^{\text{ét}}(X) \xrightarrow{\sim} \mathrm{Pic}(\mathbb{F}_q)^{\wedge}$ is the *Artin isomorphism*.

# Partial Frobenius

Unlike the topological case, we don't generally have $\pi_1^{\text{ét}}(X^I) \xrightarrow{\sim} \pi_1^{\text{ét}}(X)^I$:

**Examples**

- We have $\pi_1^{\text{ét}}(\mathbb{P}^1_{\mathbb{F}_q} \times \mathbb{P}^1_{\mathbb{F}_q}) = \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) = \pi_1^{\text{ét}}(\mathbb{P}^1_{\mathbb{F}_q})$.
- The Artin–Schreier $\mathbb{F}_p$-cover $t^p - t = xy$ of $\operatorname{Spec} \overline{\mathbb{F}}_q[x, y]$ yields a continuous homomorphism $\pi_1^{\text{ét}}(\mathbb{A}^2_{\overline{\mathbb{F}}_q}) \to \mathbb{F}_p$ that isn't a box product of homomorphisms from $\pi_1^{\text{ét}}(\mathbb{A}^1_{\overline{\mathbb{F}}_q})$.

What's the fix? For any subset $J$ of $I$, write $\mathrm{Frob}_J : X^I \to X^I$ for $\left(\prod_{j\in J} \mathrm{Frob}\right) \times \left(\prod_{i\notin J} \mathrm{id}\right)$. Write $(X^I/\partial\,\mathrm{Fr})_{\text{fét}}$ for the category of finite étale morphisms $Y \to X^I$ equipped with commuting morphisms $\mathrm{Fr}_i : \mathrm{Frob}^*_{\{i\}} Y \to Y$ whose composition is the canonical isomorphism $\mathrm{Frob}^*_I Y \xrightarrow{\sim} Y$. Note that $(X^I/\partial\,\mathrm{Fr})_{\text{fét}}$ is a Galois category, and write $\pi_1^{\text{ét}}(X^I/\partial\,\mathrm{Fr})$ for the associated profinite group.

For any $i$ in $I$, we get a functor $X_{\text{fét}} \to (X^I/\partial\,\mathrm{Fr})_{\text{fét}}$ via pullback. This induces a map $\pi_1^{\text{ét}}(X^I/\partial\,\mathrm{Fr}) \to \pi_1^{\text{ét}}(X)$.

# Drinfeld's lemma

**Lemma (Drinfeld)**

*The induced map* $\pi_1^{\text{ét}}(X^I/\partial\,\mathrm{Fr}) \to \pi_1^{\text{ét}}(X)^I$ *is an isomorphism.*

**Remark (Xue)**

The usual limit process extends this to smooth $\overline{\mathbb{Q}}_\ell$-sheaves. However, to get an analogous result for ind-smooth $\overline{\mathbb{Q}}_\ell$-sheaves, one must replace $\pi_1^{\text{ét}}(X)$ with the *Weil group* $\mathrm{Weil}(X) := \pi_1^{\text{ét}}(X) \times_{\widehat{\mathbb{Z}}} \mathbb{Z}$.

What are the partial Frobenii in our context? Consider the morphism $F_{I_1}^{(I_1,\dots,I_k)} : \mathrm{Sht}_{G,I,\underline{\omega}}^{(I_1,\dots,I_k)} \to \mathrm{Sht}_{G,I,\underline{\omega}}^{(I_2,\dots,I_1,I_k)}$ given by sending

$$((x_i)_{i\in I}, \mathcal{G}_0 \overset{\phi_1}{\dashrightarrow} \mathcal{G}_1 \overset{\phi_2}{\dashrightarrow} \cdots \overset{\phi_k}{\dashrightarrow} {}^\tau\mathcal{G}_0)$$

$$\mapsto (\mathrm{Frob}_{I_1}((x_i)_{i\in I}), \mathcal{G}_1 \overset{\phi_2}{\dashrightarrow} \cdots \overset{\phi_k}{\dashrightarrow} {}^\tau\mathcal{G}_0 \overset{{}^\tau\phi_1}{\dashrightarrow} {}^\tau\mathcal{G}_1).$$

Note that $F_{I_k}^{(I_k,I_1,\dots,I_{k-1})} \circ \cdots \circ F_{I_2}^{(I_2,\dots,I_k,I_1)} \circ F_{I_1}^{(I_1,\dots,I_k)} = \mathrm{Frob}.$

Note that $\mathrm{Bun}_Z(\mathbb{F}_q) = Z(F)\backslash Z(\mathbb{A})/Z(\mathbb{O})$ acts on $\mathrm{Sht}_{G,I,\underline{\omega}}^{(I_1,\dots,I_k)}$ via twisting. Let $\Xi$ be a discrete cocompact subgroup of $Z(F)\backslash Z(\mathbb{A})$, and consider $\mathrm{Sht}_{G,I,\underline{\omega}}^{(I_1,\dots,I_k)}/\Xi$. We're interested in the ind-constructible $\overline{\mathbb{Q}}_\ell$-sheaf

$$\mathscr{H}^0_{I,\underline{\omega},\Xi} := R^0\pi_!(\mathrm{IC}_{\mathrm{Sht}_{G,I,\underline{\omega}}^{(I_1,\dots,I_k)}/\Xi})$$

on $X^I$, where the IC-sheaf is normalized relative to $\pi : \mathrm{Sht}_{G,I,\underline{\omega}}^{(I_1,\dots,I_k)}/\Xi \to X^I$. The smallness of convolution implies that $\mathscr{H}^0_{I,\underline{\omega},\Xi}$ is independent of the ordered partition $I_1,\dots,I_k$.

