Automorphic Representations

With An Application To Measure Theory

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The Ruziewicz Problem

Let $n \geq 1$ be an integer, and consider the n-sphere S^n with the Lebesgue measure λ. Normalize λ such that $\lambda(S^n) = 1$.
Note that $\mathrm{SO}(n+1)$ preserves S^n as a subset of $\mathbb{R}^{n+1}$, and this action corresponds to rotations.
Evidently λ is a rotation-invariant, finitely-additive measure on the Lebesgue σ-algebra of S^n satisfying $\lambda(S^n) = 1$.

Question

Is λ the only one?

What's known?

Ruziewicz Problem

Is λ the only rotation-invariant, finitely-additive measure on the Lebesgue σ-algebra of S^n satisfying $\lambda(S^n) = 1$?

The answer depends on n:

- For $n = 1$, it's not! This was proved by Banach in 1921, and it uses the axiom of choice.
- For $2 \leq n \leq 3$, it is! This was proved by Drinfeld in 1984, and it uses the Ramanujan conjecture.
- For $4 \leq n$, it is! This was proved by Margulis and Sullivan in 1980, and it uses arithmetic subgroups.

Some representation theory

All our groups G are second-countable, locally-compact, Hausdorff topological groups. So they have right-invariant Haar measures μ.
We focus on *unitary representations*, i.e. where the action map

$$G \times V \to V$$

is continuous, and $(V, \langle \cdot, \cdot \rangle)$ is a Hilbert space where $\langle gv, gw \rangle = \langle v, w \rangle$ for all $g \in G$ and $v, w \in V$. Subrepresentations must be closed subspaces.

Example (Regular representation)

For any group G, we have $L^2(G)$ with right translation.

Example

For $G = \mathrm{SO}(n+1)$, write $L^2(S^n)_0 = \{f \in L^2(S^n) \mid \int \mathrm{d}\lambda\, f = 0\}$. Then

$$L^2(S^n) = \mathbb{C} \cdot \mathbf{1}_{S^n} \oplus L^2(S^n)_0.$$

Spaces of representations

Write $\widetilde{G}$ for the set of isomorphism classes of unitary representations of G.

Definition

Let V be a unitary representation of G, let $v \in V$, let $Q \subseteq G$ be compact, and let $\epsilon > 0$. Write $U(V, v, Q, \epsilon)$ for the subset

$$\left\{ W \in \widetilde{G} \;\middle|\; \begin{array}{c} \text{there exists } w_1, \ldots, w_m \in W \text{ such that} \\ \left|\langle gv, v\rangle - \sum_{i=1}^{n} \langle gw_i, w_i\rangle\right| < \epsilon \text{ for } g \in Q \end{array} \right\}.$$

The $U(V, v, Q, \epsilon)$ form a subbasis for a topology we call the *Fell topology*. Write $\widehat{G} \subset \widetilde{G}$ for the subspace of irreducible representations.

Example (Pontryagin duality)

For abelian G, the subspace $\widehat{G}$ is naturally an abelian group, and $\widehat{\widehat{G}} = G$. For instance, $\widehat{\mathbb{R}} = \mathbb{R}$, and $\widehat{S^1} = \mathbb{Z}$.

While $\widehat{G}$ is usually nice, $\widetilde{G}$ is never nice.

Induced representations

Let $H \subseteq G$ be a closed subgroup.

Definition

Let V be a unitary representation of H. The *induced representation* is

$$\mathrm{Ind}_H^G V = \left\{ \begin{array}{c} \text{measurable} \\ f : G \to V \end{array} \middle| \begin{array}{c} f(hg) = hf(g) \text{ for all } h \in H \text{ and almost all} \\ g \in G, \text{ and } \int_{H\backslash G} \mathrm{d}g \|f(g)\|^2 < \infty \end{array} \right\}$$

modulo equality almost everywhere, with right translation.

Example

For any group G, we see that $\mathrm{Ind}_{\{1\}}^G \mathbb{C}$ is the regular representation $L^2(G)$.

Example

For $G = \mathrm{SO}(n+1)$ and $H = \mathrm{SO}(n)$, we have $H\backslash G = S^n$. So we see that $\mathrm{Ind}_H^G \mathbb{C} = L^2(S^n)$.

