Notes for MATH 383 — Height Functions in Number Theory
(Winter 2018)

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These are live-TeX'd notes for a course taught at the University of Chicago in Winter 2018 by Professor Kazuya Kato. Any errors are attributed to the note-taker. If you find any such errors or have comments at large, don't hesitate to contact said note-taker at `lidansiyan@gmail.com`.

1 January 3, 2018

The subject of this course consists of Diophantine geometry and heights of numbers, or in other words, height functions in number theory. The course outline is as follows:

§1 A theorem of Siegel (a special case of §2) and the ABC conjecture.

§2 Results and conjectures in Diophantine geometry, especially finiteness results and conjectures.

§3 The Vojta conjectures, for which heights in number fields are important.

§4 Nevanlinna analogues of the above, for which heights of holomorphic functions are important. This is related to §2 and §3 by analogy.

§5 The proof of the Mordell conjecture by Faltings, the Tate conjecture for abelian varieties by Faltings, and the Tate conjecture for algebraic cycles, assuming the finitude of the number of motives with bounded heights, by Koshikawa (one of my students!).

Here are some good textbooks:

1. *Fundamentals of Diophantine Geometry* (which is the study of algebraic arithmetic geometry concerning rational points and integral points of algebraic equations) by Lang.

Diophantine geometry is named after an ancient Greek mathematician who studied rational and integral points of algebraic equations. Fermat and many others have also studied such questions.

Let us start with §1. We now introduce a theorem of Siegel and later the ABC conjecture.

1.1 Theorem (Siegel). *Let $A \supseteq \mathbb{Z}$ be an integral domain that is finitely generated over $\mathbb{Z}$. Then*

$$\{(x, y) \in A^\times \times A^\times \mid x + y = 1\}$$

is a finite set.

1.2 Examples.

- Let $A = \mathbb{Z}[\frac{1}{2}]$. Then the corresponding set is just $\{(2, -1), (-1, 2), (\frac{1}{2}, \frac{1}{2})\}$, which can be shown by any high school student.

- Let $A = \mathbb{Z}[\frac{1}{6}]$. Then the corresponding set consists of $(2,-1), (3,-2), (9,-8), (4,-3)$ and other pairs obtained from *elementary operations* (such as $x + y = 1 \implies \frac{1}{x} - \frac{y}{x} = 1$, but I shall not explicitly define these operations here), though I do not know how to prove this elementarily.

We include the ABC conjecture in this section because it implies Siegel's theorem, demonstrating that the two subjects are indeed related. The ABC conjecture was originally formulated by Osterlé and Masser in 1985, and its original form proceeds as follows:

1.3 Conjecture (ABC). [1] *Fix $\varepsilon > 0$. Then there are only finitely many triples (a, b, c) for which the a, b, c lie in $\mathbb{Z}_{>0}$, are coprime, satisfy $a + b = c$, and have*

$$\prod_{\substack{p \mid abc \\ p \text{ is prime}}} p \leq c^{1-\varepsilon}.$$

The content of this conjecture is that if $a + b = c$ and $(a, b, c) = 1$, then a, b, and c usually have many prime divisors. Slightly more precisely, we roughly have that $\prod_{p \mid abc} p$ is greater than c.

1.4 Example. For $3 + 5 = 8$, we have $3 \times 5 \times 2 = 30 > 8$.

However, we can sometimes have exceptional cases.

1.5 Examples.

- For $2^3 + 1 = 9$, we have $2 \times 3 = 6 < 9$.
- For $11^2 + 2^2 = 5^3$, we have $11 \times 2 \times 5 = 110 < 125$.
- For $1 + 80 = 81$, we have $2 \times 3 \times 5 = 30 < 81$. Here, 30 is much smaller than 81!
- For $2^5 + 7^2 = 3^4$, we have $2 \times 7 \times 3 = 42 < 81$.

While some of these exceptional cases render the ratio $\left(\prod_{p \mid abc} p\right)/c$ quite small, the ABC conjecture implies that for cofinitely many triples (a, b, c), we have

$$\log\Big(\prod_{p \mid abc} p\Big) > (1-\varepsilon)\log c \implies \frac{\log\left(\prod_{p \mid abc} p\right)}{\log c} > 1 - \varepsilon \implies \liminf_{\substack{c \to \infty \\ a+b=c \\ (a,b,c)=1}} \frac{\log\left(\prod_{p \mid abc} p\right)}{\log c} > 1 - \varepsilon.$$

Taking $\varepsilon \to 0$ yields

$$\lim_{\substack{c \to \infty \\ a+b=c \\ (a,b,c)=1}} \min\left\{\frac{\log\left(\prod_{p \mid abc} p\right)}{\log c}, 1\right\} = 1,$$

so eventually these ratios do get large.

Let's show that the ABC conjecture as stated implies the theorem of Siegel for $A = \mathbb{Z}[\frac{1}{N}]$!

[1] In 2012, Mochizuki uploaded a series of papers containing his proof of the ABC conjecture. However, it is not published yet, as it is difficult to understand the theory and check whether it's correct.

Proof of Siegel's theorem using ABC. Let (x, y) lie in the desired subset. Then $x + y = 1$, and pulling out denominators and rearranging if necessary yields an equation $a + b = c$, where the a, b, and c are positive integers satisfying $(a, b, c) = 1$. Since x and y are units in $\mathbb{Z}[\frac{1}{N}]$, they only have nonzero p-adic valuation for p dividing N. As the a, b, and c are the result of clearing denominators from x and y, we see that the primes dividing abc also divide N. Therefore

$$N \geq \prod_{p|abc} p \geq c^{1-\varepsilon}$$

for cofinitely many (a, b, c) as above. Thus the options for c are bounded, so there are finitely many altogether. □

How would we formulate ABC for more general rings A? We'll discuss one perspective this next time.

2 January 5, 2018

One can almost deduce Fermat's last theorem from the ABC conjecture. Namely, ABC implies that

$$\{(x, y, z, n) \mid x, y, z \in \mathbb{Z} \smallsetminus \{0\},\ n \geq 4 \in \mathbb{Z},\ x^n + y^n = z^n,\ (x, y, z) = 1\}$$

is finite. Of course, the actual Fermat's last theorem says that this set is empty. Furthermore, the Mordell conjecture (proved by Faltings) says that the above set is finite when you fix the value of n, because $x^n + y^n = z^n$ cuts out a projective curve in $\mathbb{P}^2$ of genus $(n-1)(n-2)/2$, which is at least 2 when $n \geq 4$, and the $\mathbb{Q}$-rational points of this curve correspond to solutions in the above set.

All the theorems we've just quoted are really strong. We've seen some of that already, and as further evidence, the ABC conjecture implies the following stronger version of the last paragraph:

2.1 Conjecture (Fermat–Catalan)**.** *The set*

$$\left\{ (a, b, c) \in \mathbb{Z}^3_{>0} \;\middle|\; \begin{array}{c} a + b = c,\ \gcd\{a, b, c\} = 1, \textit{ and there exists } x, y, z \in \mathbb{Z} \textit{ and } m, n, k \in \mathbb{Z}_{>0} \\ \textit{such that } a = x^m,\ b = y^n,\ c = z^k, \textit{ and } \frac{1}{m} + \frac{1}{n} + \frac{1}{k} < 1. \end{array} \right\}$$

is finite.

Proof of the Fermat–Catalan conjecture using ABC. Because a, b, and c have the same prime divisors as x, y, and z (respectively), we have

$$\prod_{p|abc} p \leq xyz \leq z^{k/m} z^{k/n} z = z^{k(\frac{1}{m}+\frac{1}{n}+\frac{1}{k})} = c^{(\frac{1}{m}+\frac{1}{n}+\frac{1}{k})}.$$

By choosing a sufficiently small ε, we see that this is bounded above by $c^{1-\varepsilon}$ for all choices of (m, n, k), and the ABC conjecture tells us that there are only finitely many such (a, b, c) for which $(a, b, c) = 1$. □

The Fermat–Catalan conjecture is an amalgamation of Fermat's last theorem and the following result.

2.2 Theorem (Catalan conjecture)**.** *The only solution to $x^m - y^n = 1$, for m, $n \geq 2$ and positive integers x and y, is $3^2 - 2^3 = 1$.*

Even though the ABC conjecture implies the generalized finitude that the Fermat–Catalan conjecture brings, it doesn't seem to give the precise, concrete description that the Catalan conjecture provides. The Catalan conjecture was proved by Mihăilescu in 2002, and it was a big deal.

We will now discuss a connection with $\mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$, which leads to a geometric generalization of our ABC discussion (and of our Diophantine approximation business at large). First, let $\mathcal{O}(\mathbb{C})$ denote the set of holomorphic functions $\mathbb{C} \longrightarrow \mathbb{C}$, and let us discuss a strange fundamental philosophy:

$$\text{what happens for } \mathcal{O}(\mathbb{C}) \longleftrightarrow \text{what happens for } A,$$

where A is any integral domain containing $\mathbb{Z}$ that is finitely generated over $\mathbb{Z}$. By this, I mean that (analytic) properties in $\mathcal{O}(\mathbb{C})$ correspond to (arithmetic) properties in A.

Let us begin by recasting Siegel's theorem in this setting. For potentially ultimate confusion, write $A = \mathcal{O}(\mathbb{C})$. We can consider the set

$$\{(f, g) \in A^\times \times A^\times \mid f + g = 1\}$$

just as in Siegel's theorem, and this corresponds to taking the set

$$\{f \in A \mid f(\alpha) \notin \{0, 1\} \quad \forall \alpha \in \mathbb{C}\},$$

which in turn is just the set of holomorphic functions $f : \mathbb{C} \longrightarrow \mathbb{P}^1(\mathbb{C}) \smallsetminus \{0, 1, \infty\}$. A theorem of Picard says that such functions must be constant. This constraint on such functions indicates that, in our analogy, we have

$$\text{a theorem of Picard} \longleftrightarrow \text{a theorem of Siegel.}$$

Of course, this philosophy gives no proofs, but it tells us what we ought to expect is true and is not true.

We shall pass the ABC conjecture through this analogy, and it corresponds to a certain (already proven) theorem in *Nevanlinna theory*. This could be construed as evidence for the ABC conjecture. For more on this connection, see Vojta's article "Diophantine approximation and Nevanlinna theory." The slogan of the aforementioned theorem in Nevanlinna theory is that, when we have a non-constant holomorphic map $f : \mathbb{C} \longrightarrow \mathbb{P}^1(\mathbb{C})$, it takes values in $\{0, 1, \infty\}$ at many points. On the number theory side, this shall correspond to p dividing abc for many p.

More precisely, we geometrize the statement of ABC by noticing that the set

$$\{(a, b, c) \in (\mathbb{Z} \smallsetminus \{0\})^3 \mid a + b = c,\ (a, b, c) = 1\}/\{\pm 1\}$$

bijects to $\mathbb{P}^1(\mathbb{Q}) \smallsetminus \{0, 1, \infty\}$ via the map $(a, b, c) \mapsto x := a/c$. For every prime number p, we have a map $\mathbb{P}^1(\mathbb{Q}) \longrightarrow \mathbb{P}^1(\mathbb{F}_p)$ given by reduction mod p after expressing points of $\mathbb{P}^1(\mathbb{Q})$ as $[a : c]$, where a and c are coprime integers.

As often is the case, prime numbers p correspond to points α in $\mathbb{C}$. Furthermore, in our analogy

$$\text{evaluation at } \alpha \longleftrightarrow \text{evaluating along } \mathbb{P}^1(\mathbb{Q}) \longrightarrow \mathbb{P}^1(\mathbb{F}_q),$$

and rational numbers (well, really $\mathbb{Q}$-points of $\mathbb{P}^1$) correspond to meromorphic functions, that is,

$$\Big\{\text{holomorphic maps } \mathbb{C} \longrightarrow \mathbb{P}^1(\mathbb{C})\Big\} \longleftrightarrow \mathbb{P}^1(\mathbb{Q}).$$

We see that x lands in $\{0, 1, \infty\}$ in $\mathbb{P}^1(\mathbb{F}_p)$ if and only if $p|abc$, where we still write $x = a/c$ for $(a, c) = 1$.

For now, let us define height of x to be $H(x) := \max\{|a|, |c|\}$. Additionally, write

$$N^{(1)}(x) := \prod_{\substack{p \text{ is prime} \\ x \in \{0,1,\infty\} \pmod p}} p.$$

Then the ABC conjecture becomes the statement that for all $\varepsilon > 0$, the set

$$\{x \in \mathbb{P}^1(\mathbb{Q}) \smallsetminus \{0, 1, \infty\} \mid N^{(1)}(x) \leq H(x)^{1-\varepsilon}\}$$

is finite. Given that, for any fixed $C > 0$, the set

$$\{x \in \mathbb{P}^1(\mathbb{Q}) \mid H(x) \leq C\}$$

is also finite[2], we see that the ABC conjecture in this light implies that

$$\{x \in \mathbb{P}^1(\mathbb{Q}) \smallsetminus \{0, 1, \infty\} \mid N^{(1)}(x) \leq C\}$$

is finite. From this, one can deduce Siegel's theorem.

The ABC conjecture says that we roughly have $N^{(1)}(x) \geq H(x)$. The corresponding Nevanlinna statement is that we roughly have $N^{(1)}(\alpha) \geq T_f(\alpha)$, and this latter statement is known to be true.[3]

In this geometric optic, we can replace $\mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$ with more general algebraic varieties over number fields. Let F be a number field, and let X be an algebraic variety[4] over F. Let $\overline{X}$ be a compactification of X over F, and here now $\overline{X} \smallsetminus X$ takes the role of $\{0, 1, \infty\}$. We can consider both the $\mathbb{C}$-points as well as F-points of these varieties, and this is the setting in which Vojta formulates his conjectures.

3 January 8, 2018

Recall from Lecture 1 that the ABC conjecture implies Siegel's theorem for $\mathbb{Z}[\frac{1}{N}]$. This time, let me begin by saying a few words on the work of Mochizuki. We have a map

$$\begin{aligned} \mathbb{P}^1(\mathbb{Q}) \smallsetminus \{0, 1, \infty\} &\longrightarrow \{\text{elliptic curves over } \mathbb{Q}\}/\sim \\ \lambda &\longmapsto E_\lambda : y^2 = x(x-1)(x-\lambda). \end{aligned}$$

And roughly speaking, for any prime number p,

$$\lambda \in \{0, 1, \infty\} \pmod p \iff E_\lambda \text{ has bad reduction at } p.$$

The reason for this is that the equation defining E_λ becomes singular modulo p precisely when λ becomes 0, 1, or ∞ modulo p.

Define the height of E_λ to be $H(E_\lambda) := H(\lambda)$, and recall that the latter height is defined to be $\max\{|a|, |c|\}$, where $\lambda = a/c$ for coprime a and c. Now Mochizuki was Faltings's student, the latter of whom defined the heights of abelian varieties over number fields (and used them to prove the Mordell conjecture!).

Last time, we saw that the ABC conjecture corresponds to saying we roughly have

$$\prod_{\substack{p \text{ is prime} \\ \lambda \in \{0,1,\infty\} \pmod p}} p \geq H(\lambda)$$

which hence corresponds to saying that we roughly have

$$\prod_{\substack{p \text{ is prime} \\ E_\lambda \text{ has bad reduction at } p}} p \geq H(E_\lambda),$$

which ties our story of the ABC conjecture to the story of elliptic curves.

[2]See Proposition 9.4.

[3]For the definitions of $N^{(1)}(\alpha)$ and $T_f(\alpha)$, see Lecture 17.

[4]For me, all algebraic varieties are assumed to be geometrically irreducible.

Around the 2000s, Mochizuki was studying *Hodge–Arakelov theory* (a form of "p-adic Hodge theory" for elliptic curves over number fields). He had expected this theory to be useful for proving the ABC conjecture, but he has since changed his strategy. His current strategy also studies elliptic curves, and its goal is to extract information from $\pi_1^{\text{ét}}(E \smallsetminus \{0\})$, which is related to his previous work from the 1990s on *Grothendieck's conjecture*:

3.1 Theorem (Mochizuki)**.** *Let X be a* hyperbolic *curve over a number field, that is, a smooth algebraic curve such that the universal cover of $X(\mathbb{C})$ is the upper half plane Δ. Then $\pi_1^{\acute{e}t}(X)$ determines X up to isomorphism.*

Let us now move into §2, which is concerned with various finitude results and conjectures. Let's begin with the curve case. Let X be a smooth algebraic curve over $\mathbb{C}$, which we can treat via the analytic theory. Let $\overline{X}$ be a smooth projective curve over $\mathbb{C}$ that compactifies X, and write S for the finite subset $\overline{X} \smallsetminus X$. Then $\overline{X}(\mathbb{C})$ is a compact Riemann surface, and it is topologically a donut with g holes, where $g \coloneqq g(\overline{X})$ is the genus of $\overline{X}$. We form the following table for the universal covering of $X(\mathbb{C})$, which depends on $g(\overline{X})$ and $\#S$:

$g(\overline{X})$	$\#S=0$	$\#S=1$	$\#S=2$	$\#S\geq 3$
0	$\mathbb{P}^1(\mathbb{C})$	$\mathbb{C}$	$\mathbb{C}$	Δ
1	$\mathbb{C}$	Δ		
≥ 2	Δ			

That is, X is hyperbolic if and only if $2-2g-\#S<0$. Next, let us consider holomorphic maps $\mathbb{C} \longrightarrow X$.

3.2 Proposition. *All such maps are constant if and only if the universal covering $\widetilde{X(\mathbb{C})}$ is isomorphic to Δ.*

Proof. Suppose that $\widetilde{X(\mathbb{C})}$ is Δ. Then the result follows from first lifting $\mathbb{C} \longrightarrow X(\mathbb{C})$ to a map $\mathbb{C} \longrightarrow \widetilde{X(\mathbb{C})}$ (which we can do because $\mathbb{C}$ is simply connected) and then applying the theorem of Picard.[5]

If $\widetilde{X(\mathbb{C})}$ is not Δ, then we are in one of the exceptional cases in the above table. Here, the maps

$$\mathbb{C} \hookrightarrow \mathbb{P}^1(\mathbb{C}),\ \mathbb{C} \longrightarrow \mathbb{C},\ \exp : \mathbb{C} \longrightarrow \mathbb{C}^\times,\ \mathbb{C} \longrightarrow \mathbb{C}/\Lambda$$

yield nonconstant holomorphic functions, as advertised. □

We now turn back to arithmetic. Recall from a first course in algebraic number theory that the set

$$\{(x,y) \in \mathbb{Z}^2 \mid y^2 = 2x^2 - 1\} = \{(5,7),\ (29,41), \dots\}$$

is infinite, and its members yield progressively better approximations of $\sqrt{2}$. On the other hand, the sets

$$\{(x,y) \in \mathbb{Z}^2 \mid y^3 = 2x^3 - 1\} \text{ and } \{(x,y) \in \mathbb{Z}^2 \mid y^3 = 2x^3 - 3\}$$

are finite. One point in the latter set is $(4,5)$, which corresponds to using $\frac{5}{4}$ to approximate $\sqrt[3]{2}$. In fact, both these sets correspond to rational approximations of $\sqrt[3]{2}$, and that's one reason why they're finite (use Roth's theorem). The fact that $\frac{5}{4}$ and $\sqrt[3]{2}$ are close is a crucial fact in even-tempering notes—it allows the even-tempered major third $(2^{1/12})^4$ to approximate the aesthetically pleasing ratio of frequencies $\frac{5}{4}$.

[5] Alternatively, we can just use the fact that Δ is isomorphic to the open unit disk and apply Liouville's theorem.

In the Vienna Boys' choir, they use 5:4 instead of $\sqrt[3]{2}$ for their frequency ratio of do:mi. Note that $\frac{5}{4}$ has small height, and while I don't know why it's the case because I don't know physics, this ratio of frequencies is pleasing to the ear. Similarly, the golden ratio

$$\phi := \frac{1+\sqrt{5}}{2}$$

also has small height (we haven't defined heights of arbitrary algebraic numbers yet, but we *can*[6]), and this quantity is known to be beautiful for the eyes. The moral of the story is that algebraic numbers of small height are aesthetically pleasing.

Returning to pure mathematics, let us introduce the *Mordell conjecture*, which is a theorem of Faltings:

3.3 Theorem (Faltings). *Let F be a finitely generated field over $\mathbb{Q}$ (i.e. a finite extension of some purely transcendental extension $\mathbb{Q}(T_1, \ldots, T_n)$), and suppose that X is a smooth algebraic curve over F. If $g(X) \geq 2$, then $X(F)$ is finite.*

The Mordell conjecture is usually stated for F a number field, but it has also been proved in this additional generality.

Suppose that F is a subfield of $\mathbb{C}$. Combining Proposition 3.2 with Faltings's theorem yields the following result.

3.4 Corollary. *Assume that $\overline{X} = X$. Then every holomorphic map $\mathbb{C} \longrightarrow X(\mathbb{C})$ is constant if and only if $X(A)$ is finite for all integral domains A containing $\mathbb{Z}$ that are finitely generated over $\mathbb{Z}$ and whose quotient field $Q(A)$ contains F.*

Here, we can interpret $X(A)$ in many ways—for example, by locally fixing embeddings $X \hookrightarrow \mathbb{A}^N$ and taking $X(Q(A)) \cap \mathbb{A}^N(A)$, or by using a model of X over A. They all yield a correct statement. Additionally, in Corollary 3.4, it's true that it suffices to check the finitude criterion for those A which are orders of number fields.

Today was the curve case—next time, I will introduce general conjectures for general algebraic varieties.

4 January 10, 2018

My presentation of the curve case last time was bad, so let me improve it now (as well as present the general case for algebraic varieties).

Let F be a finitely generated field over $\mathbb{Q}$, and let X be a smooth curve over F. Now suppose F is a subfield of $\mathbb{C}$, which enables us to take $\mathbb{C}$-points of X and obtain a Riemann surface $X(\mathbb{C})$. Let $\overline{X}$ be a compactification of X over F. When $X = \overline{X}$, recall that Corollary 3.4 gives us a connection between the arithmetic and analysis of X.

In the case that $X \neq \overline{X}$, our curve X must then be affine, that is, defined by polynomial equations

$$X = \{x \mid f_i(x) = 0 \text{ for all integers } 1 \leq i \leq m\}$$

for some polynomials f_i in $F[T_1, \ldots, T_n]$. In this setting, it turns out that Corollary 3.4 continues to hold, including our remark on the sufficiency of checking on A satisfying $[Q(A) : F] < \infty$:

4.1 Proposition (Siegel). *Assume that $\overline{X} \neq X$. Then the following are equivalent:*

- *Every holomorphic map $\mathbb{C} \longrightarrow X(\mathbb{C})$ is constant,*

[6] See Definition 10.3.