**Theorem (Xue)**

*The ind-constructible $\overline{\mathbb{Q}}_\ell$-sheaf $\mathscr{H}^0_{I,\underline{\omega},\Xi}$ is ind-smooth on $X^I$.*

Choosing an ordered partition with $I_1 = \{i\}$ and taking (intersection) cohomology of $F_{I_1}^{(I_1,\dots,I_k)}$ yields morphisms $\mathrm{Fr}_i : \mathrm{Frob}_{\{i\}}^* \mathscr{H}^0_{I,\underline{\omega},\Xi} \to \mathscr{H}^0_{I,\underline{\omega},\Xi}$ whose composition is the canonical isomorphism $\mathrm{Frob}_I^* \mathscr{H}^0_{I,\underline{\omega},\Xi} \xrightarrow{\sim} \mathscr{H}^0_{I,\underline{\omega},\Xi}$. Hence (the ind-smooth variant of) Drinfeld's lemma realizes $\mathscr{H}^0_{I,\underline{\omega},\Xi}$ as a continuous representation of $\mathrm{Weil}(X)^I$ over $\overline{\mathbb{Q}}_\ell$.

Write $\widehat{G}$ for the dual group. Observe that $I$-tuples $\underline{\omega}$ correspond to irreducible (algebraic) representations of $\widehat{G}^I$ over $\overline{\mathbb{Q}}_\ell$. Via the functoriality of geometric Satake, we extend $\underline{\omega} \mapsto \mathscr{H}^0_{I,\underline{\omega},\Xi}$ to a $\overline{\mathbb{Q}}_\ell$-linear functor

$$\mathscr{H}^0_{I,(-),\Xi} : \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\widehat{G}^I) \to \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\mathrm{Weil}(X)^I).$$

The *fusion* property of geometric Satake yields, for any map $\zeta : I \to J$ of finite sets, a natural 2-commutative diagram

$$\begin{array}{ccc} \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\widehat{G}^I) & \xrightarrow{\mathscr{H}^0_{I,(-),\Xi}} & \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\mathrm{Weil}(X)^I) \\ {\scriptstyle \circ\zeta^*}\downarrow & & \downarrow{\scriptstyle \circ\zeta^*} \\ \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\widehat{G}^J) & \xrightarrow{\mathscr{H}^0_{I,(-),\Xi}} & \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\mathrm{Weil}(X)^J), \end{array}$$

where $\zeta^*$ denotes $\widehat{G}^J \to \widehat{G}^I$ or $\mathrm{Weil}(X)^J \to \mathrm{Weil}(X)^I$. Note that $\mathscr{H}^0_{\mathrm{pt},\mathbf{1},\Xi}$ is the set of finitely supported $\overline{\mathbb{Q}}_\ell$-valued functions $C_c(G(F)\backslash G(\mathbb{A})/G(\mathbb{O})\Xi)$.

# Excursion algebra

We'd like to extract $\widehat{G}(\overline{\mathbb{Q}}_\ell)$-conjugacy classes of continuous homomorphisms $\mathrm{Weil}(X) \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$ from this. Hence we should study the *excursion algebra* $\mathcal{B} := \overline{\mathbb{Q}}_\ell[\mathrm{Hom}(\mathrm{Weil}(X), \widehat{G})]^{\widehat{G}}$.

**Theorem (V. Lafforgue)**

*The excursion algebra has generators $S_{I,W,x,\xi,(\gamma_i)_{i\in I}}$ indexed by a finite set $I$, an object $W$ of $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\widehat{G}^I)$, an element $x$ of $W^{\Delta(\widehat{G})}$, an element $\xi$ of $W^{*,\Delta(\widehat{G})}$, and an $I$-tuple $(\gamma_i)_{i\in I}$ of elements of $\mathrm{Weil}(X)$, satisfying certain explicit (but somewhat tedious) relations.*

Richardson's work on geometric invariant theory implies the following:

**Theorem (V. Lafforgue)**

*There exists a bijection from $\overline{\mathbb{Q}}_\ell$-algebra homomorphisms $\nu : \mathcal{B} \to \overline{\mathbb{Q}}_\ell$ to $\widehat{G}(\overline{\mathbb{Q}}_\ell)$-conjugacy classes of semisimple homomorphisms $\rho : \mathrm{Weil}(X) \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$ such that $\nu(S_{I,W,x,\xi(\gamma_i)_{i\in I}}) = \langle \xi, ((\rho(\gamma_i))_{i\in I} \cdot x\rangle$ for all such generators above.*