Weak containment

Recall that neighborhoods of $V \in \widetilde{G}$ are given by finite intersections of

$$\left\{ W \in \widetilde{G} \;\middle|\; \begin{array}{c} \text{there exists } w_1, \ldots, w_m \in W \text{ such that} \\ \left|\langle gv, v\rangle - \sum_{i=1}^{n} \langle gw_i, w_i\rangle\right| < \epsilon \text{ for } g \in Q \end{array} \right\}.$$

Definition

Let V and W be unitary representations of G. Then W *weakly contains* V if W lies in the closure of $\{V\} \subset \widetilde{G}$. We write $V \prec W$.

If $U \prec V$ and $V \prec W$, then $U \prec W$.

Example

If V is a subrepresentation of W, then $V \prec W$.

Example

If $H \subseteq G$ is a discrete closed subgroup, then $\mathbb{C} \prec \mathrm{Ind}_H^G \mathbb{C}$.

More on weak containment, and back to Ruziewicz!

Fact

The map $\mathrm{Ind}_H^G : \widetilde{H} \to \widetilde{G}$ *is continuous.*

Therefore if $V \prec W$, then $\mathrm{Ind}_H^G V \prec \mathrm{Ind}_H^G W$.
What does any of this have to do with the Ruziewicz problem? Recall that

$$L^2(S^n)_0 = \{f \in L^2(S^n) \mid \int \mathrm{d}\lambda\, f = 0\}$$

is a unitary representation of $\mathrm{SO}(n+1)$.

Fact

Let Γ *be a discrete group, and let* $\psi : \Gamma \to \mathrm{SO}(n+1)$ *be a homomorphism. Consider the representation* $L^2(S^n)_0$ *of* Γ*. If* $\mathbb{C} \not\prec L^2(S^n)_0$ *as representations of* Γ*, then the answer to the Ruziewicz problem is yes!*

The proof only uses functional analysis from Charlie Smart's Winter 2018 MATH313. (Including such gems as $L^1(X)^{**}$ being the space of finitely-additive measures, Mazur's theorem, and Goldstein's theorem.)

Work of Margulis and Sullivan

Fact

Let Γ be a discrete group, and let $\psi : \Gamma \to \mathrm{SO}(n+1)$ be a homomorphism. Consider the representation $L^2(S^n)_0$ of Γ. If $\mathbb{C} \not\prec L^2(S^n)_0$ as representations of Γ, then the answer to the Ruziewicz problem is yes!

Theorem (Margulis, Sullivan)

For $n \geq 4$, there exists an arithmetic subgroup $\Gamma \subset \mathrm{SO}(n+1)$ satisfying a stronger version of the above criterion. ("Kazhdan's property (T)")

This solves the Ruziewicz problem for $n \geq 4$. However, Kazhdan's property (T) on arithmetic subgroups won't work for $n \in \{2, 3\}$:

Theorem (Margulis)

Let G be a connected compact simple Lie group. Then G has an arithmetic subgroup satisfying Kazhdan's property (T) if and only if G is not isogenous to $\mathrm{SO}(2)$, $\mathrm{SO}(3)$, or $\mathrm{SO}(4)$.

Some algebraic groups

Let $\mathbf{G}$ be a connected reductive algebraic group over $\mathbb{Q}$.

Example

We can take $\mathbf{G} = \mathrm{GL}_2$.

Example

Let D be a quaternion algebra over $\mathbb{Q}$. We can take $\mathbf{G} = \mathbf{D}^\times$, whose functor of points is $\mathbf{D}^\times(R) = (D \otimes_{\mathbb{Q}} R)^\times$ for $\mathbb{Q}$-algebras R.

Let $A_{\mathbf{G}}$ be the identity component of the $\mathbb{R}$-points of the maximal split subtorus of the center of $\mathbf{G}$.

Example

For $\mathbf{G} = \mathrm{GL}_2$, the center is the subgroup of scalars. So $A_{\mathbf{G}} = \mathbb{R}_{>0}$. Same for $\mathbf{G} = \mathbf{D}^\times$.

L^2 spaces of automorphic representations

Write $[\mathbf{G}]$ for $\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})/A_{\mathbf{G}}$. Borel–Harish-Chandra proved that $[\mathbf{G}]$ has finite measure. Now $\mathbf{G}(\mathbb{A})$ acts on $L^2([\mathbf{G}])$ with right translation.