- *$X(A)$ is finite for all integral domains A containing $\mathbb{Z}$ that are finitely generated over $\mathbb{Z}$ and whose quotient field $Q(A)$ contains F,*
- *$X(A)$ is finite for all such integral domains A such that $[Q(A):F]$ is finite.*

Before we explain what $X(A)$ means, let us turn to the case of general algebraic varieties. They are governed by the conjectures of Lang and Vojta. Let X be an algebraic variety over F, which means that it can be covered by finitely many affine algebraic varieties. First, consider the case where X itself is affine, which once again means that it is of the form

$$X = \{x \mid f_i(x) = 0 \text{ for all integers } 1 \leq i \leq m\}$$

for some polynomials f_i in $F[T_1, \dots, T_n]$. For any commutative ring A over F, we write

$$X(A) \coloneqq \{x \in A^n \mid f_i(x) = 0 \text{ for all integers } 1 \leq i \leq m\} = \mathrm{Hom}_F(\mathcal{O}(X), A),$$

where $\mathcal{O}(X) \coloneqq F[T_1, \dots, T_n]/(f_1, \dots, f_m)$.

For general algebraic varieties X over F, consider an open cover $X = \bigcup_{j=1}^r U_j$ of X by affine algebraic varieties U_j. Then, for any integral domain A over F satisfying the condition

$$A = \bigcap_{\mathfrak{m} \in \mathrm{mSpec}(A)} A_{\mathfrak{m}} \text{ in } Q(A),$$

we have the following description of the A-points of X:

$$X(A) = \{x \in X(Q(A)) \mid \forall \mathfrak{m} \in \mathrm{mSpec}(A), \exists \text{ an integer } 1 \leq j \leq r \text{ such that } x \in \mathrm{im}(U_j(A_{\mathfrak{m}}) \longrightarrow X(Q(A))\}.$$

This also holds if we replace every instance of mSpec with Spec. In particular, we stress that $X(A)$ does not simply equal $\bigcup_{i=1}^r U_i(A)$.

4.2 Examples. Let us now examine some algebraic varieties.

- $X = \mathbb{P}^1 \smallsetminus \{0, 1\}$ is affine and corresponds to the ring $F[T, \frac{1}{T}]$, which is isomorphic to

$$F[X, Y]/(XY - 1) \text{ via } X \mapsto T,\ Y \mapsto \frac{1}{T}.$$

- $X = \mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$ is affine and corresponds to the ring $F[T, \frac{1}{T}, \frac{1}{T-1}]$, which is isomorphic to

$$F[X_1, X_2, Y_1, Y_2]/(X_1X_2 - 1, Y_1Y_2 - 1, X_1 - Y_1 + 1) \text{ via } X_1 \mapsto T,\ X_2 \mapsto \frac{1}{T},\ Y_1 \mapsto T - 1,\ Y_2 \mapsto \frac{1}{T-1}.$$

Let us now travel to the analytic side. Let X be an algebraic variety over $\mathbb{C}$, and write

$$A = \mathcal{O}^{\text{hol}}(\mathbb{C}) \coloneqq \{\text{holomorphic functions } \mathbb{C} \longrightarrow \mathbb{C}\},$$
$$X(A) \coloneqq \{\text{morphisms } \mathbb{C} \longrightarrow X(\mathbb{C}) \text{ of complex analytic spaces}\}.$$

That is, $X(A)$ is the set of holomorphic maps $\mathbb{C} \longrightarrow X(\mathbb{C})$. More explicitly, when X is affine and cut out by polynomials $f_1, \dots, f_m$ in $\mathbb{C}[T_1, \dots, T_n]$, we have

$$X(A) = \{\varphi = (\varphi_1, \dots, \varphi_n) \in \mathcal{O}^{\text{hol}}(\mathbb{C})^n \mid f_i(\varphi) = 0 \text{ for all integers } 1 \leq i \leq m\}.$$

For general algebraic varieties X over $\mathbb{C}$, we can compute $X(A)$ exactly as we did in the algebraic setting. Now $Q(A)$ is the field of meromorphic functions $\mathbb{C} \dashrightarrow \mathbb{C}$, the maximal ideals $\mathfrak{m}$ of A are in bijection with points α in $\mathbb{C}$, and under this correspondence we have

$$A_{\mathfrak{m}} = Q(A) \cap \mathcal{O}^{\mathrm{hol}}_{\mathbb{C},\alpha},$$

where $\mathcal{O}^{\mathrm{hol}}_{\mathbb{C},\alpha}$ is the stalk at α of the sheaf of holomorphic functions $\mathbb{C} \longrightarrow \mathbb{C}$. Note that $\mathcal{O}^{\mathrm{hol}}_{\mathbb{C},\alpha}$ equals the subring of convergent series in $\mathbb{C}[\![T-\alpha]\!]$, and $A_{\mathfrak{m}}$ equals the ring of meromorphic functions that are holomorphic at α. In particular, if $X = \bigcup_{j=1}^r U_j$ for some open affine algebraic subvarieties U_j, we once again have $X(A) \neq \bigcup_{j=1}^r U_j(A)$ in general.

We now need a notion of hyperbolicity for general algebraic varieties over $\mathbb{C}$.

4.3 Definition. We say that $X(\mathbb{C})$ is *(Brody) hyperbolic* if any holomorphic map $\mathbb{C} \longrightarrow X(\mathbb{C})$ is constant.

This definition is from 1978. There is an older notion of *Kobayashi hyperbolicity* (from 1967, so it's older) that is phrased in terms of differential geometry, but it only applies when $X(\mathbb{C})$ is a complex manifold (i.e. when X is smooth). I will not explain Kobayashi hyperbolicity for two reasons: it requires some differential geometry to set up, and Brody hyperbolicity fits better with our leanings toward Diophantine geometry. However, we do have the following comparison between the two notions.

4.4 Proposition. *If $X(\mathbb{C})$ is Kobayashi hyperbolic, then it is Brody hyperbolic. Furthermore, when $X(\mathbb{C})$ is compact, the converse holds as well.*

When $X(\mathbb{C})$ is not compact, the converse is known to be false.[7] A few years ago, I was wondering whether Kobayashi hyperbolicity was good or Brody hyperbolicity was good, so I wrote to Vojta. He said that Kobayashi hyperbolicity was bad and that Brody hyperbolicity was good. So from now on, let us only work with Brody hyperbolicity.

Return to the situation of a subfield F of $\mathbb{C}$ that is finitely generated over $\mathbb{Q}$, and let X be an algebraic variety over F. If X is affine and of the form

$$X = \{x \mid f_i(x) = 0 \text{ for all integers } 1 \leq i \leq m\},$$

then for any integral domain A containing $\mathbb{Z}$ and for which $Q(A)$ contains F, we set

$$X(A) \coloneqq A^n \cap X(Q(A)).$$

For general algebraic varieties X over F, we define $X(A)$ as in our previous description.

4.5 Conjecture (Lang–Vojta)**.**

1. *Assume that $X(\mathbb{C})$ is compact. Then the following are equivalent:*
   - *$X(\mathbb{C})$ is hyperbolic,*
   - *$X(K)$ is finite for any finitely generated extension K of F,*
   - *$X(K)$ is finite for any finite extension K of F.*
2. *Suppose that X is affine. Then the following are equivalent:*
   - *$X(\mathbb{C})$ is hyperbolic,*
   - *$X(A)$ is finite for any integral domain A containing $\mathbb{Z}$ that is finitely generated over $\mathbb{Z}$ and whose quotient field $Q(A)$ contains F,*

[7] See Remark 6.5.

- *$X(A)$ is finite for all such integral domains A such that $[Q(A):F]$ is finite.*

3. *Let X be* any *algebraic variety over F. Then the following are equivalent:*

   - *$X(\mathbb{C})$ is hyperbolic,*
   - *$X(A)$ is finite for any integral domain A containing $\mathbb{Z}$ that is finitely generated over $\mathbb{Z}$ and whose quotient field $Q(A)$ contains F,*
   - *$X(A)$ is finite for all such integral domains A such that $[Q(A):F]$ is finite.*

Of course, part 3 generalizes part 2. Someone also notes that the conjectures imply that open subvarieties of hyperbolic varieties remain hyperbolic, but this also follows directly from the definition of hyperbolicity. Note that hyperbolicity is equivalent to saying that $X(\mathcal{O}^{\text{hol}}(\mathbb{C})) = X(\mathbb{C})$, and on the number theory side, hyperbolicity is replaced with finitude results.

Next time, I hope to introduce a generalized version of Conjecture 4.5. Now Conjecture 4.5 itself is great, but hyperbolic varieties are quite rare as well as hard to understand. On the other hand, varieties of *general type* are much better understood as a whole, and we would like to formulate an analogous conjecture for them as well. Lang has a generalized version of Conjecture 4.5 in precisely this context.

5 January 12, 2018

Let us now introduce new conjectures that encompass the ones given last time.

5.1 Conjecture (Lang)**.** *Let F be a finitely generated field over $\mathbb{Q}$, let X be a proper algebraic variety over F, and let $Y \subseteq X$ be a Zariski closed subset. Then the following are equivalent:*

(i) *$X(K) \smallsetminus Y(K)$ is finite for any finitely generated field K over F,*

(ii) *$X(K) \smallsetminus Y(K)$ is finite for any finite extension K of F,*

(iii) *For any embedding $F \hookrightarrow \mathbb{C}$, the image of any non-constant morphism $\mathbb{C} \longrightarrow X(\mathbb{C})$ of complex analytic spaces is contained in $Y(\mathbb{C})$,*

(iv) *For any abelian variety A over $\overline{F}$, the image of any non-constant morphism $A \longrightarrow X_{\overline{F}}$ of varieties over $\overline{F}$ is contained in $Y_{\overline{F}}$.*

Last time's Conjecture 4.5 covered the case when Y is empty. In Conjecture 5.1, the implication (i) $\Longrightarrow$ (ii) is clear, but I claim we can also readily get (iii) $\Longrightarrow$ (iv).

Proof of (iii) $\Longrightarrow$ (iv). Every abelian variety A over $\mathbb{C}$ can be characterized by $A(\mathbb{C}) = \mathbb{C}^g/\Gamma$ for some discrete cocompact subgroup Γ of $\mathbb{C}^g$. By composing with $\mathbb{C}^g \longrightarrow A(\mathbb{C})$ and applying (iii), we see that any non-constant morphism $A(\mathbb{C}) \longrightarrow X(\mathbb{C})$ of complex analytic spaces lands inside $Y(\mathbb{C})$. GAGA gives you the corresponding statement for algebraic varieties over $\mathbb{C}$, and, finally, by viewing $\overline{F} \subset \mathbb{C}$ and descending from $\mathbb{C}$ to $\overline{F}$, we obtain the desired result. $\square$

These are the only statements we can prove in general. However, when X is a curve, we know Conjecture 5.1 by the theorems of Faltings and Siegel.

I was trying to phrase this all in terms of algebraic varieties last time, but it's better to do this with schemes. Usually algebraic varieties are easier than schemes, but you already saw that my presentation on Wednesday was quite messy, and someone told me that schemes are the right choice in this setting. Recall that a *semiabelian* variety is a group variety G that fits into a short exact sequence

$$1 \longrightarrow T \longrightarrow G \longrightarrow A \longrightarrow 0,$$

where T is a torus, and A is an abelian variety.

5.2 Conjecture (Lang–Vojta)**.** *Let R be an integral domain containing $\mathbb{Z}$ that is finitely generated over $\mathbb{Z}$, let X be a scheme over R of finite type, and let $Y \subseteq X$ be a Zariski closed subset. Then the following are equivalent:*

(i) $X(A) \smallsetminus Y(A)$ *is finite for any finitely generated integral domain A over R,*

(ii) $X(A) \smallsetminus Y(A)$ *is finite for any finitely generated integral domain A over R satisfying* $[Q(A) : Q(R)] < \infty$*,*

(iii) For any embedding $R \hookrightarrow \mathbb{C}$, the image of any non-constant morphism $\mathbb{C} \longrightarrow X(\mathbb{C})$ is complex analytic spaces is contained in $Y(\mathbb{C})$,

(iv) Write $F = Q(R)$. For any semi-abelian variety G over $\overline{F}$, the image of any non-constant morphism $G \longrightarrow X_{\overline{F}}$ of varieties over $\overline{F}$ is contained in $Y_{\overline{F}}$.

5.3 Remark. We equip Y with any closed subscheme structure (e.g. the reduced closed subscheme structure). Because we only consider A-points of Y for reduced rings A, this leaves our conjectures unaffected.

On Wednesday, I used varieties over $Q(A)$ instead of schemes over A, but this made the description of A-points became quite complicated.

5.4 Examples.

- In part (iv) of Conjecture 5.2, the fact that we allow semiabelian varieties instead of merely abelian one is rather necessary, as we can see from taking $X = \mathbb{A}^1$ and $Y = \varnothing$.
- Let $R = \mathbb{Z}$, and consider the situation $X = \mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$ and $Y = \varnothing$. Examples 4.2 shows that

  $$\mathbb{P}^1 \smallsetminus \{0,1,\infty\} = \operatorname{Spec} \mathbb{Z}[T, \tfrac{1}{T}, \tfrac{1}{T-1}] \text{ and thus } (\mathbb{P}^1 \smallsetminus \{0,1,\infty\})(A) \text{ is finite}$$

  by the theorem of Siegel. The theorem of Picard verifies condition (iii) as well, and one can readily show condition (iv). Altogether, this verifies the Lang–Vojta conjecture in this case. The scheme $\mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$ over $\mathbb{Z}$ is really the surface $\mathbb{P}^1_{\mathbb{Z}}$ with three horizontal curves punched out, and they're punched out in a way that results in no $\mathbb{Z}$-points.

When Y is empty, condition (iii) in Conjecture 5.2 becomes the condition that X is (Brody) hyperbolic. But there are not many hyperbolic varieties. Hyperbolicity is a differential geometric notion, and it's better to refine these conjectures for varieties of *log general type*, which is a version of *general type* for non-proper varieties. I will not define either of these in general today, but rest assured that most varieties are of log general type.

5.5 Conjecture. *Let F be a subfield of $\mathbb{C}$ that is finitely generated over $\mathbb{Q}$, and let X be an algebraic variety over F. Then the following are equivalent:*

(i) X is of log general type,

(ii) There is a proper Zariski closed subset Y of X such that every non-constant morphism $\mathbb{C} \longrightarrow X(\mathbb{C})$ of complex analytic spaces has image in $Y(\mathbb{C})$.

We have the following corollary of the above conjecture.

5.6 Conjecture. *The following are equivalent:*

(i) Any closed subvariety of X is of log general type,

(ii) $X(\mathbb{C})$ is hyperbolic.

Proof of Conjecture 5.6 using Conjecture 5.5. If $X(\mathbb{C})$ is hyperbolic, then it satisfies these condition (ii) in Conjecture 5.5 for $Y = \varnothing$. Every subvariety Z of X contains Y, so applying Conjecture 5.5 to Z shows that Z is of log general type.

Conversely, suppose that every closed subvariety of X is of log general type. By using Conjecture 5.5 to find a descending chain of algebraic varieties that every holomorphic map $\mathbb{C} \longrightarrow X(\mathbb{C})$ has to land into, we see that $X(\mathbb{C})$ is hyperbolic. $\square$

If we assume Conjecture 5.1 and Conjecture 5.6, we can deduce the following conjecture.

5.7 Conjecture (Bombieri–Lang)**.** *Let F be a finitely generated field over $\mathbb{Q}$, and let X be a proper variety of general type over F. Then $X(F)$ is not Zariski dense in X.*

In the curve case, the Bombieri–Lang conjecture is equivalent to the Mordell conjecture, because a projective curve X is of general type if and only if $g(X) \geq 2$. I should give some explanation on what general type means, shouldn't I? In the case of non-singular projective hypersurfaces $X \subset \mathbb{P}^{n+1}$ of dimension n and degree m, it is of general type if and only if $m \geq n+3$.

5.8 Examples. Note that we are working in characteristic zero for the moment.

- For an example with $n = 1$, note that the Fermat curve

$$\{[x : y : z] \mid x^m + y^m = z^m\}$$

  is of general type if and only if $m \geq 4$.

- For an example with $n = 2$, we see that

$$\{[x : y : z : w] \mid x^5 + y^5 + z^5 + w^5 = 0\}$$

  is of general type.

Hyperbolic varieties are harder to characterize. There is a conjecture of Kobayashi[8] from 1970 that characterizes hyperbolic non-singular projective hypersurfaces as those with sufficiently large m[9], given n.

6 January 17, 2018

Last time, I introduced some very general conjectures, and today I'll discuss some examples and complements of these. I hope to get to the definition of the important notion of *log general type* from last time.

6.1 Example. Over $\mathbb{C}$, consider the affine variety

$$X := \{(x_1, \ldots, x_{n+1}) \mid \sum_{i=1}^{n+1} x_i = 1,\ x_i \neq 0 \text{ for all } 1 \leq i \leq n+1\}.$$

[8]See Conjecture 7.5.

[9]In this sense, perhaps hyperbolic varieties are also quite common—we just don't understand them as well.

Note that when $n = 1$, this variety becomes $\mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$, which we have thoroughly discussed previously via Siegel's theorem, Examples 4.2, and Examples 5.4. So X is a generalized version of the $\mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$ situation. Recall that our conjectures also involve an auxiliary closed subvariety Y—here, it shall be

$$Y := \bigcup_{\substack{I \subseteq \{1,\dots,n+1\} \\ 2 \leq \#I \leq n}} Y_I, \text{ where } Y_I = \{(x_1, \dots, x_{n+1}) \in X \mid \sum_{i \in I} x_i = 0\}.$$

If $n = 1$, then X is Brody hyperbolic as we have seen before. However,for $n \geq 2$ it is not—for example, when $n = 2$ we have a non-constant holomorphic map

$$\begin{aligned} \mathbb{C} &\longrightarrow X(\mathbb{C}) \\ t &\longmapsto (e^t, -e^t, 1) \in Y_I \text{ for } I = \{1, 2\}. \end{aligned}$$

We first proceed to condition (iii) of Conjecture 5.2. Stay in the situation of Example 6.1.

6.2 Theorem (Borel (1897)). *Let $f : \mathbb{C} \longrightarrow X$ is a non-constant morphism of complex analytic spaces, and write $f = (f_1, \dots, f_{n+1})$. Then there exists an I as above such that $\sum_{i \in I} f_i = 0$ and the f_i for $i \neq I$ are constant.*

This is an improvement on the theorem of Picard, which covered the $n = 1$ case. Next, what about the algebraic aspect of Conjecture 5.2? In that situation, take $R = \mathbb{Z}$, and form the finite type $\mathbb{Z}$-scheme

$$X := \operatorname{Spec} R[T_1, \dots, T_n, \tfrac{1}{T_1}, \dots, \tfrac{1}{T_n}, \tfrac{1}{T_1 + \dots + T_n - 1}].$$

This recovers the variety X over $\mathbb{C}$ from Example 6.1, because we can simply replace x_{n+1} with $1 - x_1 - \dots - x_n$. In general, for any $\mathbb{Z}$-algebra A we have

$$\begin{aligned} X(A) &= \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[T_1, \dots, T_n, \tfrac{1}{T_1}, \dots, \tfrac{1}{T_n}, \tfrac{1}{T_1 + \dots + T_n - 1}], A) \\ &= \{a_1, \dots, a_{n+1} \in A^\times \mid \sum_{i=1}^{n+1} a_i = 1\}. \end{aligned}$$

Similarly, form $Y = \bigcup_I Y_I$, where the Y_I are closed subschemes of X cut out by $\sum_{i \in I} T_i = 0$, respectively (where $T_{n+1} = 1 - T_1 - \dots - T_n$). As Conjecture 5.2 predicts, the following result holds (but the proof is hard, and we won't describe it in class):

6.3 Theorem (Poorten–Schlickewei (1982), Evertse (1984)). *The set $X(A) \smallsetminus Y(A)$ is finite for any integral domain A containing $\mathbb{Z}$ that is finitely generated over $\mathbb{Z}$.*

However, the set $X(A)$ itself might be infinite.

6.4 Example. Let $A = \mathbb{Z}[\frac{1}{2}]$ and $n = 2$. Then $X(\mathbb{Z}[\frac{1}{2}])$ contains the infinite subset $(2^m, -2^{-m}, 1)$, so the content of Theorem 6.3 is that infinite subsets of $X(A)$ only arise from the patterns cut out by the Y_I, up to an error of finitely many points lying outside of $Y_I(A)$.

Borel's theorem and Conjecture 5.5 tell us that $X_{\mathbb{Q}}$ should be of *log general type*.

6.5 Remark. Borel's theorem shows us that $X(\mathbb{C}) \smallsetminus Y(\mathbb{C})$ is Brody hyperbolic. When $n = 2$, it turns out that the related set $X(\mathbb{C}) \smallsetminus \{(1, -1, 1), (-1, 1, 1)\}$ is Brody hyperbolic but *not* Kobayashi hyperbolic! So the converse of Proposition 4.4 is not true in general, even though we remind ourselves that Brody hyperbolicity implies Kobayashi hyperbolicity when $X(\mathbb{C})$ is compact.

6.6 Remark. Stay in the $n = 2$ case. Then $(X \smallsetminus Z)(A)$ is always finite, where Z is the union

$$Z \coloneqq \{T_1 = 1,\, T_2 = -1\} \cup \{T_1 = -1,\, T_2 = 1\}.$$

This can be shown using the theorem of Siegel. Note that that $(X \smallsetminus Z)(\mathbb{C}) = X(\mathbb{C}) \smallsetminus \{(1,-1,1),(-1,1,1)\}$, so Remark 6.5 remains consistent with Conjecture 4.5.

I hope now to explain the notion of *log general type*. Let F be a field, let X be an algebraic variety over F, and let $\overline{X}$ be a compactification of X. We begin with the case where X is smooth, and choose $\overline{X}$ such that $D \coloneqq \overline{X} \smallsetminus X$ is a divisor with normal crossings. Form the sheaf

$$\Omega^1_{\overline{X}}(\log D) \coloneqq \{\text{meromorphic Kähler differentials that are regular on } X \text{ with at worst log poles at } D\},$$

where a meromorphic differential has *at worst log poles at D* if it is locally of the form

$$g + h \cdot \frac{\mathrm{d}f}{f},$$

where g is a regular differential, h is a regular function, and f and f^{-1} are both regular outside of D.

6.7 Example. Consider the case of $X = \mathbb{A}^1 = \operatorname{Spec} F[T]$, and write $S = \frac{1}{T}$. Then $D = \{\infty\}$ is a single point. If $f\,\mathrm{d}T$ is any differential on $\overline{X}$, the quotient rule shows that

$$f\,\mathrm{d}T = f\,\mathrm{d}(\tfrac{1}{S}) = -f\tfrac{\mathrm{d}S}{S^2}.$$

Therefore here $\Omega^1_{\overline{X}}(\log D)$ has no nonzero global sections.

In general, $\Omega^1_{\overline{X}}(\log D)$ is a vector bundle of rank n on $\overline{X}$, where n is the dimension of X. Therefore $\omega \coloneqq \bigwedge^n \Omega^1_{\overline{X}}(\log D)$ is a line bundle on $\overline{X}$. We can now define the *log Kodaira dimension* of X.