# I'd love to go on an excursion—why not?

How can we give $C_c(G(F)\backslash G(\mathbb{A})/G(\mathbb{O})\Xi)$ an action of $\mathcal{B}$? Let $S_{I,W,x,\xi,(\gamma_i)_{i\in I}}$ act via

$$\begin{array}{ccccc} C_c(G(F)\backslash G(\mathbb{A})/G(\mathbb{O})\Xi) \cong \mathscr{H}^0_{\mathrm{pt},\mathbf{1},\Xi} & \xrightarrow{x} & \mathscr{H}^0_{\mathrm{pt},W|_{\Delta(\widehat{G})},\Xi} & \xrightarrow{\sim} & \mathscr{H}^0_{I,W,\Xi} \\ & & & & \downarrow{\scriptstyle (\gamma_i)_{i\in I}} \\ C_c(G(F)\backslash G(\mathbb{A})/G(\mathbb{O})\Xi) \cong \mathscr{H}^0_{\mathrm{pt},\mathbf{1},\Xi} & \xleftarrow{\xi} & \mathscr{H}^0_{\mathrm{pt},W|_{\Delta(\widehat{G})},\Xi} & \xleftarrow{\sim} & \mathscr{H}^0_{I,W,\Xi}. \end{array}$$

**Theorem (V. Lafforgue)**

*This factors through an action of $\mathcal{B}$ on $C_c(G(F)\backslash G(\mathbb{A})/G(\mathbb{O})\Xi)$.*

V. Lafforgue calls the $S_{I,W,x,\xi,(\gamma_i)_{i\in I}}$ *excursion operators* because they intuitively create new points on the curve via $x$, move them around via $(\gamma_i)_{i\in I}$, and then recombine them via $\xi$. Alternatively, one can think of the above diagram as an excursion of arrows.

# What the Hecke?

Somehow, we gave almost the whole talk without mentioning *Hecke operators*. Let $v$ be a closed point of $X$, and let $V$ be an irreducible object of $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\widehat{G})$. The classical Satake isomorphism yields a function $h_{V,v}$ in $C_c(G(\mathcal{O}_v)\backslash G(F_v)/G(\mathcal{O}_v))$, and we get a Hecke correspondence $T(h_{V,v})$ on $\mathrm{Sht}^{(I_1,\ldots,I_k)}_{G,I,\underline{\omega}}|_{(X\smallsetminus v)^I}$ and hence $\mathscr{H}^0_{I,\underline{\omega},\Xi}|_{(X\smallsetminus v)^I}$ by considering modifications at $v$ whose relative position is bounded by $V$. When all the $\omega_i = 0$, we see that this agrees with the usual Hecke operator on $C_c(G(F)\backslash G(\mathbb{A})/G(\mathbb{O})\Xi)$.

**Theorem (V. Lafforgue)**

*When all the $\omega_i = 0$, we have $T(h_{V,v}) = S_{\{1,2\},V\boxtimes V^*,\delta,\mathrm{ev},(\gamma_v,1)}$, where $\delta$ and $\mathrm{ev}$ are the canonical morphisms $\mathbf{1}\to V\otimes V^*$ and $V\otimes V^*\to\mathbf{1}$, and $\gamma_v$ is a geometric Frobenius element at $v$.*

This is called the *$S = T$ theorem*. There is a version for arbitrary $\underline{\omega}$, and this general version is crucial for the proofs of everything, but we omit it here.

## Conclusion

Note that $\langle \mathrm{ev}, (\rho(\gamma_v), 1)\cdot\delta\rangle = \mathrm{tr}(\rho(\gamma_v)|V)$. The action of $\mathcal{B}$ on $C_c(G(F)\backslash G(\mathbb{A})/G(\mathbb{O})\Xi)$ decomposes the latter into $\mathcal{B}$-eigenspaces

$$C_c(G(F)\backslash G(\mathbb{A})/G(\mathbb{O})\Xi) = \bigoplus_{\nu} \mathfrak{H}_\nu,$$

where $\nu$ runs over all $\overline{\mathbb{Q}}_\ell$-algebra homomorphisms $\nu : \mathcal{B} \to \overline{\mathbb{Q}}_\ell$. Now $T(h_{V,v})$ acts on $\mathfrak{H}_\nu$ via

$$\nu(S_{\{1,2\},V\boxtimes V^*,\delta,\mathrm{ev},(\gamma_v,1)}) = \langle \mathrm{ev}, (\rho(\gamma_v), 1)\cdot\delta\rangle = \mathrm{tr}(\rho(\gamma_v)|V),$$

where $\rho : \mathrm{Weil}(X) \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$ is the $\widehat{G}(\overline{\mathbb{Q}}_\ell)$-conjugacy class of semisimple homomorphisms corresponding to $\nu$. One can use the continuity of the $\mathrm{Weil}(X)^I$-action on the $\mathscr{H}^0_{I,\underline{\omega},\Xi}$ to show that $\rho$ is continuous, thus completing the automorphic-to-Galois direction of the Langlands correspondence for $G$ over $F$.