Definition

Write $L^2_{\text{cusp}}([\mathbf{G}]) \subseteq L^2([\mathbf{G}])$ for the set of $f : [\mathbf{G}] \to \mathbb{C}$ such that, for every proper parabolic subgroup $\mathbf{P} \subset \mathbf{G}$ with unipotent radical $\mathbf{N}$, we have $\int_{\mathbf{N}(\mathbb{Q})\backslash\mathbf{N}(\mathbb{A})} \mathrm{d}n\, f(ng) = 0$ for almost all $g \in \mathbf{G}(\mathbb{A})$.

Definition

An *automorphic representation of* $\mathbf{G}(\mathbb{A})$ is an irreducible subquotient of $L^2([\mathbf{G}])$. It is *cuspidal* if it is a subquotient of $L^2_{\text{cusp}}([\mathbf{G}])$.

Example

For $\mathbf{G} = \mathbf{D}^\times$, there are no proper parabolic subgroups. So $L^2([\mathbf{G}]) = L^2_{\text{cusp}}([\mathbf{G}])$.

Automorphic forms and factorizing

Let K be a maximal compact subgroup of $\mathbf{G}(\mathbb{A})$, factored as $K_f \times K_\infty$.

Example

For $\mathbf{G} = \mathrm{GL}_2$, we can take $K = \mathrm{GL}_2(\widehat{\mathbb{Z}}) \times \mathrm{O}(2)$.

For any automorphic representation V of $\mathbf{G}(\mathbb{A})$, write V^{sm} for its subspace of K-finite vectors. Write $\mathfrak{g}$ for the Lie algebra of $\mathbf{G}(\mathbb{R})$.

Theorem (Bernstein–Harish-Chandra)

The subspace V^{sm} consists of automorphic forms, and it's an irreducible admissible $\mathbf{G}(\mathbb{A}_f) \times (\mathfrak{g}, K_\infty)$-module.

Theorem (Flath)

Let W be an irreducible admissible $\mathbf{G}(\mathbb{A}_f) \times (\mathfrak{g}, K_\infty)$-module. Then $W = \left(\bigotimes'_p W_p\right) \otimes W_\infty$, where the W_p are irreducible admissible representations of $\mathbf{G}(\mathbb{Q}_p)$, and W_∞ is an irreducible admissible $(\mathfrak{g}, K_\infty)$-module.

Relation with modular forms

Let $f : \mathcal{H} \to \mathbb{C}$ be a cusp eigenform of weight $k \geq 2$ and level $\Gamma_1(N)$. Write $K_1(N) = \{g \in \mathrm{GL}_2(\widehat{\mathbb{Z}}) \mid g \equiv \left[\begin{smallmatrix} * & * \\ & 1 \end{smallmatrix}\right] \mod N\}$ and $\mathrm{GL}_2(\mathbb{R})^+ = \{g \in \mathrm{GL}_2(\mathbb{R}) \mid \det g > 0\}$. Since $\det K_1(N) = \widehat{\mathbb{Z}}^\times$, strong approximation implies $\mathrm{GL}_2(\mathbb{A}) = \mathrm{GL}_2(\mathbb{Q}) K_1(N) \mathrm{GL}_2(\mathbb{R})^+$.
For $g = \gamma k u \in \mathrm{GL}_2(\mathbb{A})$ with $u = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, set

$$\phi(g) = (\det u)^{k/2} (ci + d)^{-k} f\left(\frac{ai + b}{ci + d}\right).$$

Fact

We have $\phi \in L^2_{cusp}([\mathrm{GL}_2])$, and the subrepresentation V_ϕ it generates is irreducible. Furthermore, every cuspidal automorphic representation V of $\mathrm{GL}_2(\mathbb{A})$ such that V_∞^{sm} is discrete series arises this way.

Theorem (Deligne)

For such a representation and $p \nmid N$, the Hecke operator $T_p = e_{\mathrm{GL}_2(\mathbb{Z}_p)\left[\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right]\mathrm{GL}_2(\mathbb{Z}_p)}$ acts by a scalar a_p with $|a_p| \leq 2p^{(k-1)/2}$.

Global Jacquet–Langlands correspondence

Let D be a quaternion algebra over $\mathbb{Q}$ ramified at ∞.

Example

We can take the rational Hamiltonian quaternions $D = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{Q}\}$, where $i^2 = j^2 = -1$ and $ij = -ji = k$.