6.8 Definition. The *log Kodaira dimension* of X is defined to be

$$\kappa(X) \coloneqq \inf\left\{k \in \mathbb{Z} \,\middle|\, m \mapsto \frac{\dim_F \Gamma(\overline{X}, \omega^{\otimes m})}{m^k} \text{ is a bounded function}\right\}.$$

In general, we have different choices of compactification $\overline{X}$. However, the F-vector space $\Gamma(\overline{X}, \omega^{\otimes m})$ will not depend on the particular $\overline{X}$ we choose, so $\kappa(X)$ is well-defined.

Return now to the case of a general algebraic variety X over F, which might be singular. Assuming the resolution of singularities (which is known for $\operatorname{char} F = 0$), take a proper birational morphism $X' \longrightarrow X$, where X' is smooth, and define $\kappa(X)$ to be $\kappa(X')$. It turns out that $\kappa(X)$ is also independent of the X' chosen.

Finally, use log Kodaira dimension to finally define *log general type*.

6.9 Definition. We say that X is of *log general type* if $\kappa(X) = \dim X$.

6.10 Example. We begin by considering the case of $\overline{X} = \mathbb{P}^1$, using the coordinates from Example 6.7. Suppose F has characteristic not equal to 2, and form the following table of values for $\Gamma(\overline{X}, \omega^{\otimes m})$:

X	$\Gamma(\overline{X}, \omega)$	$\Gamma(\overline{X}, \omega^{\otimes m})$ for $m \geq 2$
$\mathbb{P}^1$	0	0
$\mathbb{P}^1 \smallsetminus \{\infty\}$	0	0
$\mathbb{P}^1 \smallsetminus \{0, \infty\}$	$F \cdot \frac{\mathrm{d}T}{T}$	$F \cdot (\frac{\mathrm{d}T}{T})^{\otimes m}$
$\mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$	$F \cdot \frac{\mathrm{d}T}{T} + F \cdot \frac{\mathrm{d}T}{T-1}$	$\sum F \cdot (\frac{\mathrm{d}T}{T})^{\otimes a}(\frac{\mathrm{d}T}{T-1})^{\otimes b}$
$\mathbb{P}^1 \smallsetminus \{0, 1, 2, \infty\}$	$F \cdot \frac{\mathrm{d}T}{T} + F \cdot \frac{\mathrm{d}T}{T-1} + F \cdot \frac{\mathrm{d}T}{T-2}$	something with dimension $2m + 1$

Therefore we see that the first three have log Kodaira dimension 0 and thus are not of log general type, while the last two have log Kodaira dimension 1 are thus *are* of log general type.

In general, we have either $\kappa(X) = -\infty$ or $0 \leq \kappa(X) \leq \dim X$. Next time, we'll wrap this up and then turn to the Vojta conjectures.

7 January 19, 2018

Today, let's discuss some complements to the notions of log Kodaira dimension and log general type.

7.1 Example. Let F be a field, and form the variety X as in Example 6.1 over F. As predicted by Conjecture 5.5, X is of log general type. To see this, consider the compactification $X \hookrightarrow \overline{X} := \mathbb{P}^n$ given by

$$(x_1, \dots, x_{n+1}) \mapsto [-1 : x_1 : \dots : x_n].$$

Then the divisor $D = \overline{X} \smallsetminus X$ at infinity is

$$\bigcup_{i=0}^{n} \{x_i = 0\} \cup \{x_0 + \dots + x_n = 0\},$$

so it is indeed a normal crossings divisor. Form $\omega = \bigwedge^n \Omega^1_{\overline{X}}(\log D)$ as before, and consider $\Gamma(\overline{X}, \omega^{\otimes m})$. Writing

$$e_1 := \frac{\mathrm{d}T_1}{T_1}, \dots, e_1 := \frac{\mathrm{d}T_n}{T_n},\ e_{n+1} := \frac{\mathrm{d}(T_1 + \dots + T_n - 1)}{T_1 + \dots + T_n - 1} = \frac{\mathrm{d}(T_1 + \dots + T_n)}{T_1 + \dots + T_n - 1},$$

we see that $\Gamma(\overline{X}, \omega)$ has an F-base given by

$$\alpha_j := e_{i_1} \wedge \dots \wedge e_{i_n}$$

for any integers $1 \leq i_1 < \dots < i_n \leq n+1$. In general, $\Gamma(\overline{X}, \omega^{\otimes m})$ has an F-base given by

$$\alpha_1^{a(1)} \cdots \alpha_{n+1}^{a(n+1)}$$

for any non-negative integers $a(i)$ totaling m. Calculating dimensions via stars and bars gives us

$$\dim_F \Gamma(\overline{X}, \omega^{\otimes m}) = \binom{m+n}{n} = \frac{(m+1)\cdots(m+n)}{n!},$$

which is a polynomial of degree n in m. Therefore $\kappa(X) = n$, which shows that X is of log general type.

I haven't defined what a normal crossing divisor is! In the case of a complex analytic manifold X, this notion is relatively easy to explain.

7.2 Definition. Let D be a closed complex analytic submanifold of X. We say D is a *normal crossings divisor* if we can locally identify X with an analytic submanifold of $\mathbb{C}^n$ such that, on these charts, D is of the form

$$D = \{z = (z_1, \dots, x_n) \in \mathbb{C}^n \mid \prod_{i=1}^{r} z_i = 0\} \cap X = \left(\bigcup_{i=1}^{r} \{z \in \mathbb{C}^n \mid z_i = 0\}\right) \cap X$$

for some integer $1 \leq r \leq k$.

The algebraic theory of normal crossings divisors is more complicated. For this, we begin by discussing the case of *simple normal crossings divisors*. Let F be a field, and let X be a smooth variety over F.

7.3 Definition. Let D be a closed subscheme of X. We say that D is a *simple normal crossings divisor* if, for all x in X, there exists a regular sequence $t_1, \dots, t_k$ for $\mathscr{O}_{X,x}$ such that $\mathscr{I}_{D,x}$ is generated by $t_1, \dots, t_r$ for some integer $1 \leq r \leq k$, where $\mathscr{I}_D$ denotes the quasicoherent ideal sheaf corresponding to D.

Observe that Definition 7.3 imitates Definition 7.2. However, to obtain the definition for general normal crossings divisors in the scheme-theoretic setting, we need to first pass to an étale cover.

7.4 Example. Consider a non-singular hypersurface X in $\mathbb{P}^{n+1}$ of degree m, which is the homogeneous zero set of a homogeneous polynomial f in $F[T_1, \dots, T_{n+1}]$ of degree m. Then $\dim X = n$, and the Kodaira dimension of X turns out to be

$$\kappa(X) = \begin{cases} n & \text{if } m > n+2, \\ 0 & \text{if } m = n+2, \\ -\infty & \text{if } m < n+2, \end{cases}$$

where we omit the adjective "log" since our varieties are projective. The first case visibly corresponds to general type hypersurfaces, while the second case is referred to as *Calabi–Yau* varieties. When $n = 1$, this corresponds precisely to elliptic curves, and when $n = 2$ this corresponds precisely to *K3 surfaces*.

In this setting, we have the following conjecture of Kobayashi from 1970.

7.5 Conjecture (Kobayashi). *A* general *hypersurface of dimension n and degree m is hyperbolic if $m \geq 2n + 1$.*

By *general*, I mean in the sense that it's true for a Zariski dense subset of $\mathbb{P}^N$, after we fix n and m and then parameterize such hypersurfaces via their coefficients. Some partial results are known, but we do not know Kobayashi's conjecture in any great generality.

We now move into §3, which concerns heights of numbers, the Vojta conjectures, the heights of functions, and Nevanlinna theory. These are related via analogy. We plan to start by discussing the case of an open subset of $\mathbb{P}^1$ (on both the Vojta and Nevanlinna sides) and its relation to Roth's theorem. Later, I'll discuss Koshikawa's work on algebraic cycles as well as Faltings's proof of the Mordell conjecture and Tate conjecture for abelian varieties.

Let's begin with the $\mathbb{P}^1$ situation. Write $\overline{X} := \mathbb{P}^1_{\mathbb{Q}}$, and let X be a non-empty proper open subvariety of $\overline{X}$. Similarly, form $\overline{\mathfrak{X}} := \mathbb{P}^1_{\mathbb{Z}}$, and let $\mathfrak{D}$ be a closed subscheme of $\overline{\mathfrak{X}}$ such that $D := \overline{X} \smallsetminus X$ equals $\mathfrak{D}_{\mathbb{Q}}$.

7.6 Conjecture (Vojta). *Fix a positive ε. Then there are only finitely many x in $X(\mathbb{Q})$ such that*

$$\prod_{\substack{\text{primes } p \\ x \in \mathfrak{D}(\mathbb{F}_p) \pmod p}} p \leq H(x)^{\deg D - 2 - \varepsilon}.$$

7.7 Example. In the case when $X = \mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$, we have $\deg D = 3$. Therefore Vojta's conjecture becomes precisely the ABC conjecture.

I have to explain more about this, but I'm out of time. We'll continue next time! It's known that hyperbolicity for hypersurfaces is stable under small analytically open subsets of $\mathbb{P}^N$, but it's not know that we can take a Zariski open subset (which are all quite big) instead.

8 January 22, 2018

Let $\overline{X}$ be $\mathbb{P}^1_{\mathbb{Q}}$, let $\overline{\mathfrak{X}}$ be $\mathbb{P}^1_{\mathbb{Z}}$, and let $\mathfrak{D}$ be a closed proper subset of $\overline{\mathfrak{X}}$. Write D for $\mathfrak{D}_{\mathbb{Q}}$, write X for $\overline{X} \smallsetminus D$, and write $\mathfrak{X}$ for $\overline{\mathfrak{X}} \smallsetminus \mathfrak{D}$. We denote our affine coordinate for $\mathbb{P}^1$ using T.

8.1 Examples.

1. Consider the case where $\mathfrak{D}$ consists of the $\mathbb{Z}$-points $\{0, 1, \infty\}$. Then this recovers the setting $X = \mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$ to which we have become accustomed.
2. If $\mathfrak{D} = \operatorname{Spec} \mathbb{Z}[T]/(T^3 - 2)$ in $\operatorname{Spec} \mathbb{Z}[T] \subset \mathbb{P}^1_{\mathbb{Z}}$, then D is the closed point in $\mathbb{P}^1_{\mathbb{Q}}$ corresponding to the Galois orbit of $\sqrt[3]{2}$.
3. This example subsumes the previous one—fix a monic polynomial $f(T)$ in $\mathbb{Z}[T]$, and form $\mathfrak{D} = \operatorname{Spec} \mathbb{Z}[T]/(f)$. Then D is the set of Galois orbits of the roots of f in $\mathbb{P}^1_{\mathbb{Q}}$. We don't exactly need $f(T)$ to be monic, but assuming otherwise would change the setup a bit.

By viewing $\mathfrak{D}$ as scheme over $\mathbb{Z}$[10], we can form $\mathfrak{D}(\mathbb{F}_p)$, which is defined by the same equation. Recall that we have a reduction map $\mathbb{P}^1(\mathbb{Q}) \longrightarrow \mathbb{P}^1(\mathbb{F}_p)$. Vojta's conjecture for Example 8.1.2 implies that the sets

$$\{(a, b) \in \mathbb{Z}^2 \mid a^3 - 2b^3 = 1\} \text{ and } \{(a, b) \in \mathbb{Z}^2 \mid a^3 - 2b^3 = -3\}$$

are finite. To see this, the conjecture itself says that

$$\prod_{\substack{\text{primes } p \\ x \in \mathfrak{D}(\mathbb{F}_p) \pmod p}} p \leq H(x)^{1-\varepsilon}$$

for only finitely many x in $X(\mathbb{Q})$, since $\deg D = 3$. Writing $x = a/b$ for coprime integers a and b, we see that

$$x \pmod p \in \mathfrak{D}(\mathbb{F}_p) \iff (a/b)^3 \equiv 2 \pmod p \iff a^3 - 2b^3 \equiv 0 \pmod p.$$

Now the (a, b) lying in our sets of interest only have the prime 3 (at most) dividing $a^3 - 2b^3$. Therefore the quantity

$$\prod_{\substack{\text{primes } p \\ x \in \mathfrak{D}(\mathbb{F}_p) \pmod p}} p$$

is bounded from above by 3. Recalling from Lecture 2 there are only finitely many x with bounded height, we see that Vojta's conjecture concludes the proof.

Note that this also shows that $\mathfrak{X}(\mathbb{Z}[\frac{1}{N}])$ is finite for $\deg D > 2$, since

$$\mathfrak{X}(\mathbb{Z}[\tfrac{1}{N}]) = \{x \in X(\mathbb{Q}) \mid x \pmod p \notin \mathfrak{D}(\mathbb{F}_p) \text{ for } p \nmid N\}.$$

Thus the Vojta conjecture encapsulates our earlier conjecture conjectures and results.

Let us discuss the connection with geometry. In our $\mathbb{P}^1$ setting, recall from Proposition 3.2 that $X(\mathbb{C})$ is hyperbolic if and only if $\deg D \geq 3$. In turn, one can extend Example 6.10 to show that this occurs if and only if X is of log general type. The $\deg D \geq 3$ condition here coincides with the condition that $\deg D - 2 = \deg(\Omega^1_{\overline{X}}(\log D)) > 0$, which is an important criterion in the general Vojta conjectures.

The following theorem of Roth also follows from Vojta's conjecture:

[10] With any scheme structure we like—see Remark 5.3.

8.2 Theorem (Roth). *Let α be an algebraic number, which we view as a complex number, and fix a positive number ε. Then there are only finitely many rational numbers x satisfying*

$$|x-\alpha| \leq \frac{1}{H(x)^{2+\varepsilon}}.$$

This is a statement that limits how well we can approximate algebraic numbers using rational numbers.

8.3 Example. While this example can be proven rather elementarily, we *could* use Roth's theorem to show that

$$\alpha = \sum_{n=0}^{\infty} \frac{1}{10^{n!}}$$

is transcendental, because the series $\sum_{n=m}^{\infty} \frac{1}{10^{n!}}$ approximates α too well for α to be algebraic.

Proof sketch of Roth's theorem using Vojta's conjecture. One can reduce to the case where α is an algebraic integer, so assume this is the case. Let $f(T)$ be the monic irreducible polynomial of α over $\mathbb{Q}$. Furthermore, assume that α lies in $\mathbb{R}$, since otherwise of course we cannot approximate α by rational numbers.Integrality implies that $f(T)$ has coefficients in $\mathbb{Z}$. Applying Vojta's conjecture to the situation of Example 8.1.3 tells us that cofinitely many rational numbers x satisfy

$$\prod_{x \in \mathfrak{D}(\mathbb{F}_p) \pmod p} p \geq H(x)^{\deg D - 2 - \varepsilon},$$

where $\mathfrak{D}$ and D are as in Example 8.1. Now write

$$f(T) = T^n + c_1 T^{n-1} + \cdots + c_n$$

for some integers c_i. Writing $x = a/b$ for coprime integers a and b, emulating the above calculation for $f(T)$ in place of $T^3 - 2$ shows that

$$N_x \coloneqq |a^n + a^{n-1}c_1 b + \cdots + c_n b^n| \geq \prod_{x \pmod p \in \mathfrak{D}(\mathbb{F}_p)} p.$$

Next, form $M_x \coloneqq \max\{|f(x)|^{-1}, 1\}$. Then M_x is large if and only if $f(x)$ is close to zero, which occurs if and only if x is close to some conjugate of α. It turns out that we roughly have

$$M_x N_x \sim H(x)^{\deg D},$$

where $\sim$ means that both sides bound one another up to a constant factor. We should view M_x as an archimedean contribution and N_x as a nonarchimedean contribution. Combining our estimate for N_x with our estimate from Vojta's conjecture yields

$$N_x \gtrsim H(x)^{\deg D - 2 - \varepsilon} \implies M_x \lesssim H(x)^{2+\varepsilon} \implies |x - \alpha| \sim M_x^{-1} \gtrsim \frac{1}{H(x)^{2+\varepsilon}}.$$

I will flesh out this argument next time. □

The weaker estimate $N_x \gtrsim H(x)^{\dim D - 2 - \varepsilon}$ that we used above is actually known unconditionally to be true, so this yields Roth's theorem unconditionally. The main difference between this weaker estimate and the Vojta conjecture itself is that N_x might not just contain the primes p involved in

$$\prod_{x \pmod p \in \mathfrak{D}(\mathbb{F}_p)} p$$

with multiplicity 1 but rather with some very high multiplicity. We can already see this phenomenon in the above situation, as N_x is the sum of many high powers.

9 January 24, 2018

Last time, I started explaining how Roth's theorem is a special case of Vojta's conjecture. I was using some bad notation, so let me now use some better notation. Keeping everything else from Lecture 8 unless otherwise specified, write

$$\begin{aligned} H_{D,\infty}(x) &\coloneqq \max\{|f(x)|^{-1}, 1\} \\ H_{D,f}(x) &\coloneqq |a^n + c_1 a^{n-1} b + \cdots + c_n b^n| \\ H_D(x) &\coloneqq H_{D,\infty}(x) H_{D,f}(x), \end{aligned}$$

where our notation is now much more suggestive of the archimedean and nonarchimedean contributions. We have the following three key points:

(1) Note that

$$\prod_{\substack{p \text{ is prime} \\ p \mid H_{D,f}(x)}} p = \prod_{\substack{p \text{ is prime} \\ x \in \mathfrak{D}(\mathbb{F}_p) \pmod p}} p \text{ divides } H_{D,f}(x),$$

divides $H_{D,f}(x)$. However, the above product of primes does not contain the multiplicity of the primes dividing $H_{D,f}(x)$, i.e. it is the radical of $H_{D,f}(x)$.

(2) We roughly have $H_D(x) \sim H(x)^{\deg D}$, where we note that $\deg D = \deg f$. More precisely:

9.1 Proposition. *There exists positive numbers C_1 and C_2 such that*

$$C_1 H(x)^{\deg D} \leq H_D(x) \leq C_2 H(x)^{\deg D}$$

for all x in $\mathbb{Q}$ such that $f(x) \neq 0$.

I will not prove this, but the key step is

$$\begin{aligned} H_D(x) &= \max\left\{\left|\left(\frac{a}{b}\right)^n + \cdots + c_n\right|^{-1}, 1\right\} |a^n + \cdots + c_n b^n| \\ &= \max\left\{\frac{|b|^n}{|a^n + \cdots + c_n b^n|}, 1\right\} |a^n + \cdots + c_n b^n| = \max\{|b|^n, |a^n + \cdots + c_n b^n|\}. \end{aligned}$$

One immediately sees that we can set $C_2 \coloneqq \max\{|c_1|, \ldots, |c_n|, 1\}$, since $H(x) = \max\{|a|, |b|\}$. The other direction also becomes relatively straightforward.

(3) When $|x - \alpha|$ is small enough, we have $|x - \alpha| \sim H_{D,\infty}(x)^{-1}$. This is the case because

$$f(x) = \prod_{i=1}^{\deg f} (x - \alpha_i),$$

where the α_i range over the conjugates of α over $\mathbb{Q}$. When $|x - \alpha|$ is small, the value $|f(x)|^{-1}$ blows up from the $|x - \alpha|^{-1}$ contribution (while the other factors stay bounded), so this term is what $H_{D,\infty}(x)$ equals. The constants involved in $\sim$ comes from considering these other factors

$$\prod_{\alpha_i \neq \alpha} |\alpha - \alpha_i|.$$

9.2 Remark. If $f(T)$ equals the polynomial T, then $H_D(x)$ equals the height $H(x)$ of x as defined before. To see this, take

$$H_D(x) = \max\{|x|^{-1}, 1\}|a| = \max\{|\tfrac{b}{a}|, 1\} = \max\{|b|, |a|\} = H(x).$$

Now that we have elucidated these three key points, the proof of Roth's theorem of Vojta's conjecture follows as in Lecture 8.

We can also interpret $H_{D,f}(x)$ as

$$H_{D,f}(x) = \prod_{p \text{ is prime}} H_{D,p}(x),$$

where $H_{D,p}(x) \coloneqq \max\{|f(x)|_p^{-1}, 1\}$ for the normalized p-adic norm $|\cdot|_p$. Therefore $H_D(x)$ can be written as

$$H_D(x) = \prod_{v \text{ a place of } \mathbb{Q}} H_{D,v}(x) \implies H(x) = \prod_{v \text{ a place of } \mathbb{Q}} \max\{|x|_v^{-1}, 1\} = \prod_{v \text{ a place of } \mathbb{Q}} \max\{1, |x|_v\}, \quad (\star)$$

where the last equality is due to the product formula. This segues nicely into our next topic: heights on $\mathbb{P}^n$!

9.3 Definition. Let x lie in $\mathbb{P}^n(\mathbb{Q})$, and write x as $[a_0 : \cdots : a_n]$ for some integers a_i such that $(a_0, \ldots, a_n) = 1$. Then the *height* of x, which we denote using $H(x)$, is defined to be

$$H(x) \coloneqq \max\{|a_0|, \cdots, |a_n|\}.$$

9.4 Proposition. *For all real B, the set*

$$\{x \in \mathbb{P}^n(\mathbb{Q}) \mid H(x) \leq B\}$$

is finite.

I don't have time to discuss heights in further detail today, so we'll conclude by giving examples of interesting questions about heights you could study instead. Let $\overline{X}$ be a closed subvariety of $\mathbb{P}^n$ over $\mathbb{Q}$, and let X be a dense open subset of $\overline{X}$. The following conjecture was studied by Manin and Tschinkel, where Manin studied the a part and Tschinkel studied the b part.[11]

9.5 Conjecture (Manin–Tschinkel)**.** *There exist real numbers a, b, and c such that*

$$\#\{x \in X(\overline{\mathbb{Q}}) \mid H(x) \leq B\} = cB^a \log(B)^b(1 + o(1))$$

as $B \to \infty$.

For another interesting question, we also have a more general conjecture of Vojta. Let $\overline{\mathfrak{X}}$ be a closed subset of $\mathbb{P}^1_{\mathbb{Z}}$, and let $\mathfrak{X}$ be a dense open subset of $\overline{\mathfrak{X}}$. Write $\mathfrak{D} \coloneqq \overline{\mathfrak{X}} \smallsetminus \mathfrak{X}$, and form their respective base changes $\overline{X}$, X, and D to $\mathbb{Q}$. There is a version of Conjecture 7.6 that continues to use heights, but this time the exponent of $H(x)$ is related to $\bigwedge^m \Omega^1_{\overline{X}}(\log D)$ (as we remarked in Lecture 8), where m is the dimension of X. I'll explain this in detail next time.

[11]For further discussion, see Lecture 13.

10 January 26, 2018

Today, I'll continue talking about heights in $\mathbb{P}^n$ over a number field. In the situation of a general number field F, it's not generally possible to take a representative of x in $\mathbb{P}^n(F)$ of the form $[a_0 : \cdots : a_n]$ with $(a_0, \ldots, a_n) = 1$, by the failure of unique factorization. In this situation, we adopt the perspective of Equation ($\star$), and define heights using the following product over all places.