Theorem (Jacquet–Langlands)

There exists an injection

$$\begin{aligned}\mathrm{JL} :&\{\textit{cuspidal automorphic representations of } \mathbf{D}^\times(\mathbb{A})\} \\ &\hookrightarrow \{\textit{cuspidal automorphic representations of } \mathbf{GL}_2(\mathbb{A})\}\end{aligned}$$

such that

- *if* $V_\infty^{sm} = \mathbb{C}$, *then* $\mathrm{JL}(V)_\infty^{sm}$ *is the discrete series* σ_2,
- *if* D *is split at* p, *then* $\mathrm{JL}(V)_p^{sm} = V_p^{sm}$.

Back to Ruziewicz!

Fact

Let Γ be a discrete group, and let $\psi : \Gamma \to \mathrm{SO}(n+1)$ be a homomorphism. Consider the representation $L^2(S^n)_0$ of Γ. If $\mathbb{C} \not\prec L^2(S^n)_0$ as representations of Γ, then the answer to the Ruziewicz problem is yes!

Let $n \in \{2, 3\}$. Let D be a quaternion algebra over $\mathbb{Q}$ ramified at ∞. Then $\mathbf{D}^\times(\mathbb{R}) = \mathbb{H}^\times$, so $\mathbf{D}^\times(\mathbb{R})/A_{\mathbf{D}^\times} = \mathrm{SU}(2) = S^3$.

- For $n = 2$, conjugation on purely imaginary quaternions yields $\mathbf{D}^\times(\mathbb{R})/A_{\mathbf{D}^\times} \twoheadrightarrow \mathrm{SO}(n+1)$.
- For $n = 3$, right translation yields $\mathbf{D}^\times(\mathbb{R})/A_{\mathbf{D}^\times} \to \mathrm{SO}(n+1)$. Note that $\mathbf{D}^\times(\mathbb{R})/A_{\mathbf{D}^\times}$ acts transitively on S^3.

In both cases, consider the discrete group $D^\times$ with the homomorphism

$$D^\times \to \mathbf{D}^\times(\mathbb{R}) \to \mathbf{D}^\times(\mathbb{R})/A_{\mathbf{D}^\times} \to \mathrm{SO}(n+1).$$

Work of Drinfeld

Consider the discrete group $D^\times$ with the homomorphism

$$D^\times \to \mathbf{D}^\times(\mathbb{R}) \to \mathbf{D}^\times(\mathbb{R})/A_{\mathbf{D}^\times} \to \mathrm{SO}(n+1).$$

For contrapositive's sake, suppose $\mathbb{C} \prec L^2(S^n)_0$. Because $\mathbf{D}^\times(\mathbb{R})/A_{\mathbf{D}^\times}$ acts transitively on S^n, we have $L^2(S^n)_0 \hookrightarrow L^2(\mathbf{D}^\times(\mathbb{R})/A_{\mathbf{D}^\times})_0$. Therefore $\mathbb{C} \prec L^2(\mathbf{D}^\times(\mathbb{R})/A_{\mathbf{D}^\times})_0$. Since D is ramified at ∞, we see $D^\times$ is discrete in $\mathbf{D}^\times(\mathbb{A}_f)$. Therefore $\mathbb{C} \prec \mathrm{Ind}_{D^\times}^{\mathbf{D}^\times(\mathbb{A}_f)} L^2(\mathbf{D}^\times(\mathbb{R})/A_{\mathbf{D}^\times})_0$. This induced representation is a subrepresentation of $L^2([\mathbf{D}^\times])$ where $\mathrm{SU}(2)$ acts trivially. As it weakly contains $\mathbb{C}$, for any $\epsilon > 0$ and large enough p we can find an automorphic representation V of $\mathbf{D}^\times(\mathbb{A})$ with $|a_p - (p+1)| < \epsilon$. The Jacquet–Langlands correspondence yields an automorphic representation $\mathrm{JL}(V)$ of $\mathrm{GL}_2(\mathbb{A})$ with the same property and $\mathrm{JL}(V)_\infty^{\mathrm{sm}}$ isomorphic to the discrete series σ_2. By Deligne's theorem, this does not exist. Thus $\mathbb{C} \not\prec L^2(S^n)_0$, so the answer to the Ruziewicz problem is yes! □

Closing remarks

We could use a weaker version of the Ramanujan conjecture instead:

Theorem (Rankin)

We have $|a_p| = O(p^{k/2-1/5})$.

References:

- B. Bekka, P. de la Harpe, and A. Valette, *Kazhdan's Property (T)*.
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Thank you!