10.1 Definition. Let $x = [a_0 : \cdots : a_n]$ be a point of $\mathbb{P}^n(F)$. We say the *height* of x (relative to F) is

$$H_F(x) := \prod_{v \text{ a place of } F} \max\{|a_0|_v, \ldots, |a_n|_v\},$$

where we normalize the absolute values to be the "big" ones, i.e. the ones that make the complex places the square of the usual absolute value on $\mathbb{C}$. By the product formula, $H_F(x)$ is well-defined.

In the case of $F = \mathbb{Q}$, Equation ($\star$) shows that we recover our previous definition of $H(x)$.

10.2 Proposition. *For any finite extension F' of F and x in F, we have $H_{F'}(x) = H_F(x)^{[F':F]}$.*

Proof. For all places v' of F' lying over a place v of F, we have $|x|_{v'} = |x|_v^{[F'_{v'}:F_v]}$. Therefore this equality follows from the sum formula $\sum_{v'|v}[F'_{v'} : F_v] = F$. ☐

10.3 Definition. Let x be a point of $\mathbb{P}^n(\overline{\mathbb{Q}})$. We say the (absolute) *height* of x is $H(x) := H_F(x)^{1/[F:\mathbb{Q}]}$, where F is any number field over which x is defined.

Proposition 10.2 shows that the map $H : \mathbb{P}^n(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}_{>0}$ is well-defined. Now this H map is very complicated—for instance, it's certainly far from being continuous for the complex topology on $\mathbb{P}^1(\overline{\mathbb{Q}})$, but that makes sense because that's just the topology induced from one place. Similar statements would be expected and indeed true at the nonarchimedean places, for instance.

10.4 Example. Consider the $n = 1$ situation. Here, we define the height of any x in $\overline{\mathbb{Q}}$ via taking the height of $[x : 1]$ in $\mathbb{P}^1(\overline{\mathbb{Q}})$. The algebraic numbers α satisfying $[\mathbb{Q}(\alpha) : \mathbb{Q}] \leq 2$ of smallest height are

α	$H(\alpha)$
$0, \pm 1, \pm 2, \pm\zeta_3^2$	1
$\pm\frac{1\pm\sqrt{5}}{2}$	$\sqrt{\frac{1+\sqrt{5}}{2}}$

Note that the second row includes the golden ratio, and its low height encapsulates its simplicity. This simplicity appears to our eyes, even if we don't explicitly compute its height. Small numbers give us some beauty.

Our eyes can enjoy 2-dimensional beauty, but our ears are not so good for this. Our ears detect sound, which comes from time and hence is 1-dimensional.

do	re	mi	fa	so
1	$\sqrt[12]{2}$	$\sqrt[12]{2^2}$	$\sqrt[12]{2^3}$	$\sqrt[12]{2^4}$

These ratios are good when used by pianos, but for the harmony of human voices, it may be better to use

do:mi	do:fa	do:so
5:4	4:3	3:2

On the other hand, the piano needs even-tempering for the changing of keys. The problem of which system is better is probably why ancient Greek mathematicians discovered irrational numbers. To them, mathematics and music were the same subject.

The following theorem is an important generalization of Proposition 9.4.

10.5 Theorem. *Fix n, fix d, and fix a positive number C. Then*

$$\{x \in \mathbb{P}^n(\overline{\mathbb{Q}}) \mid H(x) \leq C \text{ and } x \text{ is defined over a field of degree at most } d\}$$

is a finite set. In particular, $\{x \in \mathbb{P}^n(F) \mid H_F(x) \leq C\}$ is finite for any number field F.

I will not prove it, but the idea is to bound the possible coefficients for the minimal polynomial of any x in the above set using our constraints.

10.6 Example. The sequence $H(2^{1/n})$ is bounded, which does not contradict Theorem 10.5 because the degrees of $2^{1/n}$ are unbounded.

Let's now turn to a version of Vojta's conjecture for (Brody) hyperbolic spaces. Embed F into $\mathbb{C}$, and let $\mathfrak{X}$ be a locally closed subscheme of $\mathbb{P}^n_{\mathcal{O}_F}$, i.e. a quasi-projective scheme over $\mathcal{O}_F$. Assume that $\mathfrak{X}(\mathbb{C})$ is Brody hyperbolic.

10.7 Conjecture (Vojta)**.** *There exists a positive number a such that, for almost all x in $\mathfrak{X}(F)$, we have*

$$\prod_{v \in \Sigma(x)} \#\mathbb{F}_v \geq H(x)^a,$$

where $\Sigma(x)$ denotes the set of primes of F for which $x \pmod v$ does not lie in $\mathfrak{X}(\mathbb{F}_v)$, and $\mathbb{F}_v$ denotes the residue field at v.

Conjecture 10.7 does not seem to explicitly appear in the literature, but I believe many experts think it is true.

10.8 Example. Because $\mathbb{P}^n(F) = \mathbb{P}^n(\mathcal{O}_F)$, it follows from Conjecture 10.7 that $\mathfrak{X}(\mathcal{O}_{F,S})$ is finite for any finite subset S of places containing the places at infinity, where $\mathcal{O}_{F,S}$ denotes the S-integers of F. This is the case because for any x in $\mathfrak{X}(F)$, this point x lies in $\mathfrak{X}(\mathcal{O}_{F,S})$ if and only if $x \pmod v$ lies in $\mathfrak{X}(\mathbb{F}_v)$ for all places v lying outside of S, which follows from the description of schemes given in Lecture 5. Conjecture 10.7 bounds the heights of cofinitely many such x, and Theorem 10.5 yields the result.

10.9 Example. Take $F = \mathbb{Q}$, and let $\mathfrak{X}$ be $\mathbb{P}^1_{\mathbb{Z}} \smallsetminus \{0, 1, \infty\}$ as usual. Then the ABC conjecture predicts that Conjecture 10.7 holds for $\mathfrak{X}$ with $a = 1 - \varepsilon$ for any positive ε.

11 January 29, 2018

We now turn to the *(Batyrev)–Manin* conjecture, beginning with an extended example for motivation.

11.1 Example. Let X denote $\mathbb{P}^1_{\mathbb{Q}} \smallsetminus D$, where D is either $\varnothing$, $\{\infty\}$, $\{0, \infty\}$, or $\{0, 1, \infty\}$. Let $\mathfrak{X}$ be an open subscheme of $\mathbb{P}^1_{\mathbb{Z}}$ whose generic fiber is X, which could be given by $\mathbb{P}^1_{\mathbb{Z}}$ or a subscheme analogous to one given in Examples 4.2. For any positive integer M and positive number B, write

$$N(\mathfrak{X}(\mathbb{Z}[\tfrac{1}{M}]), B) \coloneqq \#\{x \in \mathfrak{X}(\mathbb{Z}[\tfrac{1}{M}]) \mid H(x) \leq B\}.$$

We shall consider the limit

$$\lim_{B \to \infty} \frac{\log N(\mathfrak{X}(\mathbb{Z}[\frac{1}{M}]), B)}{\log B},$$

often testing it out on small values of M.

- In the $D = \varnothing$ case, we have $\mathfrak{X}(\mathbb{Z}[\frac{1}{M}]) = \mathbb{P}^1(\mathbb{Q})$. Therefore $N(\mathfrak{X}(\mathbb{Z}[\frac{1}{M}]), B)$ becomes the number of points in $\mathbb{P}^1(\mathbb{Q})$ with height at most B. After forgetting scalars and the relative primeness of a lowest terms fraction expression, this is roughly the set
$$\{(a, b) \in \mathbb{Z}^2 \mid |a| \leq B,\ |b| \leq B\},$$
which has size B^2. Thus here the limit becomes 2.

- In the $D = \{\infty\}$ case, we have $\mathfrak{X}(\mathbb{Z}) = \mathbb{Z}$. Therefore $N(\mathfrak{X}(\mathbb{Z}), B)$ roughly has size 2B, and
$$N(\mathfrak{X}(\mathbb{Z}[\tfrac{1}{2}], B) = \#\{n = \tfrac{m}{2^r} \in \mathbb{Z}[\tfrac{1}{2}] \mid |m| \leq B,\ |2^r| \leq B, \text{ where } (m, 2^r) = 1\}$$
roughly has size $2(B + \frac{B}{2} + \frac{B}{4} + \cdots) = 4B$. In both these situations, we see that the limit becomes 1.

- In the $D = \{0, \infty\}$ case, we have have
$$N(\mathfrak{X}(\mathbb{Z}[\tfrac{1}{2}]), B) = \#\{\pm 2^m \mid 2^{|m|} \leq B\} \sim 4\frac{\log B}{\log 2}.$$
Therefore the limit here is 0.

- In the $D = \{0, 1, \infty\}$ case, the set $\mathfrak{X}(\mathbb{Z}[\frac{1}{M}])$ is finite by Siegel's theorem. Therefore the limit here is 0.

This behavior is related to the sheaf $\Omega^1_{\mathbb{P}^1_{\mathbb{Q}}}(\log D)$ on $\mathbb{P}^1_{\mathbb{Q}}$. Recall that we have an isomorphism $\mathscr{O}_{\mathbb{P}^1_{\mathbb{Q}}}(-2(\infty)) \xrightarrow{\sim} \Omega^1_{\mathbb{P}^1_{\mathbb{Q}}}$ given by $f \mapsto f\,\mathrm{d}T$. We can see this from the descriptions
$$\Omega^1_{\mathbb{P}^1_{\mathbb{Q}}}(U) = \{f\,\mathrm{d}T \mid f \text{ has no poles on } U\} \text{ and } \mathscr{O}_{\mathbb{P}^1_{\mathbb{Q}}}(-2(\infty)) = \{f \mid f + 2(\infty) \geq 0 \text{ restricted to } U\},$$
where U is any open subset of $\mathbb{P}^1_{\mathbb{Q}}$, because the coordinate $S = \frac{1}{T}$ at infinity satisfies $\mathrm{d}S = -\mathrm{d}T/T^2$. Therefore we have $\Omega^1_{\mathbb{P}^1_{\mathbb{Q}}}(\log D) \cong \mathscr{O}_{\mathbb{P}^1_{\mathbb{Q}}}(D - 2(\infty))$. Furthermore, note in our examples that
$$\lim_{B\to\infty} \frac{\log N(\mathfrak{X}(\mathbb{Z}[\frac{1}{M}]), B)}{\log B} = \max\{2 - \deg D, 0\}.$$

We are now ready to present the Manin conjecture in general. Let F be a number field, let $\overline{X}$ be an irreducible smooth closed subscheme of $\mathbb{P}^n_F$, let D be a normal crossings divisor of $\overline{X}$, and write $X := \overline{X} \smallsetminus D$. Furthermore, let $\mathfrak{X}$ be a locally closed subscheme of $\mathbb{P}^n_{\mathcal{O}_F}$ whose generic fiber is X. Set
$$\alpha := \{s \in \mathbb{Q}_{\geq 0} \mid \mathrm{cl}(\omega) + s \cdot \mathrm{cl}(H) \text{ is effective}\},$$
where ω denotes $\bigwedge^{\dim \overline{X}} \Omega^1_{\overline{X}}(\log D)$, and H is the hyperplane associated to our choice of height function H.[12] I will explain these notions next time—for now, let us just proceed to the conjecture.

11.2 Conjecture (Batyrev–Manin)**.** *There exists a finite extension F_1 of F and a finite set of places S_1 of F_1 containing the archimedean places such that, for any finite extension F' of F_1 and finite set S' of places of F' containing those lying above S_1, then there exists a proper closed subscheme Y of X such that*
$$\lim_{B\to\infty} \frac{\#\{x \in \mathfrak{X}(\mathcal{O}_{F',S'}) \mid x \notin Y(F'),\ H(x) \leq B\}}{\log B} = \alpha.$$

In the case when X has dimension 1, the subscheme Y has dimension 0, so the $x \notin Y(F')$ condition does not change this limit, as it only affects finitely many points. However, this may not be true in general.

[12] This definition of α is incorrect as given here—see Lecture 12.

12 January 31, 2018

We shall continue discussing the Batyrev–Manin conjecture today, and for this we begin with some preliminaries on divisors and line bundles. Let k be a field, and let V be a connected proper smooth algebraic variety over k.

12.1 Definition. A *divisor* is a formal finite sum $\sum_P a_P \cdot P$, where the a_P are integers and P ranges over closed irreducible codimension 1 subvarieties of V. We write $\mathrm{Div}(V)$ for the abelian group of divisors on V, which is a free abelian group on the set of P.

For any f in $k(V)^\times$, we write

$$\mathrm{Div}(f) := \sum_P \mathrm{ord}_P(f) P,$$

where $\mathrm{ord}_P(f)$ is the *order* of f at P. We define $\mathrm{ord}_P(f)$ via the normalized valuation induced from the discrete valuation ring $\mathscr{O}_{V,v}$, where v is the generic point of P. We define the divisor class group $\mathcal{C}\ell(V)$ via the exact sequence

$$k(V)^\times \xrightarrow{\mathrm{Div}} \mathrm{Div}(V) \longrightarrow \mathcal{C}\ell(V) \longrightarrow 0.$$

On the other hand, for any $D = \sum_P a_P P$ in $\mathrm{Div}(V)$, write $\mathscr{O}_V(D)$ for the sheaf of $\mathscr{O}_V$-modules given by

$$U \mapsto \{f \in k(V) \mid \mathrm{ord}_P(f) + a_P \geq 0 \text{ for all } v \in U\}.$$

Then $\mathscr{O}_V(D)$ is a line bundle, i.e. it is locally isomorphic to $\mathscr{O}_V$ as an $\mathscr{O}_V$-module. Form the set

$$\mathrm{Pic}(V) := \{\text{isomorphism classes of line bundles on V}\},$$

which becomes an abelian group under tensor products. We get an isomorphism $\mathcal{C}\ell(V) \xrightarrow{\sim} \mathrm{Pic}(V)$ given by sending $D \mapsto \mathscr{O}_V(D)$.

12.2 Example. Let $V = \mathbb{P}^1_k$ with the coordinate T, and consider the sheaf Ω^1_V on V. It is a line bundle on V because it is $\mathscr{O}_V\, \mathrm{d}T$ when restricted to $\operatorname{Spec} k[T]$ and $\mathscr{O}_V\, \mathrm{d}(\frac{1}{T})$ when restricted to $\operatorname{Spec} k[\frac{1}{T}]$. However, it is not the trivial line bundle. In fact, we can show that $\Omega^1_V \cong \mathscr{O}_V(-2(\infty))$.

For simplicity in defining the *Néron–Severi group*, assume that k lies in $\mathbb{C}$. Then we can form the complex manifold $V(\mathbb{C})$, and we define the Néron–Severi group to be

$$\mathrm{NS}(V) := \mathrm{im}(\mathcal{C}\ell(V) \longrightarrow H^2(V(\mathbb{C}), \mathbb{Z})),$$

where the map is given by sending P to the cohomology class of $P(\mathbb{C})$. In particular, this map factors through $\mathcal{C}\ell(V)$. This may be an analytic construction, but it can be made algebraic in some way that I won't specify.

12.3 Example. When $\dim V = 1$, the Néron–Severi group embeds into $H^2(V(\mathbb{C}), \mathbb{Z}) = \mathbb{Z}$, and it's the image of the map sending P to $\deg(P) := [k(P) : k]$. Therefore if k is algebraically closed, we see that $\mathrm{NS}(V)$ equals the entirety of $H^2(V(\mathbb{C}), \mathbb{Z})$.

In the setting of general V, the sheaf of Kähler differentials Ω^1_V is a vector bundle of rank n, where $n := \dim V$. Therefore the *canonical class* $\omega := \bigwedge^n \Omega^1_V$ is a line bundle on V.

12.4 Definition. We say a divisor K on V is *canonical* if its corresponding line bundle is isomorphic to ω. For any normal crossings divisor D of V, we say that $K + D$ is *log canonical* for (V, D) or for $V \smallsetminus D$.

It can be shown that the line bundle corresponding to $K+D$ is isomorphic to $\bigwedge^n \Omega^1_V(\log D)$.

12.5 Example. Suppose that $V = \mathbb{P}^n_k$, and let $H \subset \mathbb{P}^n_k$ be the hyperplane corresponding to $[a_0 : \cdots : a_n]$ in $\mathbb{P}^n(k)$. Then the isomorphism class of the line bundle corresponding to H is independent of H, and we denote it using $\mathscr{O}_{\mathbb{P}^n_k}(1)$. The isomorphism class corresponding to K equals $-(n+1)$ times $\mathscr{O}_{\mathbb{P}^n_k}(1)$.

Next, suppose that V is now a closed subscheme of $\mathbb{P}^n_k$. If H is a hyperplane that does not contain V, then we say $V \cap H$ is a *hyperplane section* of V. In this situation, the line bundle associated to $V \cap H$ is independent of H, and it equals the pullback of $\mathscr{O}_{\mathbb{P}^n_k}(1)$ to V.

Next, let F be a number field, which we view as a subfield of $\mathbb{C}$. Let $\overline{X}$ be a connected smooth closed subvariety of $\mathbb{P}^N_F$. Let D be a normal crossings divisor on $\overline{X}$, and write $X \coloneqq \overline{X} \smallsetminus D$. Write n for the dimension of X. Here comes my first mistake—the definition of α should have been

$$\alpha \coloneqq \inf\{s \in \mathbb{R}_{\geq 0} \mid \mathrm{cl}(K+D) + s\cdot \mathrm{cl}(H) \in \overline{\mathrm{NS}(\overline{X})}_{\mathbb{R}\text{-eff}}\},$$

where $\overline{\mathrm{NS}(\overline{X})}_{\mathbb{R}\text{-eff}}$ is the closure in $\mathrm{NS}(\overline{X}) \otimes_{\mathbb{Z}} \mathbb{R}$ of the subset

$$\mathrm{NS}(\overline{X})_{\mathbb{R}\text{-eff}} \coloneqq \{\sum_{i=1}^m a_i\,\mathrm{cl}(E_i) \mid a_i \geq 0,\ E_i \text{ is an effective divisor on } \overline{X}\}.$$

We remark that $\mathrm{NS}(\overline{X})_{\mathbb{R}\text{-eff}}$ might be generated by rays of of slope tending towards a limit, so $\mathrm{NS}(\overline{X})_{\mathbb{R}\text{-eff}}$ itself might not be closed. I'm not sure how Vojta came up with α in the formulation of his conjectures, but in any case it's a very geometric (rather than arithmetic) quantity.

Finally, let $\mathfrak{X}$ be a locally closed subscheme of $\mathbb{P}^N_{\mathcal{O}_F}$ whose generic fiber is X, and let $\overline{\mathfrak{X}}$ be the closure of $\overline{X}$ in $\mathbb{P}^N_{\mathcal{O}_F}$. Now that we have the right setup, we can state the Batyrev–Manin conjecture, which actually comes with two parts:

12.6 Conjecture (Batyrev–Manin)**.**

(1) Assume that α is positive. Then if $F' \supseteq F$ is a sufficiently large number field, then if S is a sufficiently large finite set of places of F' containing the archimedean ones, then if Y is a sufficiently large proper closed subset of X, then

$$\lim_{B\to\infty} \frac{\#\{x \in \mathfrak{X}(\mathcal{O}_{F',S}) \mid x \notin Y(F'),\ H(x) \leq B\}}{\log B} = \alpha.$$

In the Batyrev–Manin, I'm not sure whether the S or Y should come first, that is, which "sufficiently large" condition depends on which. I'll get to the next part of the conjecture next time.

13 February 2, 2018

Last time, I was stating the conjecture of Batyrev and Manin. Recall that α is some real number that was defined in a highly geometric manner. I stated the conjecture for positive α, but I didn't get to finish.

13.1 Conjecture (Batyrev–Manin)**.** *Keeping our setup from before,*

(2) If α is zero, then for any finite set of places S of F containing the archimedean places and any positive ε, there exists a proper closed subset Y of X such that

$$\frac{\#\{x \in \mathfrak{X}(\mathcal{O}_{F,S}) \mid x \notin Y(F),\ H(x) \leq B\}}{B^{\varepsilon}}$$

is a bounded function of B.

More on the Batyrev–Manin conjecture can be found in their article "Sur le nombre des points rationnels de hauteur borné." In the case when $X = \overline{X}$ is proper, we see that $\mathfrak{X}(\mathcal{O}_{F,S}) = X(F)$, so we can replace $\mathfrak{X}(\mathcal{O}_{F,S})$ with $X(F)$ in the Batyrev–Manin conjecture.

13.2 Example. Consider the case $X = \overline{X} = \mathbb{P}^n_F$. In this case, $\alpha = n+1$, because in this case $\operatorname{Pic}(\mathbb{P}^n)$ is isomorphic to $\mathbb{Z}$, where any integer a corresponds to the isomorphism class of the line bundle corresponding to any hyperplane H. Thus we see that $\overline{\mathrm{NS}(\mathbb{P}^n_F)}_{\mathbb{R}\text{-eff}}$ equals $\mathbb{R}_{\geq 0}$. Furthermore, the canonical divisor K of $\mathbb{P}^n_F$ has class equal to $-(n+1)$ times the class of H, and here $D = \varnothing$. Altogether we see that $\alpha = n+1$.

The case of a general number field F is difficult to understand, so let us specialize to $F = \mathbb{Q}$. Then

$$\#\{x = [a_0 : \cdots : a_n] \in \mathbb{P}^n(\mathbb{Q}) \mid H(x) = \max\{|a_0|, \ldots, |a_n|\} \leq B\} \sim B^{n+1},$$

where we choose $a_0, \ldots, a_n$ satisfying $(a_0, \ldots, a_n) = 1$. Therefore we see that the Batyrev–Manin conjecture indeed holds for $\mathbb{P}^n_{\mathbb{Q}}$.

13.3 Example. Let $X = \overline{X}$ be a projective (which is the same thing as proper in this case, and I'll move between these readily) smooth curve over F, and embed $\overline{X}$ as a subvariety of $\mathbb{P}^m_F$. Example 12.3 shows that $\overline{\mathrm{NS}(X)}_{\mathbb{R}\text{-eff}}$ consists of the preimage of $\mathbb{R}_{\geq 0} \subset H^2(X(\mathbb{C}), \mathbb{R}) = \mathbb{R}$ in $\mathrm{NS}(X)_{\mathbb{R}}$. Therefore

$$\overline{\mathrm{NS}(X)}_{\mathbb{R}\text{-eff}} = \mathbb{R}_{\geq 0},$$

and because here the degree of K is $2g-2$ (where g is the genus of X), we see that

$$\alpha = \begin{cases} 0 & \text{if } g \geq 1, \\ \left\lfloor \frac{2}{\deg H} \right\rfloor & \text{if } g = 0, \end{cases}$$

where H is a hyperplane section (that is, a closed point) of X with minimal degree.

Now suppose that $g = 1$, and write E for X. Then the theory of elliptic curves tells you that

$$\#\{x \in E(F) \mid H(x) \leq B\} = c \log(B)^{r/2}(1 + o(1)),$$

where r is the Mordell–Weil rank of $E(F)$. Taking the limit from the Batyrev–Manin conjecture indeed yields 0, because the extra log factor kills the rank term regardless.

For $g \geq 2$, the Mordell conjecture tells us that $\#X(F)$ is finite, which verifies the conjecture too, and we have dealt with the $g = 0$ case in Example 13.2 (assuming Example 13.2 extends to general F), since here $X = \mathbb{P}^1_F$ if and only if it has rational points at all.

13.4 Example. If X is of log general type, then the class of $K + D$ always already lies in $\overline{\mathrm{NS}(\overline{X})}_{\mathbb{R}\text{-eff}}$, so here $\alpha = 0$. Recall that the definition of X being of log general type is equivalent to $\bigwedge^n \Omega^1_{\overline{X}}(\log D)$ being *big*, whose definition we give below.

13.5 Definition. Let $\mathscr{L}$ be a line bundle on $\overline{X}$. We say that $\mathscr{L}$ is *big* if

$$\kappa(\mathscr{L}) \coloneqq \inf\left\{ s \in \mathbb{Z} \mid m \mapsto \frac{\dim \Gamma(X, \mathscr{L}^{\otimes m})}{m^s} \text{ is a bounded function} \right\}$$

equals $\dim \overline{X}$.

The notion of *height zeta functions* are important in this subject, so we review it now.

13.6 Definition. Let Λ be a set, and let $H : \Lambda \longrightarrow \mathbb{R}_{>0}$ be a map. Their *height zeta function* is defined to be

$$Z_{\Lambda,H}(s) := \sum_{x\in\Lambda} \frac{1}{H(x)^s},$$

ignoring convergence issues for now.

13.7 Example. In our setup, we take Λ to be $\mathfrak{X}(\mathcal{O}_{F,S})$ and H to be the usual absolute height function:

- When $\mathfrak{X} = \mathbb{P}^1_{\mathbb{Z}}$, then the resulting height zeta function is

  $$Z_{\mathbb{P}^1(\mathbb{Q}),H}(s) = \frac{4\zeta(s-1)}{\zeta(s)},$$

  where ζ is the Riemann zeta function. To see this, note that

  $$Z_{\mathbb{P}^1(\mathbb{Q}),H}(s) = \sum_{n=1}^{\infty} \frac{\#\{x \in \mathbb{P}^1(\mathbb{Q}) \mid H(x) = n\}}{n^s} = \sum_{n=1}^{\infty} \frac{4\varphi(n)}{n^s} = 4\frac{\zeta(s-1)}{\zeta(s)},$$

  where φ denotes the Euler phi function.

In general, suppose that we can write $Z_{\Gamma,H}$ as

$$Z_{\Gamma,H}(s) = \frac{c}{(s-a)^b} + \frac{g(s)}{(s-a)^{b-1}}$$

for some non-negative real number a, positive integer b, positive real number c, and holomorphic function $g(s)$ on the domain $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > a - S\}$ for some positive S. The $Z_{\Gamma,H}(s)$ absolutely converges for $\mathrm{Re}(s) > a$, and we have

$$\#\{x \in \Gamma \mid H(x) \le B\} = \frac{c}{a(b-1)!} B^a \log(B)^{b-1}(1 + o(1)),$$

where our asymptotic is taken with respect to B.

13.8 Example. Consider the situation of Example 13.7. As

$$\zeta(s) = \frac{1}{s-1} + \text{ a holomorphic function}$$

around $s = 1$, we see that

$$Z_{\mathbb{P}^1(\mathbb{Q}),H}(s) = 4\frac{\zeta(s-1)}{\zeta(s)} = \frac{4}{\zeta(2)}\frac{1}{s-2} + \text{ a holomorphic function}$$

around $s = 2$. Then $a = 2$, $b = 1$, and $c = 4/\zeta(2)$, so we have

$$\#\{x \in \mathbb{P}^1(\mathbb{Q}) \mid H(x) \le B\} = \frac{4}{2\zeta(2)} B^2(1 + o(1)) = \frac{12}{\pi^2} B^2(1 + o(1)).$$

The method of zeta functions is very powerful. Batyrev–Manin further conjectured that $a = \alpha$. They also have some conjectures concerning b, but they are very complicated, so I do not plan to discuss them in class.

14 February 5, 2018

Let us now discuss the height functions associated to line bundles. Next time, we'll investigate a question that Olivier brought up after class last time, since it's an interesting question that I spent a lot of time thinking about. I *would* do it now, but he's not here, so that'd be weird.

Let F be a number field, and let $\overline{X}$ be a smooth closed subvariety of $\mathbb{P}^n_F$. Let D be a normal crossings divisor of $\overline{X}$, and form $X := \overline{X} \smallsetminus D$. As usual, let $\mathfrak{X}$ be a locally closed subscheme of $\mathbb{P}^n_{\mathcal{O}_F}$ whose generic fiber is X. We have the following general conjecture of Vojta.

14.1 Conjecture (Vojta)**.** *Let ε be positive. Then there exists a proper closed subset Y of X such that, for cofinitely many x in $X(F) \smallsetminus Y(F)$, we have*

$$\left(\prod_{\substack{v \text{ a prime} \\ x \in \mathfrak{X}(\mathbb{F}_v) \pmod{v}}} \#\mathbb{F}_v \right)^{1/[F:\mathbb{Q}]} \geq H_{K+D}(x) H(x)^{-\varepsilon},$$

where H denotes our absolute height as before, and H_{K+D} has not yet been explained yet.

For any proper algebraic variety V over F and line bundle $\mathscr{L}$ on V, we shall obtain a *height function*

$$H_{\mathscr{L}} : V(\overline{F}) \longrightarrow \mathbb{R}_{>0},$$

via constructing a height function $H_{F,\mathscr{L}} : V(F) \longrightarrow \mathbb{R}_{>0}$ and setting $H_{\mathscr{L}} := H_{F,\mathscr{L}}^{1/[F:\mathbb{Q}]}$. The exponent's role is to ensure that we can repeat this for all finite extensions F'/F and get a well-defined function $H_{\mathscr{L}}$.

14.2 Example. If $V = \overline{X}$ and $\mathscr{L} = \bigwedge^{\dim X} \Omega^1_{\overline{X}}(\log D)$, then $H_{\mathscr{L}}$ yields the desired H_{K+D} in the statement of Vojta's conjecture.

14.3 Example. If $V = \mathbb{P}^n_F$ and $\mathscr{L}$ is the line bundle associated to a hyperplane H, then the resulting $H_{\mathscr{L}}$ is the usual absolute height on $\mathbb{P}^n_F$.

Well, these examples are not exact on the nose—the height function $H_{\mathscr{L}}$ depends on certain choices made. However, for any such choices, the resulting height functions $H_{\mathscr{L}}$ and $H'_{\mathscr{L}}$ shall satisfy $H_{\mathscr{L}} \sim H'_{\mathscr{L}}$, that is, there exists $C \geq 1$ such that

$$C^{-1} H_{\mathscr{L}}(x) \leq H'_{\mathscr{L}}(x) \leq C H_{\mathscr{L}}(x)$$

for all x in $V(\overline{F})$. What exactly are these choices? Well, let us fix

- a *metric* $|\cdot|_v$ on $\mathscr{L}$ for each archimedean place v,
- an *integral structure* of $\mathscr{L}$ (which can be replaced by a norm $|\cdot|_v$ on $\mathscr{L}$ for each non-archimedean place v, indicating that this is an analog of the first choice).

14.4 Example. Let us first consider the case when $V = \operatorname{Spec} F$ is a point. Then line bundles $\mathscr{L}$ on V are just 1-dimensional vector spaces. Let L is a 1-dimensional F-vector space. Then an integral structure on L is just the choice of a finitely-generated $\mathcal{O}_F$-submodule $L_{\mathcal{O}_F}$ of L, and a metric $|\cdot|_v$ for archimedean v is a map $L \otimes_v \mathbb{C} \longrightarrow \mathbb{R}_{\geq 0}$ such that

- $|ax|_v = |a|_v |x|_v$ for all a in F and x in L, where now we use the "small" absolute value for $|a|_v$,
- $|x|_v \neq 0$ for $x \neq 0$.

Then the associated height function (which is just a single value, because $V(F) = V(\overline{F})$ is just a point) is

$$H_F(L) \coloneqq \#(L_{\mathcal{O}_F}/\mathcal{O}_F x) \prod_{v|\infty} |x|_v^{-1}$$

for any nonzero x in $L_{\mathcal{O}_F}$, which is independent of the choice of x.

14.5 Example. Let $F = \mathbb{Q}$ in Example 14.4, let $|\cdot|_\infty$ be the usual absolute value, let $L = \mathbb{Q}$, and let $L_\mathbb{Z}$ be $n^{-1}\mathbb{Z}$ for any $n \geq 1$ in $\mathbb{Z}$. Taking $x = 1$ shows that $H_\mathbb{Q}(L) = n$.

For another presentation of Example 14.4, let v be a non-archimedean place of F. Choose a basis e of the $\mathcal{O}_{F,(v)}$-module $L_{\mathcal{O}_{F,(v)}} \coloneqq \mathcal{O}_{F,(v)} \otimes_{\mathcal{O}_F} L_{\mathcal{O}_F}$, and let $|x|_v \coloneqq |x/e|_v$, where the right-hand side denotes the usual "small" absolute value. We see that

$$H_F(L) = \prod_{v \text{ a place of } F} |x|_v^{-1},$$

which is independent of the choice of x because any two choices differ by an element of $F^\times$, and the product formula implies that this doesn't change $H_F(L)$.

Now let's tackle case of general varieties, where we begin by discussing metrics at the archimedean place. Let V now be a proper algebraic variety over $\mathbb{C}$, and let $\mathscr{L}$ be a line bundle on V.

14.6 Definition. A *metric* on $\mathscr{L}$ is a morphism of sheaves

$$|\cdot|_\mathscr{L} : \mathscr{L}^{\text{top}} \longrightarrow \mathcal{C}_{\mathbb{R}_{\geq 0}},$$

where $\mathscr{L}^{\text{top}}$ is the sheaf of topological sections of the (geometrically realized) line bundle $\mathscr{L}$ on $V(\mathbb{C})$, and $\mathcal{C}_{\mathbb{R}_{\geq 0}}$ is the sheaf of $\mathbb{R}_{\geq 0}$-valued continuous functions on $V(\mathbb{C})$, such that

- $|f\lambda|_\mathscr{L} = |f||\lambda|_\mathscr{L}$ for any sections f of $\mathcal{C}_{\mathbb{R}_{\geq 0}}$ and λ of $\mathscr{L}$,
- if λ is a nonvanishing section of $\mathscr{L}$, then $|\lambda|_\mathscr{L}$ is a nonvanishing section of $\mathcal{C}_{\mathbb{R}_{\geq 0}}$.

I only have five minutes left, so I can't treat the nonarchimedean condition yet.

14.7 Example. For $V = \mathbb{P}^1_\mathbb{C}$ and $\mathscr{L} = \mathscr{O}_V((\infty))$, the assignment

$$|f|_\mathscr{L}(x) \coloneqq \frac{|f(x)|}{\max\{1, |x|\}}$$

yields a metric on $\mathscr{L}$. This shall be the metric that we use to obtain the usual absolute height on $\mathbb{P}^1_F$.

I didn't get time to do so today, but next time I'll discuss the nonarchimedean component of this height function construction.

15 February 7, 2018

Last time, we were trying to set up general height functions on proper varieties V over a number field F, but it's easier when V is actually projective, so we make this assumption from now on. Let $\mathscr{L}$ be a line bundle on V. Recall that we were trying to construct an associated height function

$$H_\mathscr{L} : V(\overline{F}) \longrightarrow \mathbb{R}_{>0},$$

and that while this construction depends on some choices, these choices yield equivalent functions under the relation $\sim$. The choices we need to make are:

- a metric $|\cdot|_{\mathscr{L}}$ in the sense of Definition 14.6 on $\mathscr{L}$ over $V(\overline{F}_v)$ for all archimedean place v of F,
- a projective scheme $V_{\mathcal{O}_F}$ over $\mathcal{O}_F$ whose generic fiber is isomorphic to V and a coherent $\mathcal{O}_F$-module on $V_{\mathcal{O}_F}$ whose generic fiber, under the previous identification, is isomorphic to $\mathscr{L}$.

Recall that, for a locally Noetherian scheme, a coherent module is one that correspond to finitely generated modules on affine patches.

15.1 Example. Such objects *do* exist. For instance, we can take the Zariski closure of V in $\mathbb{P}^n_{\mathcal{O}_F}$ for $V_{\mathcal{O}_F}$, where we view V as living inside $\mathbb{P}^n_F$.

15.2 Definition. Using the above choices, we construct

$$H_{F,\mathscr{L}} : V(F) \longrightarrow \mathbb{R}_{>0}$$

as follows. For any x in $V(F)$, form the fiber $\mathscr{L}(x)$ of $\mathscr{L}$ at x, which is a 1-dimensional F-vector space. Then set

$$H_{F,\mathscr{L}}(x) := H_F(\mathscr{L}(x)),$$

where H_F denotes the function from Example 14.4 constructed using the pullbacks of the norms $\|\cdot\|_{\mathscr{L}}$ and the integral structures $V_{\mathcal{O}_F}$ and $\mathscr{L}_{\mathcal{O}_F}$ to $\operatorname{Spec} F$ via x.

15.3 Example. Let V be $\mathbb{P}^n_F$, and let $\mathscr{L}$ be $\mathscr{O}_{\mathbb{P}^n_F}(1)$. Let's choose $V_{\mathcal{O}_F}$ to be $\mathbb{P}^n_{\mathcal{O}_F}$, and let $\mathscr{L}_{\mathcal{O}_F}$ be $\mathscr{O}_{\mathbb{P}^n_{\mathcal{O}_F}}(1)$, which is formed in the same way as $\mathscr{O}_{\mathbb{P}^n_F}(1)$, except that you use $\mathcal{O}_F$ throughout instead of F. Write $T_0, \dots, T_n$ for the homogeneous coordinates of $\mathbb{P}^n$, and let U_j be the affine patch where T_j is nonzero. We write

$$x = [x_1 : \cdots : x_{j-1} : 1 : x_{j+1} : \cdots : x_n]$$

for any F-point of U_j, where the $x_1, \dots, \widehat{x}_j, \dots, x_n$ lie in F. Note that x is an $\mathcal{O}_F$-point if and only if these $x_1, \dots, \widehat{x}_j, \dots, x_n$ actually lie in $\mathcal{O}_F$. In this situation, we have

$$\mathscr{L}_{\mathcal{O}_F}(x) = \mathcal{O}_F x_1 + \cdots + \mathcal{O}_F x_{j-1} + \mathcal{O}_F x_{j+1} + \cdots + \mathcal{O}_F x_n,$$

so for any nonarchimedean place v of F we get $\mathscr{L}_{\mathcal{O}_{F,v}}(x) = \mathcal{O}_{F,v} e$, where e is any element of F satisfying

$$|e|_v = \max\{|x_1|_v, \dots, |x_{j-1}|_v, 1, |x_{j+1}|_v, \dots, |x_n|_v\}.$$

Of course, for general F-points x, we have $\mathscr{L}(x) = F$.

Let's now turn to the archimedean contribution. For any section f of $\mathscr{L}$, let $|\cdot|_{\mathscr{L}}$ on $V(\overline{F}_v)$ be the metric given by sending

$$|f|_{\mathscr{F}}(x) := \frac{|f(x)|}{\max\{|x_1| \dots, |x_{j-1}|, 1, |x_{j+1}|, \dots, |x_n|\}}.$$

With this metric in hand, we see that $H_{F,\mathscr{L}}(x) = H_F(\mathscr{L}(x))$ is well-defined.

15.4 Definition. We extend our height functions to $V(\overline{F})$ by defining

$$H_{\mathscr{L}}(x) := H_{F',\mathscr{L}'}(x)^{1/[F':F]},$$

where F' is any number field for which x lies in $V(F')$, $\mathscr{L}'$ is the base change of $\mathscr{L}$ to $V_{F'}$, and we construct $H_{F',\mathscr{L}'}$ using the pullbacks of our choices for V and $\mathscr{L}$ over F.

Now that we have completed a description of the theory of height functions, Conjecture 14.1 makes sense.

16 February 12, 2018

Given all the conjectures we've introduced in the course, it's always nice to see what these conjectures concretely mean.

16.1 Example. Let $n \geq 4$ be an integer, and consider the situation where

$$\mathfrak{X} = \mathbb{P}^2_{\mathbb{Z}} \smallsetminus \{x_0^n + x_1^n + x_2^n = 0\} = \operatorname{Spec} \mathbb{Z}[T_1, T_2]/(T_1^n + T_2^n + 1)$$

Then $X \coloneqq \mathfrak{X}_{\mathbb{Q}}$ is of log general type, and we see that

$$\mathfrak{X}(\mathbb{Z}[\tfrac{1}{N}]) = \{[x_0 : x_1 : x_2] \in \mathbb{P}^2(\mathbb{Q}) \mid \gcd\{x_0, x_1, x_2\} = 1,\ x_0^n + x_1^n + x_2^n \in \mathbb{Z}[\tfrac{1}{N}]^\times\}.$$

Conjecture 5.2 and Conjecture 5.5 together predict that there exists a proper closed subscheme Y of X such that

$$\{x \in \mathfrak{X}(\mathbb{Z}[\tfrac{1}{N}]) \mid x \notin Y(\mathbb{Q})\}$$

is finite. Furthermore, these conjectures predict that this Y can be taken to be the smallest closed subset of X such that any non-constant holomorphic map $f : \mathbb{C} \longrightarrow X(\mathbb{C})$ has image in $Y(\mathbb{C})$. I expect this Y equals

$$Y = \{x_0^n + x_1^n = 0\} \cup \{x_0^n + x_2^n = 0\} \cup \{x_1^n + x_2^n = 0\},$$

as we have a holomorphic map $\mathbb{C} \longrightarrow X(\mathbb{C})$ given by $t \mapsto [t : \alpha t : 1]$, where α is an n-th root of -1. In the case when $n = 5$ and $N = 2$, we also see that $\mathfrak{X}(\mathbb{Z}[\frac{1}{N}])$ is infinite, because it contains the elements

$$[2^m : -2^m : 1]$$

for all integers m. This infinitude would also be blocked by punching out Y. If my description of Y is correct, then $\mathfrak{X}(\mathbb{Z}[\frac{1}{N}])$ should be finite when $n = 4$, for arbitrary N. More strongly, I expect that

$$\{(x, y, z) \in \mathbb{Z}^3 \mid \gcd\{x, y, z\} = 1,\ x^n + y^n + z^n \in \mathbb{Z}[\tfrac{1}{3}]^\times\} = \{\pm(0, 0, 1),\ \pm(1, 0, 0),\ \pm(0, 1, 0),\ \pm(1, 1, 1)\}$$

for any $n \geq 4$. My evidence for believing this is that it holds for $\max\{|x|, |y|, |z|\} \leq 4$ and $n = 4$.

Let's use the Vojta conjectures as not evidence (as they're not proven), but rather support for this conjecture. Suppose that $n \geq 16$ and that $4|n$. As background motivation, consider the following result of Legendre, which was originally stated by Fermat.

16.2 Theorem (Legendre)**.** *If m is an integer that cannot be written as $4^a(8b + 7)$ or any integers a and b, then m can be written as a sum of three perfect squares.*

Returning to Example 16.1, rewrite our desired condition as

$$(x^{n/4})^4 + (y^{n/4})^4 + (z^{n/4})^4 \in \mathbb{Z}[\tfrac{1}{3}]^\times.$$

At this point, let X be the $n = 4$ case of the original setup (not our new $n \geq 16$ setup), and let Y be the proper closed subset of X given to us by Conjecture 14.1.

16.3 Remark. In the case when X is of log general type, it is conjectured that the Y appearing in Conjecture 5.2 and Conjecture 5.5 can be taken to be the same as the one appearing in Conjecture 14.1. I am not sure if this conjecture is written down in Vojta's papers, however.

Anyways, choosing a small ε and using Conjecture 14.1 shows that cofinitely many of our desired triples (x, y, z) satisfy

$$\prod_{p|(x^n+y^n+z^n)} p \geq C \max\left\{|x|^{n/4}, |y|^{n/4}, |z|^{n/4}\right\}^{1-\varepsilon}$$

for some constant C. Our $N = 3$ condition implies that the left-hand side is at most 3, which would narrowly bound the possibilities for (x, y, z). In general, this problem seems very hard—we can think of it as a second Fermat's last theorem. It's can't be the first Fermat's last theorem, because that's already been solved, but this seems just as hard. Furthermore, you can readily obtain many second Fermat's last theorems, by changing N from 3 to 2 or 5.

Returning to general principles, let's discuss the height functions appearing in Conjecture 14.1 and the line bundles appearing in their formulation. Recall the situation of Definition 13.5. Let V be a projective variety over a field F, and let $\mathscr{L}$ be a line bundle on V.

16.4 Definition. We say that $\mathscr{L}$ is

- *very ample* if it is the pullback of the canonical bundle on $\mathbb{P}^M$ for some closed embedding $V \hookrightarrow \mathbb{P}^M$,
- *ample* if $\mathscr{L}^{\otimes k}$ is very ample for some positive integer k.

For any divisor D on V, we say that D is *very ample*, *ample*, or *big* if the line bundle $\mathscr{O}_V(D)$ has the corresponding property.

We relate bigness to ampleness.

16.5 Proposition. *Ample line bundles are also big. Furthermore, a divisor B is big if and only for any ample divisor A, there exists positive integers b and a and an effective divisor E such that*

$$b \cdot \mathscr{O}_V(B) = a \cdot \mathscr{O}_V(A) + \mathscr{O}_V(E)$$

in $\mathrm{Pic}(V)$*, and this is also equivalent to saying that there exists* some *positive integer b, ample divisor A, and effective divisor E satisfying*

$$b \cdot \mathscr{O}_V(B) = \mathscr{O}_V(A) + \mathscr{O}_V(E).$$

16.6 Example. Let X be a smooth quasiprojective variety, let $\overline{X} \hookrightarrow \mathbb{P}^n$ be a compactification of X such that $D := \overline{X} \smallsetminus X$ is a normal crossings divisor of $\overline{X}$, and let K be a canonical divisor of $\overline{X}$. Recall from Example 13.4 that X being of log general type is equivalent to saying that $K + D$ is big, so Proposition 16.5 indicates that we can find positive integers b and a such that

$$b(K + D) \sim aA + E,$$

where A is the ample divisor of a hyperplane section of $\overline{X}$, and E is some effective divisor.

We can combine Example 16.6 with the following statement on heights to decompose H_{K+D} in terms of the standard height on $\mathbb{P}^n$.

16.7 Proposition. *Suppose everything is over a number field F. For any divisors D_1 and D_2 on $\overline{X}$, we have $H_{D_1+D_2} \sim H_{D_1} H_{D_2}$.*

In the situation of Example 16.6, we see that H_A corresponds to the standard height on $\mathbb{P}^n$. Therefore we can replace the

$$H_{K+D}(x)H(x)^{-\varepsilon}$$

on the right-hand side of Conjecture 14.1 with

$$H(x)^{a/b-\varepsilon}H_E(x)^{1/b}.$$

Outside of E, there exists a constant C such that $H_E \geq C$, because its archimedean contribution is always bounded above by some constant on $V(\overline{F})$, and its non-archimedean contribution is bounded above by some constant on $(V \smallsetminus E)(\overline{F})$.

Next time, I'll probably introduce Nevanlinna theory, and on Friday I'll probably discuss the Mordell conjecture and its proof.

17 February 14, 2018

I'll introduce *Nevanlinna theory* today. I hope you can appreciate the mysterious connection between number theory and analysis that it helps to formulate. Recall from Lecture 2 that

$$\text{a theorem of Siegel} \longleftrightarrow \text{a theorem of Picard},$$

where Siegel says that $\{x \in \mathbb{Q} \mid x, 1-x \in \mathbb{Z}[\frac{1}{N}]^\times\}$ is finite, and Picard says that any holomorphic map $\mathbb{C} \longrightarrow \mathbb{P}^1(\mathbb{C}) \smallsetminus \{0, 1, \infty\}$ is constant. A finer version of the above analogy is given by

$$\text{Vojta's conjecture (namely, Conjecture 14.1)} \longleftrightarrow \text{a conjecture of Griffiths.}$$

17.1 Example. In the case of $X = \mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$, for which Conjecture 14.1 becomes the ABC conjecture, Griffith's conjecture asks the following question. For any non-constant holomorphic map $f : \mathbb{C} \longrightarrow \mathbb{P}^1(\mathbb{C})$, how often does $f(\alpha)$ lie in $\{0, 1, \infty\}$?

In the case of general varieties, let $\overline{X}$ be a projective smooth variety over a number field F as usual, let D be a normal crossings divisor of $\overline{X}$, and form $X := \overline{X} \smallsetminus D$ as before. We'll also fix models $\overline{\mathfrak{X}}$ and $\mathfrak{D}$ for these objects over $\mathcal{O}_F$. Then the question of Griffiths becomes: given a holomorphic function $f : \mathbb{C} \longrightarrow \overline{X}(\mathbb{C})$ that is nonconstant, how often do we have $f(\alpha)$ in $D(\mathbb{C})$? Recall that on the number theory side, Conjecture 14.1 relates the quantity

$$\prod_{\substack{v \text{ is prime} \\ x \in \mathfrak{D}(\mathbb{F}_v) \pmod v}} \#\mathbb{F}_v$$

with the height of x associated to certain line bundles. We can construct a similar notion of *height functions* in the analytic setting. To remove uninteresting cases, suppose that the image of $\mathbb{C}$ under f does not lie in $D(\mathbb{C})$.

17.2 Definition. The analog of the above product of primes on the Nevanlinna side is the quantity

$$N_{f,D}^{(1)}(r) := \sum_{\substack{\alpha \in \mathbb{C} \\ |\alpha| < r \\ f(\alpha) \in D(\mathbb{C})}} \begin{cases} \log \frac{r}{|\alpha|} & \text{if } \alpha \neq 0, \\ \log r & \text{if } \alpha = 0, \end{cases}$$

where r is any non-negative number. Since D is a normal crossings divisor and $f(\mathbb{C})$ does not lie in $D(\mathbb{C})$, we see that $f^{-1}(D(\mathbb{C}))$ has codimension 1 in $\mathbb{C}$, that is, it is discrete. Thus its intersection with $\{|z| < r\}$ is finite, so $N_{f,D}^{(1)}(r)$ is indeed a finite sum.

17.3 Example. In the setting of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, we could consider functions $f : \mathbb{C} \longrightarrow \mathbb{P}^1(\mathbb{C})$ like $f(z) = e^z$ or $f(z) = z^n$ for integers $n \geq 1$.

As for the analog of the arithmetic height function, let $\mathscr{L}$ be a holomorphic line bundle on $\overline{X}(\mathbb{C})$. Just like the arithmetic situation, our height function (which we denote using $f \mapsto T_{f,\mathscr{L}}(r)$) depends on a choice of metric on $\mathscr{L}$, but different metrics yield functions that are equivalent under $\sim$, where the implied constant is independent of f and r. Griffiths's conjecture then becomes that we roughly always have

$$N^{(1)}_{f,D}(r) \geq T_{f,K+D}(r).$$

17.4 Example. Recall the situation of Example 17.3. In this situation, line bundles are classified by degree, and $\deg K = -2$ and $\deg D = 3$. Therefore $K + D$ is a hyperplane class.

(1) For $f(z) = e^z$, note that f never hits 0 nor ∞. Then we only need to consider 1, and for this we see that

$$N^{(1)}_{f,D}(r) = \left(2\left\lfloor \frac{r}{2\pi} \right\rfloor + 1\right)\log r - 2\log\left(\left\lfloor \frac{r}{2\pi} \right\rfloor !\right) - 2\left\lfloor \frac{r}{2\pi} \right\rfloor \log 2\pi.$$

We will see in Example 18.2 that

$$T_{f,K+D}(r) = T_{f,H}(r) = \frac{r}{\pi},$$

so the Griffiths conjecture indeed holds in this case. However, it barely holds—Stirling's formula implies that

$$N^{(1)}_{f,D}(r) - T_{f,K+D}(r) \to 0$$

as $r \to \infty$.

(2) For $f(z) = z^n$ when n is a positive integer, note that $f(z) = 0$ only at $z = 0$ and that $f(z) = 1$ precisely for the n-th roots of unity. Therefore for $r > 1$, we have

$$N^{(1)}_{f,D}(r) = (n + 1)\log r,$$

and it turns out in Example 18.2

$$T_{f,H}(r) = n \log r$$

in this case, which once again verifies Griffiths's conjecture.

We shall now turn to actually defining $T_{f,\mathscr{L}}(r)$. Let V be a projective smooth complex manifold, let $\mathscr{L}$ be a holomorphic line bundle on V equipped with a metric $|\cdot|_{\mathscr{L}}$, and let $f : \mathbb{C} \longrightarrow V$ be a holomorphic map. Write $\mathscr{L} = \mathscr{O}_V(D)$ for some divisor D on V such that $f(\mathbb{C})$ is not contained in the support of D.

17.5 Definition. Let r be positive. The function $T_{f,\mathscr{L}}(r)$ is defined to be

$$T_{f,\mathscr{L}}(r) := m_{f,D}(r) + N_{f,D}(r),$$

where $m_{f,D}(r)$ is the analog of the archimedean component of arithmetic heights, and $N_{f,D}(r)$ is the analog of the non-archimedean component. We shall define these two components below.

GAGA tells us that our complex analytic setup is actually algebraic, and we can choose a generator 1 for $\mathscr{L}$ on the Zariski open subset $U := V \smallsetminus \operatorname{supp} D$ of V. Write $W = \log|1|_{\mathscr{L}}^{-1}$, which is a real-valued function on U. Then we get a function

$$W \circ f : \mathbb{C} \smallsetminus \text{some discrete set}(= f^{-1}(D)) \longrightarrow \mathbb{R}$$

because the image of f is not contained in the support of D. We define

$$m_{f,D}(r) := \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta\,(W \circ f)(re^{\theta i}),$$

which converges because the singularities of $W \circ f$ when considered as a function on all of $\mathbb{C}$ are "mild" in a manner we shall not describe. Finally, we define

$$N_{f,D}(r) := \sum_{\substack{\alpha\in\mathbb{C}\\ |\alpha|<r}} \operatorname{ord}_\alpha(f^*D)\begin{cases}\log\frac{r}{|\alpha|} & \text{if } \alpha \neq 0,\\ \log r & \text{if } \alpha = 0.\end{cases}$$

Note that $N_{f,D}(r)$ is almost exactly equal to $N^{(1)}_{f,D}(r)$, except now we count the multiplicity with which f reaches a value α rather than just always weight it as 1.

18 February 16, 2018

Let's recall the setup of Nevanlinna theory. Let V be a smooth projective algebraic variety over $\mathbb{C}$, and let $f : \mathbb{C} \longrightarrow V(\mathbb{C})$ be an analytic morphism. Let $\mathscr{L}$ be a line bundle on $V(\mathbb{C})$, which we take to be algebraic, although GAGA tells us this is the same as taking a holomorphic vector bundle on $V(\mathbb{C})$. Then we have defined a height function $T_{f,\mathscr{F}}(r)$ for all positive r, and while this function depends on the choice of a generator of $\mathscr{L}(U)$ for $U = V \smallsetminus \operatorname{supp} D$ as well as a metric $|\cdot|_{\mathscr{L}}$ on $\mathscr{L}$, different choices give equivalent functions under the relation $\sim$.

Today, we'll cover some examples of these functions. Write D for a divisor of V such that $\mathscr{L} = \mathscr{O}_V(D)$, and choose D such that $f(\mathbb{C})$ is not contained in the support of D. As the absolute value of the generator $|1|_{\mathscr{L}}$ on U goes to ∞ as we approach $\operatorname{supp} D$, we see that $W = \log|1|_{\mathscr{L}}^{-1}$ goes to $-\infty$. Therefore the $m_{f,D}(r)$ term measures how close f gets to $\operatorname{supp} D$, where this value is more negative when f gets closer to $\operatorname{supp} D$. This is analogous to the inversion of the archimedean sizes in, say, Example 14.4, and this is also why we say that $m_{f,D}(r)$ is an analog of the archimedean contribution.

The philosophy on $N_{f,\mathscr{L}}$ is that it measures how often the function f crosses $\operatorname{supp} D$ on $\mathbb{C}$, whereas $m_{f,\mathscr{L}}$ measures how f crosses $\operatorname{supp} D$ at infinity. Our motivation is further solidified by *Jensen's formula*:

18.1 Proposition (Jensen). *For any meromorphic function $g : \mathbb{C} \dashrightarrow \mathbb{C}$, we have*

$$\sum_{|\alpha|<r} \operatorname{ord}_\alpha(g)\begin{cases}\log\frac{r}{|\alpha|} & \textit{if } \alpha \neq 0,\\ \log r & \textit{if } \alpha = 0,\end{cases} = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\theta\,\log|g(re^{i\theta})| - \text{“}\log|g(0)|\text{”},$$

where "$\log|g(0)|$" is the leading coefficient of the Laurent expansion of g.

Thus we can also interpret $m_{f,D}(r)$ as trying to count hits of f on D on the neighborhood of ∞ whose boundary is the circle of radius r around the origin. We write $U(0)$ for "$\log|g(0)|$". You can also use Jensen's formula to show that $T_{f,\mathscr{L}}(r)$ is independent of the choice of D.

18.2 Example. Suppose that $V = \mathbb{P}^n_{\mathbb{C}}$, where we use the homogeneous coordinates $z_0, \dots, z_n$. Identify $\mathbb{C}^n$ as an open subset of $V(\mathbb{C})$ via the 0-th coordinate, and suppose that $f(\mathbb{C})$ lies in $\mathbb{C}^n$. (In the $n = 1$ case, this just amounts to asking that f is actually a holomorphic function $\mathbb{C} \longrightarrow \mathbb{C}$.)

Let D be $\mathbb{P}^n(\mathbb{C}) \smallsetminus \mathbb{C}^n$, and choose the metric from Example 15.3. Then $N_{f,D}(r)$ is constantly zero, because $f^*(D) = 0$. A generator for our bundle of choice $\mathscr{L} = \mathscr{O}_V(D)$ on $\mathbb{C}^n$ is given by z_0, so we have

$$T_{f,\mathscr{L}}(r) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta\, \log\max\left\{|f_1(re^{i\theta})|, \dots, |f_n(re^{i\theta})|, 1\right\},$$

where we write $f = (f_1, \dots, f_n)$. Pulling out the maximum yields

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta\, \max\left\{\log|f_1(re^{i\theta})|, \dots, \log|f_n(re^{i\theta})|, 0\right\}.$$

Let $n = 1$, and try out the functions from Example 17.3. When $f(z) = e^z$, we see that

$$T_{f,\mathscr{L}}(r) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta\, \max\{\underbrace{\mathrm{Re}(re^{i\theta})}_{r\cos\theta}, 0\} = \frac{r}{\pi},$$

and if $f(z) = z^m$ for an integer $m \geq 1$, then

$$T_{f,\mathscr{L}}(r) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta\, \max\{m\log r, 0\} = m\log r$$

for $r \geq 1$. This finishes a computation that we advertised in Example 17.4.

We may now give the *Griffiths conjecture*. Let $\overline{X}$ be a smooth closed subvariety of $\mathbb{P}^n_{\mathbb{C}}$, let D be a normal crossings divisor on $\overline{X}$, set $X := \overline{X} \smallsetminus D$, and let H be a hyperplane section of $\overline{X}$.

18.3 Conjecture (Griffiths)**.** *Let ε be positive. Then there exists a closed proper subvariety Y of $\overline{X}$ such that, for any real c and holomorphic function $f : \mathbb{C} \longrightarrow \overline{X}(\mathbb{C})$ with image in $Y(\mathbb{C})$, we have*

$$N^{(1)}_{f,D}(r) \geq T_{f,K+D}(r) - \varepsilon T_{f,H}(r) + c$$

for r in a subset of $\mathbb{R}_{\geq 0}$ with cofinite Lebesgue measure.

18.4 Example. Take $\overline{X} = \mathbb{P}^1_{\mathbb{C}}$ and D to be any finite subset. Here $K + D \sim (\deg D - 2)H$, so Griffiths's conjecture becomes the statement that the right Y exists such that

$$\sum_{\substack{|\alpha|<r \\ f(\alpha)\in D}} \begin{cases} \log\frac{r}{|\alpha|} & \text{if } \alpha \neq 0, \\ \log r & \text{if } \alpha = 0, \end{cases} \geq \frac{(\deg D - 2)}{2\pi}\int_0^{2\pi} \mathrm{d}\theta\, \max\{\log|f(re^{i\theta})|, 0\}$$

for r in a subset of cofinite Lebesgue measure. This statement is known to be true.

More generally, the Griffiths conjecture known for curves. However, very little known for general varieties. Apologies for the lack of any discussion on the Mordell conjecture—we'll discuss the Mordell conjecture as well as the Tate conjecture for abelian varieties next time!

19 February 19, 2018

We now move to §4 today: how Faltings proved the *Mordell conjecture*. He used the theory of heights. We'll also discuss how one could hope to prove the Tate conjecture for algebraic cycles using the theory of heights. Let's start with the following historical picture. Faltings proved four big theorems in 1983:

(1) the *Tate conjecture* for abelian varieties,

(2) the *Shafarevich conjecture* for abelian varieties,

(3) the *Shafarevich conjecture* for curves,

(4) the *Mordell conjecture*.

Actually, he proceeded as (1) $\implies$ (2) $\implies$ (3) $\implies$ (4). Part (1) is where he used the notion of heights of abelian varieties over number fields, and I hope to discuss this today. I also want to describe how to travel from one implication to another, beginning with part (1): the *Tate conjecture*!

Let F be a finitely generated field over its prime subfield (i.e. $\mathbb{Q}$ or $\mathbb{F}_p$ for some prime p, depending on $\operatorname{char} F$). Let A and B be abelian varieties over F, and let p be a prime number not equal to $\operatorname{char} F$.

19.1 Theorem (Tate conjecture for abelian varieties)**.** *The map*

$$\operatorname{Hom}(A,B)\otimes_{\mathbb{Z}}\mathbb{Z}_p \longrightarrow \operatorname{Hom}_{G_F}(T_pA, T_pB),$$

that sends a morphism to the induced morphism on p-adic Tate modules, where $G_F := \operatorname{Gal}(\overline{F}/F)$ for a fixed separable closure $\overline{F}$ of F, is an isomorphism.

Tate proved his conjecture in the case when F is a finite field, and in general it's a theorem of Faltings. I guess I should discuss abelian varieties a bit first.

19.2 Definition. Let k be a field. An *abelian variety* over k is a projective smooth algebraic variety over k with the structure of an algebraic group.

It turns out that all abelian varieties are commutative algebraic groups, and hence the name. I won't explain so much. Over $\mathbb{C}$, we can interpret abelian varieties A as complex analytic varieties via taking $A(\mathbb{C})$ with the analytic topology, and in this case $A(\mathbb{C})$ is a complex Lie group that is isomorphic to $\mathbb{C}^g/L$ for some discrete cocompact subgroup L of $\mathbb{C}^g$ satisfying a *polarizability* condition. I won't elaborate upon polarizability, but without this condition, such a $\mathbb{C}^g/L$ is just called a *complex torus*. Polarizability is always satisfied for $g=1$, but it's not satisfied in general—there exist complex tori that are not abelian varieties when $g\geq 2$.

19.3 Example. An abelian variety of dimension 1 is precisely an elliptic curve E. When $\operatorname{char} k \neq 2$, we can always describe E as a projective plane curve of the form

$$E = \{(x,y)\in\mathbb{A}^2 \mid y^2 = ax^3+bx^2+cx+d\}\cup\{\infty\}$$

for some a, b, c, and d in k, where ax^3+bx^2+cx+d has no multiple roots. In projective coordinates x_0, x_1, x_2 that correspond to our affine coordinates via $(x,y)\leftrightarrow[1:x:y]$, this is amounts to

$$E = \{[x_0:x_1:x_2] \mid x_2^2x_0 = ax_1^3+bx_1^2x_0+cx_1x_0^2+dx_0^3\},$$

where ∞ becomes $[0:0:1]$. In this presentation for E, ∞ is always the identity for the group operation, and the map $(x,y)\mapsto(x,-y)$ is always the the inversion map.

For a specific example, consider the elliptic curve E given by the equation $y^2=x^3+1$. Then the group of $\mathbb{R}$-points $E(\mathbb{R})$ is isomorphic to $\mathbb{R}/\mathbb{Z}$, and the group of $\mathbb{C}$-points is isomorphic to $\mathbb{C}/\mathbb{Z}[\zeta_3]$. Note that we get a nice automorphism on $\mathbb{C}/\mathbb{Z}[\zeta_3]$ from multiplication by ζ_3, which corresponds to $(x,y)\mapsto(\zeta_3x,y)$ on the algebraic side.

Let A be an abelian variety over F of dimension g. For any n such that $\operatorname{char} F$ does not divide n, it is a fact that its n-torsion subgroup satisfies

$$A(\overline{F})[n] := \ker(n : A(\overline{F}) \longrightarrow A(\overline{F})) \cong (\mathbb{Z}/n\mathbb{Z})^{2g}.$$

In the case when $F \hookrightarrow \mathbb{C}$, we can see this from the lattice description via

$$A(\mathbb{C})[n] = \ker(n : \mathbb{C}^g/L \longrightarrow \mathbb{C}^g/L) = (\tfrac{1}{n}L)/L \cong (\mathbb{Z}/n\mathbb{Z})^{2g}.$$

In the setting of general fields F, we naturally have a Galois action on $A(\overline{F})$.

19.4 Example. Return to our concrete elliptic curve from Example 19.3. When $n = 2$, its 2-torsion is

$$E(\overline{\mathbb{Q}})[2] = \{\infty, (-\zeta_3^a, 0)\}$$

as a ranges through $\{0, 1, 2\}$. Thus we get a nontrivial Galois action already, and this Galois action only gets bigger (i.e. the subgroup acting trivially gets smaller) as we increase n.

19.5 Definition. The *p-adic Tate module* of A, denoted using T_pA, is defined to be

$$T_pA = \varprojlim_n A(\overline{F})[p^n],$$

where the transition maps are given by multiplication by p.

If p is a prime such that $\operatorname{char} F$ does not divide p, our previously stated fact indicates that T_pA is isomorphic to $\mathbb{Z}_p^{2g}$ as a $\mathbb{Z}_p$-module. In the situation where $F \hookrightarrow \mathbb{C}$, our description of $A(\mathbb{C})[n]$ yields

$$T_pA = \varprojlim_n A(\mathbb{C})[p^n] = \varprojlim_n (\tfrac{1}{p^n}L)/L = \varprojlim\nolimits_n L/p^nL = L \otimes_{\mathbb{Z}} \mathbb{Z}_p.$$

Furthermore, note that $H_1(\mathbb{C}^g/L, \mathbb{Z}) = L$, so we can rewrite the Tate module as $H_1(\mathbb{C}^g/L, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_p = H_1(\mathbb{C}^g/L, \mathbb{Z}_p)$. This reformulation leads to the Tate conjecture for general algebraic varieties:

19.6 Conjecture (Tate conjecture for algebraic cycles)**.** *Let F be a field that is finitely generated over its prime subfield, and let X be a proper smooth algebraic variety over F. Then for all primes p such that* $\operatorname{char} F$ *does not divide p, the* cycle map

$$\mathrm{CH}^r(X) \otimes_{\mathbb{Z}} \mathbb{Q}_p \longrightarrow \left(H^{2r}_{ét}(X_{\overline{F}}, \mathbb{Q}_p)(r)\right)^{G_F},$$

where $\mathrm{CH}^r(X)$ *denotes the r-th* Chow group *of codimension r algebraic cycles modulo rational equivalence, and (r) denotes a Tate twist, is surjective.*

For the remainder of my life, I hope to study this conjecture via studying the "heights of the motive $H^{2r}(X)(r)$." Maybe you can pick it up too—you're young. Koshikawa has shown that this general conjecture of Tate follows from some finitude statements for heights on motives.

Now how does this conjecture of Tate relate to the Tate conjecture for abelian varieties from earlier? Injectivity is already known for the Tate module map, so the whole statement lies in surjectivity. By taking $X = A \times B$ and sending h in $\operatorname{Hom}(A, B)$ to its graph, we obtain an element of $\mathrm{CH}^g(X) \otimes_{\mathbb{Z}} \mathbb{Q}_p$. Interpreting p-adic Tate modules as $H^1_{\text{ét}}((-)_{\overline{F}}, \mathbb{Q}_p)^\vee$ and using the Künneth formula to decompose $H^{2g}_{\text{ét}}(X_{\overline{F}}, \mathbb{Q}_p)$ allows us to obtain the Tate conjecture for abelian varieties as stated earlier.

Let's now turn to the *Shafarevich conjecture* for abelian varieties. Let F be a number field, and let S be a finite set of places of F containing the archimedean ones. Let g be a positive integer.

19.7 Theorem (Shafarevich conjecture for abelian varieties)**.** *There are only finitely many isomorphism classes of abelian varieties over F of dimension g with good reduction outside of S.*

19.8 Example. For an example of what bad reduction means, consider the equation $y^2 = x^3 + 1$ from Example 19.3. Then modulo 3, this equation becomes singular, because then $y^2 = (x+1)^3$. Therefore we say that E has *bad reduction* at $p = 3$.

Let's now state the Shafarevich conjecture for curves too.

19.9 Theorem (Shafarevich conjecture for curves)**.** *There are only finitely many isomorphism classes of smooth projective curves over F with good reduction outside of S.*

These conjectures are a natural follow-up to our discussion of heights in arithmetic geometry up to now, for the following reason. In brief, we can deduce Shafarevich's conjecture for abelian varieties by applying Conjecture 4.5 to moduli spaces. Let $\mathfrak{h}$ be the complex upper half plane, and recall that we have a map

$$\begin{aligned} \mathfrak{h} &\twoheadrightarrow \{\text{elliptic curves over } \mathbb{C}\}/\cong \\ \tau &\longmapsto \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau). \end{aligned}$$

We can generalize this to general abelian varieties by forming the *Siegel upper half space*

$$\mathfrak{h}_g \coloneqq \{\tau \in \mathrm{M}_g(\mathbb{C}) \mid \tau \text{ is symmetric, and } \operatorname{Im}(\tau) \text{ is positive definite}\},$$

and we similarly get a map

$$\begin{aligned} \mathfrak{h}_g &\twoheadrightarrow \{\text{dimension } g \text{ abelian varieties over } \mathbb{C}\}/\cong \\ \tau &\longmapsto \mathbb{C}^g/\left(\sum_{i=1}^{g} \mathbb{Z}e_i + \mathbb{Z}\tau e_i\right), \end{aligned}$$

where $e_1, \ldots, e_g$ is the standard basis of $\mathbb{C}^g$. We shall prove next time that $\mathfrak{h}_g$ is (Brody) hyperbolic, and it turns out that there exists a discrete group Γ with a left action on $\mathfrak{h}_g$ such that $\Gamma\backslash\mathfrak{h}_g$ consists of the $\mathbb{C}$-points of an algebraic variety X over $\mathbb{Q}$.

19.10 Example. In the $g = 1$ case of the usual complex upper half plane, this discrete group is just

$$\Gamma(2) \coloneqq \left\{ \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{Z}) \mid \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \equiv \left[\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right] \pmod 2 \right\}.$$

Note that $X(\mathbb{C}) = \Gamma\backslash\mathfrak{h}_g$ must also be hyperbolic, by taking lifts to its universal cover $\mathfrak{h}_g$. Furthermore, roughly speaking, there exists an algebraic variety $\mathfrak{X}$ over $\mathbb{Z}$ whose generic fiber is X and for which

$$\mathfrak{X}(\mathcal{O}_{F,S}) = \{\text{dimension } g \text{ abelian varieties over } F \text{ with good reduction outside of } S\}/\cong.$$

Then applying Conjecture 4.5 to $\mathfrak{X}$ immediately yields Shafarevich's conjecture for abelian varieties. I'll talk about how to move between these conjectures next time.

20 February 21, 2018

I will now explain the proofs of the Tate conjecture for abelian varieties, the Mordell conjecture, etc. by Faltings. What I hope to show is that the ideas are natural and the proofs are easy, once the ideas come. To this end, I hope I don't spend too much time explaining complicated things—some of the details regarding moduli spaces shall be rough, but I hope you find it natural nonetheless.

Recall our labeling of the Tate conjecture for abelian varieties, the Shafarevich conjecture for abelian varieties, the Shafarevich conjecture for curves, and the Mordell conjecture as (1)–(4), respectively. There should also be an even deeper level:

(0) finitude of isomorphism classes of abelian varieties in any given isogeny class,

as well as the following variant of the Tate conjecture:

(1)' for any two abelian varieties A and A', if V_pA is isomorphic to V_pA', then A and A' are isogeneous.

I will explain these additional ingredients. It's not easy to discern all these points (as well as the implications between them) from papers written on their proofs, and I hope to make this all clearer. The proofs go like (0) $\implies$ (1) $\implies$ (1)' and (0)+(1)' $\implies$ (2). Let's now backtrack and define the terms involved.

20.1 Definition. Let A and A' be abelian varieties over a field k. We say that A and A' are *isogeneous* and write $A \sim A'$ if there exist homomorphisms $f : A \longrightarrow A'$ and $g : A' \longrightarrow A$ and a positive natural number n such that $g \circ f = f \circ g = n$.

20.2 Example. Let $k = \mathbb{C}$, let A be $\mathbb{C}/\mathbb{Z}[\zeta_3]$, and let $A' = \mathbb{C}/(\mathbb{Z} + n\mathbb{Z}[\zeta_3])$. There is a canonical quotient map $g : A' \longrightarrow A$, and the map $\mathbb{C} \longrightarrow \mathbb{C}$ given by multiplication by n induces a map $f : A \longrightarrow A'$. We see then that f and g witness the fact that A and A' are isogeneous.

We can now make sense of (0):

20.3 Theorem. *Let F be a field that is finitely generated over $\mathbb{Q}$ or $\mathbb{F}_\ell$, and let A be an abelian variety over F. Then the set $\{A'$ abelian variety over $F \mid A' \sim A\}/\cong$ is finite.*

Part (0) is actually the hardest part of Faltings's proofs, and he proved it using the theory of heights. It was known beforehand the case when F is finite by Tate, and the implication (0) $\implies$ (1) was also known by Tate.[13] The second implication (1) $\implies$ (1)' is also relatively accessible. While it's not easy to extract this from the literature, I want to give the feeling that everything is easy. It's a nice feeling.

Sketch of (1) $\implies$ (1)'. We can turn an isomorphism $V_pA \cong V_pA'$ into an injective morphism $T_pA \hookrightarrow T_pB$ with finite cokernel, after multiplying by some power of p. The Tate conjecture then yields the map $A \longrightarrow B$ up to $\mathbb{Z}_p$-coefficients, and multiplying out the previous factor of p (along with passing down to $\mathbb{Z}$-coefficients somehow) gives the desired result. $\square$

We shall focus on the cases when F is actually finite over its prime subfield, that is, F is either a number field or a finite field. In the latter case, part (0) is evident because the entire set

$$\{\text{abelian varieties of dimension } g \text{ over } F\}/\cong\, = X(F)$$

is finite, where X is the moduli space of abelian varieties of dimension g. This is true because X is something like an algebraic variety, roughly speaking. When F more generally is finitely generated over a finite field, a similar argument works by considering F geometrically. On the other hand, when F is a number field, it contains no "base" field, so we cannot play such tricks.

Therefore let us move to the situation when F is a number field F, and let's begin our rough discussion of the moduli space of abelian varieties of dimension g. Our $\mathfrak{h}_g$ has another description:

$$\begin{aligned} \mathfrak{h}_g &\xrightarrow{\sim} \{z \in \mathrm{M}_g(\mathbb{C}) \mid z \text{ is symmetric, } 1 - {}^t\overline{z} \cdot z \text{ is positive definite}\} \\ \tau &\mapsto z := (\tau - i)(\tau + i)^{-1}. \end{aligned}$$

The inverse of the above map sends $z \mapsto \tau := i(1 + z)(z - z)^{-1}$.

20.4 Example. When $g = 1$, this alternative description of $\mathfrak{h}$ is $\{z \in \mathbb{C} \mid |z| < 1\}$, which is precisely the bounded open disk model of the Poincaré upper half plane.

[13] We shall discuss (0) $\implies$ (1) in Lecture 22.

Our bounded realization of $\mathfrak{h}_g$ shows that it is (Brody) hyperbolic, since Liouville's theorem indicates that bounded holomorphic functions are constant.

20.5 Example. In the $g = 1$ case, the map realizing complex tori as elliptic curves over $\mathbb{C}$ yields an isomorphism

$$\Gamma(2)\backslash\mathfrak{h} \xrightarrow{\sim} \{\text{elliptic curves over } \mathbb{C}\}/\cong = \mathbb{P}^1(\mathbb{C}) \smallsetminus \{0, 1, \infty\},$$

where the last identification is given by the parameter λ in the Legendre form $y^2 = x(x-1)(x-\lambda)$ of an elliptic curve over $\mathbb{C}$. Now $\mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$ is a nice convenient scheme $\mathfrak{X}$ over $\mathbb{Z}$, and for any ring R, we roughly have

$$\{\text{elliptic curves over } R\}/\cong = \mathfrak{X}(R) = \operatorname{Hom}(\mathbb{Z}[\lambda, \tfrac{1}{\lambda(1-\lambda)}], R) = \{\lambda \in R^\times \mid 1 - \lambda \in R^\times\}.$$

I will not explain what an elliptic curve over a general ring is. However, for any number field F and finite set of places S of F containing the archimedean ones, we do have

$$\{\text{elliptic curves over } \mathcal{O}_{F,S}\}/\cong = \{\text{elliptic curves over } F \text{ with good reduction outside } S\}/\cong$$

via taking Néron models. Our above description (along with Siegel's theorem) tells you that this set is finite.

Returning to the situation of arbitrary g, we can play a similar game, which we indeed played last time. This showed us that, if we knew Conjecture 4.5, then we would have (2) as a consequence. So (2) is natural.

It's a fact that isogeneous abelian varieties have good reduction at the same places, which follows from the converse of (1)' (which is immediate) and the *Néron–Ogg–Shafarevich criterion*:

20.6 Proposition (Néron–Ogg–Shafarevich criterion)**.** *Let A be an abelian variety over F, and let $v \nmid p$ be a prime. Then the Galois representation V_pA is unramified at v if and only if A has good reduction at v.*

In the general setting of abelian varieties, the Néron–Ogg–Shafarevich criterion is a theorem of Serre–Tate. Thus if the abelian variety A over F of dimension g is given, and if S is the finite set of bad reduction places of A (together with the infinite places), we see that

$$\begin{aligned}&\{A' \text{ abelian variety over } F \mid A' \sim A\}/\cong \\ \hookrightarrow &\{A' \text{ abelian variety over } F \mid \dim A = g,\ A' \text{ has good reduction outside of } S\}.\end{aligned}$$

The latter set is finite by (2), so we see one could actually deduce (0) from (2) as well.

What actually happened was that we eventually deduced (2) from (0). And Faltings proved (0) itself roughly as follows: he took the moduli space X of abelian varieties of dimension g (well, there's all sorts of additional data like polarizations, endomorphisms, and level structure involved, but we'll pretend that all doesn't exist), which is a projective variety $X \hookrightarrow \mathbb{P}^n$, and he studied heights on X. He found that if $A' \sim A$, then most of the time $H(A') = H(A)$, where $H(-)$ denotes the naive height of the F-point on X corresponding to A, and he proved that

$$\{H(A') \mid A' \sim A\}$$

is finite. Coupled with Theorem 10.5, we'd obtain (0).

. . . Well, the above sketch is not *quite* true. What Faltings *actually* did was define the *Faltings height* $H_{\mathrm{Fal}}(A)$, which is an improved version of $H(A)$ that's roughly the equal to $H(A)$. He then he used H_{Fal} to carry out the above strategy. We've run out of time now, but I'll continue discussing this next time.

21 February 23, 2018

Let us restrict to the case where F is a number field for today. I hope to finish this course by the end of next week! Recall that we cursorily described (1) $\Longrightarrow$ (1)' last time.

Sketch of (0)+(1)' $\Longrightarrow$ (2). By (0), it suffices to prove that

$$\{A \text{ abelian variety over } F \mid \dim A = g,\ A \text{ has good reduction outside of } S\}/\sim$$

is finite. Now (1)' indicates that A modulo $\sim$ is determined by V_pA, and we may assume that the primes above p lie in S, because this only enlarges the above set of interest. By the Néron–Ogg–Shafarevich criterion, our condition becomes equivalent to asking that V_pA is unramified outside of S. Thus it would suffice to show that only finitely many isomorphism classes (that is, conjugacy classes) of continuous Galois representations $G_F \longrightarrow \mathrm{GL}_{2g}(\mathbb{Q}_p)$ are unramified outside of S.

By taking Tate twists, we see that this preliminary statement is utterly false. However, we know the weight of V_pA, as it is a Galois representation of weight -1. That is, for any v not in S, the characteristic polynomial of geometric Frobenius has coefficients in $\mathbb{Z}$, and its roots have size $\#\mathbb{F}_v^{1/2}$ for all complex absolute values on $\mathbb{Q}_p$. Once we restrict the weight of our Galois representations, the number of such objects is finite. The proof of this uses the Chebotarev density theorem as well as reduction modulo p from $\mathrm{GL}_{2g}(\mathbb{Z}_p)$ to $\mathrm{GL}_{2g}(\mathbb{Z}/p\mathbb{Z})$. I shall not explain more here, but this is a standard technique in number theory. $\square$

The proof of (2) $\Longrightarrow$ (3) follows immediately from the fact that

$$\begin{aligned} &\{\text{smooth projective curves of genus } g \text{ with good reduction outside of } S\}/\cong \\ \hookrightarrow\ &\{\text{abelian varieties of dimension } g \text{ with good reduction outside of } S\}/\cong, \end{aligned}$$

where the map sends C to its *Jacobian* J_C. And what is the Jacobian, you ask? The general construction over arbitrary fields is complicated, but for the situation over $\mathbb{C}$, we can proceed with complex geometry as follows:

21.1 Definition. Let C be a smooth projective curve over $\mathbb{C}$ of genus g. Its *Jacobian*, as a complex manifold, is defined by

$$J_C(\mathbb{C}) := \frac{\mathrm{Hom}_{\mathbb{C}}(\Gamma(\mathbb{C}, \Omega^1_C), \mathbb{C})}{H_1(C(\mathbb{C}), \mathbb{Z})} \approx \frac{\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^g, \mathbb{C})}{\mathbb{Z}^{2g}} = \mathbb{C}^g/\mathbb{Z}^{2g},$$

where the embedding $H_1(C(\mathbb{C}), \mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{C}}(\Gamma(\mathbb{C}, \Omega^1_{\mathbb{C}}), \mathbb{C})$ sends γ to the map $\omega \mapsto \int_\gamma \omega$. It turns out that J_C has a natural structure of an abelian variety over $\mathbb{C}$.

We could try to redo this for more general singular curves C, but it's not clear what "good reduction" means for a variety that is already singular itself. I think I used to think about this problem, but I can't recall now, and I'm no expert in the field. Apologies.

Finally, let us deduce the Mordell conjecture from (3).

Proof of (3) $\Longrightarrow$ (4). Let C be a proper smooth curve of genus g over F, for $g \geq 2$. We want to prove that $C(F)$ is finite. It was already known by Manin (and probably others) there exists an integer $c \geq 2$ and a finite set of places S containing the archimedean ones such that we can construct a map

$$C(F) \longrightarrow \left\{ \varphi : C' \longrightarrow C \;\middle|\; \begin{array}{l} \varphi \text{ is a surjective map from a proper smooth curve over } F, \\ \varphi \text{ is ramified at exactly one point, which is } F\text{-rational}, \\ \text{the genus of } C' \text{ lies in } [2, c], \text{ and} \\ C' \text{ has good reduction outside of } S. \end{array} \right\} /\cong .$$

This construction and strategy were known well before Faltings. It's known that the map in the opposite direction that sends $\varphi : C' \longrightarrow C$ to the corresponding ramified point is a retract, which implies that the above map is an injection. Then it suffices to show that the set on the right-hand side is finite.

Now the finitude of isomorphism classes of C' appearing in the above set follows from (3). And it's a standard piece of geometry that, for any proper smooth curves C_1 and C_2 over a field k satisfying $g(C_1) \geq 2$, the number of surjective morphisms $C_1 \longrightarrow C_2$ is finite. This finishes the proof. □

While I may not be giving all the details, I want to give the impression that most of the deductions should be considered straightforward, after one proves (0). As for deducing (0) itself, the method for doing so shall be to construct the Faltings height $H_{\mathrm{Fal}}(A)$, which will be some positive number. Embedding

$$X = \text{“the moduli space of abelian varieties of dimension } g\text{”} \hookrightarrow \mathbb{P}^N$$

yields a naive height function $A \mapsto H(A)$, but this depends on the choice of embedding. It turns out that there exists a positive C such that $H_{\mathrm{Fal}}(A) \leq H(A)^C$ for all abelian varieties A over F. Our goal will be to define $H_{\mathrm{Fal}}(A)$ and prove that

$$\{H_{\mathrm{Fal}}(A') \mid A' \sim A\}$$

is a bounded set. By applying our comparison $H_{\mathrm{Fal}}(A) \leq H(A)^C$ with naive heights as well as Theorem 10.5, this would imply (0). There was an expert on this topic who spoke last week at the number theory seminar last week!

21.2 Definition. Let A be an abelian variety over F. We define its *Faltings height* to be

$$H_{\mathrm{Fal}}(A) := H(L),$$

where L is the 1-dimensional F-vector space $\bigwedge^g \Gamma(A, \Omega^1_A)$, and the integral structure $L_{\mathcal{O}_F}$ and metrics $|\cdot|_v$ are given as follows:

- Choose an F-basis $\omega_1, \ldots, \omega_g$ of $\Gamma(A, \Omega^1_A)$. Then $\omega_1 \wedge \cdots \wedge \omega_g$ is an F-basis of L, and we define its absolute value to be

  $$\int_{A(\overline{F}_v)} \omega_1 \wedge \cdots \wedge \omega_g \wedge \overline{\omega}_1 \wedge \cdots \wedge \overline{\omega}_g$$

  for complex v, and the square root of this quantity for real v.

- We take $L_{\mathcal{O}_F}$ to be $\bigwedge^n \Gamma(\mathcal{A}, \Omega^1_{\mathcal{A}})$, where $\mathcal{A}$ is the *Néron model* of A over $\mathcal{O}_F$.

We still need to compare Faltings heights with the naive heights as well as bound the set of Faltings heights itself in an isogeny class, which will need to spill over to next time.

22 February 26, 2018

I want to get to generalizations of our discussion to motives, so we will not have the time to discuss the proof of (0) any further. Many apologies!

The proof that (0) $\Longrightarrow$ (1) was carried out by Tate when F is a finite field, and the number field case is merely a slight modification. We give the proof below:

Proof of (0) $\Longrightarrow$ (1). We break this up into five steps:

1. "Easy" (background) things:

   - For any abelian varieties A and B over a field k, the group $\mathrm{Hom}(A, B)$ is a finitely generated abelian group. One can readily see this when $k = \mathbb{C}$ using the analytic theory, for which we can deduce the general case when $\operatorname{char} k = 0$ (and hence for number fields). This result is also immediate when k is finite, but the argument for general k is harder.
   - When $\operatorname{char} k$ does not divide p, the map $\mathrm{Hom}(A, B) \otimes_{\mathbb{Z}} \mathbb{Z}_p \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(T_pA, T_pB)$ is injective. We have already used this fact earlier, when we discussed the relationship between the Tate conjectures for abelian varieties and algebraic cycles. Thus we may focus on surjectivity.
   - The proof of the Tate conjecture can be reduced to the case $A = B$ by using the decomposition
   $$\mathrm{Hom}(A \oplus B, A \oplus B) = \mathrm{Hom}(A, A) \oplus \mathrm{Hom}(A, B) \oplus \mathrm{Hom}(B, A) \oplus \mathrm{Hom}(B, B),$$
   as the $A = B$ case implies isomorphisms for the left-hand side as well as the outer terms of the right-hand side, and dimension counting yields the result for $\mathrm{Hom}(A, B)$ and $\mathrm{Hom}(B, A)$, where we use the fact that these are finitely generated $\mathbb{Z}$-modules.
   - It turns out that it suffices to prove that $\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}_p \longrightarrow \mathrm{End}_{G_F}(V_p(A))$ is an isomorphism.

2. Basic things about isogenies:

   - If T' is a finite index subrepresentation of T_pA over $\mathbb{Z}_p$, then there exists an isogeny $A' \longrightarrow A$ such that $T_pA' \hookrightarrow T_pA$ has image T'. We can immediately see this by quotienting out the subgroup scheme corresponding to $(T_pA)/T'$.

     More precisely, for any positive integer N such that $\operatorname{char} k$ does not divide N and any G_k-stable subgroup H of $A[n](\overline{k})$, there exists an abelian variety $A' := A/H$ over k and a map $g : A' \longrightarrow A$ such that its composition with the canonical quotient map $f : A \longrightarrow A'$ both ways equals N. Any T' as above lies in p^nT_pA for some positive integer n, so applying the $N = p^n$ case yields the desired result.

     22.1 Example. Return to our concrete elliptic curve E from Example 19.3, whose complex points are given by $E(\mathbb{C}) = \mathbb{C}/\mathbb{Z}[\zeta_3]$. Then
     $$E[3](\mathbb{Q}) = \{\infty, (0, \pm 1)\},$$
     so we can take $H = \{\infty, (0, \pm 1)\}$. Multiplication by ζ_3 in the complex picture corresponds to $(x, y) \mapsto (\zeta_3 x, y)$ in the plane curve picture, so for z in $\mathbb{C}$, the condition $(\zeta_3 - 1)z \equiv 0 \pmod{\mathbb{Z}[\zeta_3]}$ becomes equivalent (on the affine patch) to asking that $x = \zeta_3 x$ and hence $x = 0$. Thus $y = \pm 1$ for such a point, so altogether we see that H is the kernel of multiplication by $\zeta_3 - 1$. Therefore $(E/H)(\mathbb{C}) = \mathbb{C}/(\zeta_3 - 1)^{-1}\mathbb{Z}[\zeta_3]$ under the identification of $E(\mathbb{C})$ with $\mathbb{C}/\mathbb{Z}[\zeta_3]$, and in this situation we actually get an isomorphism $E/H \xrightarrow{\sim} E$ given by multiplication by $\zeta_3 - 1$ on $\mathbb{C}$-points.

     On the other hand, if we take $H = \{\infty, (-1, 0)\}$ inside $E[2]$, then $(E/H)(\mathbb{C})$ is sandwiched between $\mathbb{C}/\mathbb{Z}[\zeta_3]$ and $\mathbb{C}/2^{-1}\mathbb{Z}[\zeta_3]$ for similar reasons. While $\mathbb{C}/2^{-1}\mathbb{Z}[\zeta_3]$ is isomorphic to E, here E/H is not isomorphic to E.

3. We shall use (0) in the following proposition.

22.2 Proposition. *Let F be a number field, and let A be an abelian variety over F. For any subrepresentation W of V_pA over $\mathbb{Q}_p$, there exists a θ in $\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}_p$ such that $\theta(V_pA) = W$.*

Proof. Let T' be the intersection $W \cap T_pA$. Now in general T' will not have finite index in T_pA, but we can create an object that must have finite index in T_pA: the sum $T' + p^nT_pA$. This sum is also Galois-stable, so Step 2 gives us an abelian variety $A^{(n)}$ over F and an isogeny $f_n : A^{(n)} \longrightarrow A$ such that $T_pA^{(n)}$ is identified with $T' + p^nT_pA$ under f_n. Now (0) implies that these $A^{(n)}$ range over finitely many isomorphism classes, so we can find an increasing sequence $\{n_i\}_i$ such that the $\{A^{(n_i)}\}_i$ are all isomorphic. Because $T' + p^{n_0}T_pA$ contains $T' + p^{n_i}T_pA$, we can replace f_{n_i} with its factor $f_{n_i} : A^{(n_i)} \longrightarrow A^{(n_0)}$.

Choose isomorphisms $h_i : A^{(n_0)} \xrightarrow{\sim} A^{(n_i)}$, and form the endomorphism $\theta_i := f_{n_i} \circ h_i$. Then θ_i sends $T' + p^{n_0}T_pA$ to $T' + p^{n_i}T_pA$, and because $\operatorname{End}(A) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is compact, some subsequence $\{i_k\}_k$ of $\{\theta_i\}_i$ converges. Letting θ be the limit of this subsequence, we see that θ sends $T' + p^{n_0}T_pA$ to

$$\bigcap_{k=1}^{\infty} T' + p^{n_{i_k}}T_pA = T'$$

and hence $(T' + p^{n_0}T_pA)[\frac{1}{p}] = V_pA$ to $T'[\frac{1}{p}] = W$, as desired. □

The proof of Proposition 22.2 is a nice slick argument.

4. (More) basic things:

   - It is known that $\operatorname{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a (finite-dimensional) semisimple algebra over $\mathbb{Q}$, which indicates that $\operatorname{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}_p$ is a semisimple algebra over $\mathbb{Q}_p$. Now finite-dimensional semisimple algebras over a field k are all of the form

     $$\bigoplus_{i=1}^{r} \mathrm{M}_{n_i}(D_i),$$

     where the D_i are finite-dimensional skew fields (that is, division algebras) over k.

5. Recall the *double centralizer theorem*, which says the following.

   22.3 Theorem (Double centralizer)**.** *Let R be a semisimple k-subalgebra of $\mathrm{M}_n(k)$. Then we have*

   $$C(C(R)) = R,$$

   where $C(A)$ denotes the centralizer $C(A) := \{x \in \mathrm{M}_n(k) \mid xy = yx$ for all y in $S\}$ of A.

   Our goal is to apply this to $\operatorname{End}(A)\otimes_{\mathbb{Z}}\mathbb{Q}_p$ inside $\operatorname{End}_{\mathbb{Q}_p}(V_pA)$, in order to show that any β in $\operatorname{End}_{G_F}(V_pA)$ lies inside $\operatorname{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}_p$. The double centralizer theorem tells us that is it enough to prove that

   $$\beta c = c\beta$$

   for all c in $C(\operatorname{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}_p)$, and we'll pick up on this next time.

23 February 28, 2018

Recall from Step 1 that we reduced the Tate conjecture to showing that $\operatorname{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}_p \longrightarrow \operatorname{End}_{G_F}(V_pA)$ is an isomorphism, where A is an abelian variety over a number field F. Let β be in $\operatorname{End}_{G_F}(V_pA)$, and consider the graph

$$W := \{(x, \beta x) \mid x \in V_pA\} \subset (V_pA)^2 = V_p(A^2)$$

of β. It's a subrepresentation of $V_p(A^2)$, so Proposition 22.2 (which is a consequence of (0)) yields a u in $\operatorname{End}(A^2) \otimes_{\mathbb{Z}} \mathbb{Q}_p$ for which $W = u(V_p(A^2))$. The semisimplicity of $\operatorname{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}_p$ and the double centralizer theorem indicate that it suffices to prove

$$c\beta = \beta c$$

for all c in $C(\operatorname{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}_p)$. The fact that $\operatorname{End}(A^2) \otimes_{\mathbb{Z}} \mathbb{Q}_p = \mathrm{M}_2(\operatorname{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}_p)$ implies that the map

$$(x, y) \mapsto (cx, cy)$$

preserves $u((V_pA)^2)$, because c commutes with the matrix entries of u. As $u((V_pA)^2) = W$, we see that this map preserves W. In other words, for all x in V_pA, the pair $(cx, c\beta x)$ lies in W, indicating that $c\beta x = \beta c x$. Hence $c\beta = \beta c$, completing our proof. □

This proof of (0) $\Longrightarrow$ (1) may seem like a series of slick tricks, but this material was well-known and relatively straightforward in Faltings's time. The real difficulty lies in proving (0), which is the core of the argument.

Today, I want to begin discussing how one could possibly attack the Tate conjecture for algebraic cycles in general. We shall describe the analog of (0) $\Longrightarrow$ (1) in this setting, where (1) now denotes the general Tate conjecture for algebraic cycles. Let k be a field contained in $\mathbb{C}$. We now give the definition of a *motive*, and while it's different from the original definition of Grothendieck, our description is simpler and not too different.

23.1 Definition.

- A *motive* over k is a formal symbol

$$\bigoplus_{i=1}^{t} H^{m_i}(X_i)(r_i),$$

  where the X_i are smooth projective varieties over k, and the m_i and r_i are integers.

- For any motive $M = \bigoplus_{i=1}^{t} H^{m_i}(X_i)(r_i)$, we write M_B for its *Betti realization*

$$M_B := \bigoplus_{i=1}^{t} H^{m_i}_{\text{sing}}(X_i(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}(2\pi i)^{r_i}.$$

  We also frequently denote this using $\bigoplus_{i=1}^{t} H^{m_i}_B(X_i)(r_i)$.

- For any two motives of the form $H^m(X)(r)$ and $H^n(Y)(s)$, we define the set of *morphisms* of motives to be

$$\operatorname{Hom}(H^m(X)(r), H^n(Y)(s)) := \begin{cases} 0 & \text{if } m - 2r \neq n - 2s, \\ \operatorname{Hom}_{\mathbb{Q}}(H^m_B(X)(r), H^n_B(Y)(s)) & \text{otherwise.} \end{cases}$$

  We say that $m - 2r$ is the *weight* of $H^m(X)(r)$.

- For any two motives $A = \bigoplus_{i=1}^{t} H^{m_i}(X_i)(r_i)$ and $B = \bigoplus_{j=1}^{u} H^{n_j}(Y_j)(s_j)$, we define the set of *morphisms* from A to B to be

$$\operatorname{Hom}(A, B) := \bigoplus_{i=1}^{t} \bigoplus_{j=1}^{u} \operatorname{Hom}(H^{m_i}(X_i)(r_i), H^{n_j}(Y_j)(s_j)),$$

Here's an important idea we have already used to deduce the Tate conjecture for abelian varieties from the general Tate conjecture: the maps on cohomology arising from morphisms of varieties come from algebraic cycles. More specifically, for any smooth projective variety Z over k, we have a cycle map $\mathrm{CH}^i(Z) \longrightarrow H^{2i}_{\mathrm{sing}}(Z(\mathbb{C}),\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}(2\pi i)^i$.[14] Tensoring with $\mathbb{Q}$ and setting $Z = X \times Y$ yields a map

$$\mathrm{CH}^i(X \times Y)_{\mathbb{Q}} \longrightarrow H^{2i}_B(X \times Y)(i)$$

and the graph Γ_f of a morphism $f : X \longrightarrow Y$ yields a codimension $\dim Y$ cycle in $X \times Y$, whose images in

$$\begin{aligned} H^{2\dim Y}_B(X \times Y)(\dim Y) &= \bigoplus_{j=0}^{2\dim Y} H^j_B(X) \otimes_{\mathbb{Q}} H^{2\dim Y - j}_B(Y)(\dim Y) = \bigoplus_{j=0}^{2\dim Y} H^j_B(X) \otimes_{\mathbb{Q}} H^j_B(Y)^\vee \\ &= \bigoplus_{j=0}^{2\dim Y} \mathrm{Hom}_{\mathbb{Q}}(H^j_B(Y), H^j_B(X)) \end{aligned}$$

via the Künneth formula and Poincaré duality correspond to the induced maps $f^* : H^j_B(Y) \longrightarrow H^j_B(X)$.

23.2 Example. Consider the motive $H^0(\operatorname{Spec} k)$. Its Betti realization is the 1-dimensional $\mathbb{Q}$-vector space $\mathbb{Q}$, so we will denote this motive using $\mathbb{Q}$. It turns out that

$$\mathrm{Hom}(\mathbb{Q}, H^{2r}(X)(r)) = \mathrm{CH}^r(X)_{\mathbb{Q}}/(\text{homological equivalence}),$$

which is also the image of $\mathrm{CH}^r(X)_{\mathbb{Q}} \longrightarrow H^{2r}_B(X)(r)$. Our linear-algebraic definition of motives indicates that this implies the Tate conjecture for when the source is $\mathbb{Q}$.

The theory of étale cohomology shows that

$$H^m_B(X)(r) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} = H^m_{\text{ét}}(X_{\mathbb{C}}, \mathbb{A}_f)(r) = H^m_{\text{ét}}(X_{\overline{k}}, \mathbb{A}_f)(r) = H^m_{\text{ét}}(X_{\overline{k}}, \mathbb{A}_f) \otimes_{\mathbb{A}_f} \mathbb{A}_f(r)$$

has a continuous action of $G_k \coloneqq \mathrm{Gal}(\overline{k}/k)$, if we give this space the discrete topology. Here, we define $\mathbb{A}_f(r) \coloneqq \mathbb{A}_f(1)^{\otimes r}$ and $\mathbb{A}_f(1) \coloneqq \mathbb{Q} \otimes \varprojlim_n \mu_n$. Similarly, we can define the *p-adic realization* M_p of a motive M using étale cohomology. With this, we can now reformulate the Tate conjecture for algebraic cycles in terms of motives.

23.3 Conjecture (Tate conjecture for motives)**.** *Suppose that k is finitely generated over $\mathbb{Q}$. If M and N are two motives over k, then $\mathrm{Hom}(M,N) \otimes_{\mathbb{Q}} \mathbb{Q}_p \xrightarrow{\sim} \mathrm{Hom}_{G_k}(M_p, N_p)$ for any prime p.*[15]

We can relate this to our previous formulation of the Tate conjecture using graphs of morphisms and the cycle map as in Lecture 19.

As for the analog of (0) $\Longrightarrow$ (1), let $k = F$ be a number field, let X be a smooth projective variety over F, and consider the motive $M \coloneqq \mathbb{Q} \oplus H^{2r}(X)(r)$. Let β lie in $H^{2r}_{\text{ét}}(X_{\overline{F}}, \mathbb{Q}_p)(r)^{G_F}$. Then the motivic analog of (0), which is something like "finitude of isomorphism classes of motives in an isogeny class," shall imply that if we set

$$W \coloneqq \{(x, x\beta) \mid x \in \mathbb{Q}_p\} \subseteq M_p,$$

then there exists a u in $\mathrm{End}(M) \otimes_{\mathbb{Q}} \mathbb{Q}_p$ such that $W = uM_p$. This is the motivic version of Proposition 22.2, and I'll finish this up on Friday.

[14] The i in the parentheses is the imaginary unit, whereas the i in the superscript is the codimension of the cycles.

[15] The note-taker thinks that, given our definition of $\mathrm{Hom}(M,N)$, this is not correct. For instance, if $M = H^1(A)$ and $N = H^1(B)$ for two abelian varieties A and B of dimension g, then the left-hand side has dimension $4g^2$, whereas the right-hand side has dimension at most $4g^2$, with equality not generally attained. However, if we use the actual definition of $\mathrm{Hom}(A,B)$ instead, this conjecture is stated correctly.

24 March 2, 2018

Fix a field k inside $\mathbb{C}$. Recall that a motive over k, in our sense, is just a symbol

$$M = \bigoplus_{i=1}^{t} H^{m_i}(X_i)(r_i),$$

where the X_i are smooth projective varieties over k. We want a notion of integral structures for motives, which will be defined as follows.

24.1 Definition. Let M be a motive over k. A *$\mathbb{Z}$-structure on M* is a free $\mathbb{Z}$-module $T_{\mathbb{Z}} \subseteq M_B$ of finite rank such that $M_B = T_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$, and $T := T_{\mathbb{Z}} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} \subseteq M_B \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ is stable under the action of $\mathrm{Gal}(\overline{k}/k)$. For any two motives $(M, T_{\mathbb{Z}})$ and $(M', T'_{\mathbb{Z}})$ over k with $\mathbb{Z}$-structures, a *morphism* $h : (M, T_{\mathbb{Z}}) \longrightarrow (M', T'_{\mathbb{Z}})$ is just a morphism $h : M \longrightarrow M'$ for which $h(T_{\mathbb{Z}})$ lies in $T'_{\mathbb{Z}}$.

As we play with all these $\mathbb{Z}$-lattices and $\widehat{\mathbb{Z}}$-lattices, we should observe the following fact: for any finite-dimensional $\mathbb{Q}$-vector space V, there is a bijection

$$\{\mathbb{Z}\text{-lattices in } V\} \longleftrightarrow \{\widehat{\mathbb{Z}}\text{-lattices in } V \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}\}$$

given by $T_{\mathbb{Z}} \mapsto T_{\mathbb{Z}} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ in one direction and $T \mapsto T \cap V$ in the other. Therefore, we can interpret a $\mathbb{Z}$-structure equivalently as the data of a $\mathrm{Gal}(\overline{k}/k)$-stable $\widehat{\mathbb{Z}}$-lattice T in $M_B \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$.

If we don't write the twist r, it means that $r = 0$. Furthermore, recall that for any morphism $f : Y \longrightarrow X$ of smooth projective varieties over k, we obtain an induced map

$$H^m(X)(r) \longrightarrow H^m(Y)(r),$$

and this map comes from the graph Γ_f of f, which is a codimension $\dim Y$ cycle in $X \times Y$.

24.2 Example. Let A and B be abelian varieties over k, let $M = H^1(A)$, and let $N = H^1(B)$. Then $M_B \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} = \mathrm{Hom}_{\mathbb{A}}(VA, \mathbb{A}_f)$, where VA denotes the adelic Tate module

$$VA := \mathbb{Q} \otimes_{\mathbb{Q}} \varprojlim_n A[n](\overline{k}).$$

of A. Therefore $T := \mathrm{Hom}_{\widehat{\mathbb{Z}}}(TA, \widehat{\mathbb{Z}})$ yields a $\mathbb{Z}$-structure on M, where T denotes the integral adelic Tate module of A. Upon carrying out the same construction for B and B, we see that the Tate conjecture says that the natural map is an isomorphism

$$\mathrm{Hom}(B, A) \xrightarrow{\sim} \mathrm{Hom}((M, T), (N, S)).$$

With the notion of $\mathbb{Z}$-structures in hand as well as the motivating Example 24.2, we can now state the analog of (0) for motives.

24.3 Conjecture. *Let F be a finitely generated field over $\mathbb{Q}$, and let M be a motive over F. Then the set*

$$\{(M, T) \mid T \textit{ is a } \mathbb{Z}\textit{-structure on } M\}/\cong$$

is finite.

24.4 Example. In the case of an abelian variety A, where the motive M equals $H^1(A)$, we have seen that $\mathbb{Z}$-structures on M are the same thing as isomorphism classes of abelian varieties isogeneous to A. Therefore (0) for motives is equivalent to (0) for abelian varieties, which was proved by Faltings.

We'll now prove the analog of (0) $\Longrightarrow$ (1) in the motivic context, where now (1) is the Tate conjecture for algebraic cycles. This shall be similar to our original (0) $\Longrightarrow$ (1) in the case of abelian varieties. And while (0) is still wildly conjectural in the setting of general motives, this might prove to be a fruitful angle of attack on (1).

24.5 Remark. Any motive $M = \bigoplus_{i=1}^t H^{m_i}(X_i)(r_i)$ always has a $\mathbb{Z}$-structure. For example, we can take

$$T_{\mathbb{Z}} = \bigoplus_{i=1}^t H^{m_i}(X_i(\mathbb{C}), \mathbb{Z})/\text{torsion} \otimes_{\mathbb{Z}} \mathbb{Z}(2\pi i)^{r_i} \subseteq M_B$$

on the Betti realization side, or we could (equivalently, by comparison theorems) take

$$T = \bigoplus_{i=1}^t H^{m_i}_{\text{ét}}(X_{\overline{k}}, \widehat{\mathbb{Z}})/\text{torsion} \otimes_{\widehat{\mathbb{Z}}} \widehat{\mathbb{Z}}(r_i) \subseteq M_B \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$$

on the étale realization side.

Proof of (0) $\Longrightarrow$ (1) for motives. We work with our algebraic cycle formulation of the Tate conjecture. Write M for the motive $\mathbb{Q} \oplus H^{2r}(X)(r)$, and let β lie in $(H^{2r}_{\text{ét}}(X_{\overline{F}}, \mathbb{Q}_p)(r))^{G_F}$. Form the $\mathbb{Q}_p$-vector space

$$W \coloneqq \mathbb{Q}_p(1, \beta) \subseteq M_p = M_B \otimes_{\mathbb{Z}} \mathbb{Z}_p = \mathbb{Q}_p \oplus H^{2r}_{\text{ét}}(X_{\overline{F}}, \mathbb{Q}_p)(r).$$

Now fix a $\mathbb{Z}$-structure T on M, and decompose it as $T = \prod_\ell T_p$, where the T_ℓ is the $\mathbb{Z}_\ell$-module, corresponding to the decomposition $\widehat{\mathbb{Z}} = \prod_\ell \mathbb{Z}_\ell$. Set $W_{\mathbb{Z}_p} \coloneqq W \cap T_p$, and consider the $\mathbb{Z}$-structure

$$T^{(n)} \coloneqq \prod_{\ell \neq p} T_\ell \times (W_{\mathbb{Z}_p} + p^n T_p).$$

By (0), the set

$$\{(M, T^{(n)}) \mid n \geq 0\}/\cong$$

is finite, and the same argument as in Proposition 22.2 generalizes to show that $W = uM_p$ for some u in $\operatorname{End}_{\mathbb{Q}_p}(M_p) = \operatorname{End}(M) \otimes_{\mathbb{Q}} \mathbb{Q}_p$. Using the decomposition $M_p = \mathbb{Q}_p \oplus H^{2r}_{\text{ét}}(X_{\overline{F}}, \mathbb{Q}_p)(r)$, we break up u into the matrix form

$$u = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix},$$

where

$$u_{11} : \mathbb{Q}_p \longrightarrow \mathbb{Q}_p,\ u_{12} : H^{2r}_{\text{ét}}(X_{\overline{F}}, \mathbb{Q}_p)(r) \longrightarrow \mathbb{Q}_p,\ u_{21} : \mathbb{Q}_p \longrightarrow H^{2r}_{\text{ét}}(X_{\overline{F}}, \mathbb{Q}_p)(r),$$
$$\text{and } u_{22} : H^{2r}_{\text{ét}}(X_{\overline{F}}, \mathbb{Q}_p)(r) \longrightarrow H^{2r}_{\text{ét}}(X_{\overline{F}}, \mathbb{Q}_p)(r)$$

are $\mathbb{Q}_p$-linear maps. Example 23.2 indicates that u_{21} comes from the image of $\mathrm{CH}^r(X)_{\mathbb{Q}_p}$ in $H^{2r}_{\text{ét}}(X_{\overline{F}}, \mathbb{Q}_p)(r)$, and because Poincaré duality yields

$$\begin{aligned} \operatorname{Hom}_{\mathbb{Q}_p}(H^{2r}_{\text{ét}}(X_{\overline{F}}, \mathbb{Q}_p)(r), \mathbb{Q}_p) &= \operatorname{Hom}_{\mathbb{Q}_p}(\mathbb{Q}_p^\vee, (H^{2r}_{\text{ét}}(X_{\overline{F}}, \mathbb{Q}_p)(r))^\vee) \\ &= \operatorname{Hom}_{\mathbb{Q}_p}(\mathbb{Q}_p, H^{2(\dim X - r)}_{\text{ét}}(X_{\overline{F}}, \mathbb{Q}_p)(\dim X - r)), \end{aligned}$$

we also see that u_{12} comes from the image of $\mathrm{CH}^{\dim X - r}(X)_{\mathbb{Q}_p}$ in $H^{2(\dim X - r)}_{\text{ét}}(X_{\overline{F}}, \mathbb{Q}_p)(\dim X - r)$.

For all x in $\mathbb{Q}_p$ and y in $H^{2r}_{\text{ét}}(X_{\overline{F}}, \mathbb{Q}_p)(r)$, we have

$$u\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u_{11}x + u_{12}y \\ u_{21}x + u_{22}y \end{pmatrix} = \begin{pmatrix} z \\ \beta z \end{pmatrix}$$

for some z in $\mathbb{Q}_p$. Setting $y = 0$ and $x = 1$ yields

$$\begin{pmatrix} u_{11} \\ u_{21}(1) \end{pmatrix} = \begin{pmatrix} z \\ \beta z \end{pmatrix} \implies u_{21}(1) = \beta u_{11}.$$

If $u_{11} \neq 0$, then $\beta = u_{21}/u_{11}$ already comes from $\mathrm{CH}^r(X)_{\mathbb{Q}_p}$, as desired. If instead $u_{11} = 0$, this implies that $u_{21} = 0$ as well, so altogether we have

$$\left\{ \begin{pmatrix} u_{12}(y) \\ u_{22}(y) \end{pmatrix} \in \mathbb{Q}_p \oplus H^{2r}_{\text{ét}}(X_{\overline{F}}, \mathbb{Q}_p) \,\middle|\, y \in H^{2r}_{\text{ét}}(X_{\overline{F}}, \mathbb{Q}_p)(r) \right\} = uM_p = W = \mathbb{Q}_p \begin{pmatrix} 1 \\ \beta \end{pmatrix}.$$

This shows us that $u_{22}(y) = u_{12}(y)\beta$ for all y in $H^{2r}_{\text{ét}}(X_{\overline{F}}, \mathbb{Q}_p)(r)$. Now if $u_{12} = 0$, then $u_{22} = 0$ as well and hence $u = 0$, which cannot be the case. Therefore $u_{12} \neq 0$. Somehow, we know that u_{22} comes from the image of $\mathrm{CH}^{\dim X}(X \times X)_{\mathbb{Q}_p}$, and if we allow ourselves the conjecture that the intersection pairing on Chow groups modulo homological equivalence is non-degenerate, we can pair u_{12} with its dual to show that β comes from an algebraic cycle too. □

Thus the Tate conjecture for algebraic cycles can be reduced to a generalized version of (0). Concerning (0) itself, Koshikawa has defined a height function $H_{\text{Kos}}(M, T)$ on motives with $\mathbb{Z}$-structure. The goal would be to show, for a fixed motive M, that

$$\{H_{\text{Kos}}(M, T) \mid T \text{ is a } \mathbb{Z}\text{-structure on } M\}$$

is bounded, in analogy with Faltings's result on H_{Fal} for abelian varieties. However, to conclude a proof of (0), the issue is that one also needs a variant of Faltings's rough comparison between H_{Fal} and naive heights on the moduli space of abelian varieties. In the motivic setting, we have no moduli space of motives to use for a similar strategy.

I wanted to spend the rest of my time defining $H_{\text{Kos}}(M, T)$, but I only have three minutes left. So I will just describe it a little. The construction of H_{Kos} is similar to that of H_{Fal}, where we use the de Rham realization of M and the graded pieces of its natural filtration to obtain the line bundle whose height we'd compute, á la Definition 21.2.

I don't have time to do the rest, and I refer you to Koshikawa's paper "On heights of motives with semistable reduction" for more details. The construction involves a seemingly strange tensor product, but it's not *that* strange—it can be motivated[16] by the formula

$$\sum_{r=0}^{m} r \dim \mathrm{gr}^r H^m_{\mathrm{dR}}(M) = \frac{m}{2} \dim H^m_{\mathrm{dR}}(M),$$

so it's not too weird.

24.6 Example. In the case of an abelian variety A, where the motive is $M = H^1(A)$, we have $\mathrm{gr}^1 H_{\mathrm{dR}}(M) = \Gamma(A, \Omega^1_A)$. This will end up showing that Koshikawa height is a natural generalization of Faltings height.

[16] Get it?