Notes for MATH 372 — Geometric Satake
(Spring 2018)

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These are live-TeX'd notes for a course taught at the University of Chicago in Spring 2018 by Professor Victor Ginzburg. Any errors are attributed to the note-taker. If you find any such errors or have comments at large, don't hesitate to contact said note-taker at `lidansiyan@gmail.com`.

1 March 27, 2018

This class is about geometric Satake, which is one of the basic problems in geometric Langlands. When Googling earlier, I discovered that while there are many references on geometric Satake, there are only a few on classical Satake. Geometric Satake is something Drinfeld came up with, and to understand it, it's helpful to consider the classical setup he had in mind when developing it.

Today, we will focus on GL_n, as general reductive groups only present technical changes. We shall begin with the function field analogy:

$$\mathbb{Z} \longleftrightarrow \mathbb{F}_q[\Sigma^{\mathrm{aff}}],$$

where Σ^{aff} is a smooth affine curve over $\mathbb{F}_q$. Taking fraction fields in this analogy yields

$$\mathbb{Q} \longleftrightarrow \mathbb{F}_q(\Sigma^{\mathrm{aff}}),$$

and taking completions at prime ideals yields

$$\mathbb{Z}_p \longleftrightarrow \widehat{\mathcal{O}}_x,$$

where $\widehat{\mathcal{O}}_x$ is the completed local ring at a closed point x of a smooth completion Σ of Σ^{aff}. Note that we can form $\widehat{\mathcal{O}}_x$ even for x in $\Sigma \smallsetminus \Sigma^{\mathrm{aff}}$, which corresponds on the left-hand side to completions at archimedean places (e.g. the completion $\mathbb{R}$ in the case of $\mathbb{Q}$).

Number theorists are also interested in automorphic forms, for which we offer the following provisional definition.

1.1 Definition. An *automorphic function* is a $\mathbb{C}$-valued function on

$$\mathrm{GL}_n(\widehat{\mathbb{Z}})\backslash \mathrm{GL}_n(\mathbb{A}) / \mathrm{GL}_n(\mathbb{Q}).$$

On the geometric side, we can form the analogous set

$$\prod_{x\in\Sigma_0} \mathrm{GL}_n(\widehat{\mathcal{O}}_x)\Big\backslash {\prod_{x\in\Sigma_0}}' \mathrm{GL}_n(K_x)\Big/ \mathrm{GL}_n(\mathbb{F}_q(\Sigma)),$$

where Σ_0 denotes the set of closed points of Σ, and K_x is the fraction field of $\widehat{\mathcal{O}}_x$. We can reinterpret this double coset space in a geometric manner, via the following simple yet important observation of Weil:

1.2 Proposition (Weil)**.** *We have a natural bijection*

$$\mathrm{Bun}_n(\Sigma)(\mathbb{F}_q) \xrightarrow{\sim} \prod_{x\in\Sigma_0} \mathrm{GL}_n(\widehat{\mathscr{O}}_x) \Big\backslash {\prod_{x\in\Sigma_0}}' \mathrm{GL}_n(K_x) \Big/ \mathrm{GL}_n(\mathbb{F}_q(\Sigma)),$$

where $\mathrm{Bun}_n(\Sigma)(\mathbb{F}_q)$ *denotes the set of isomorphism classes of rank* n *vector bundles on* Σ *over* $\mathbb{F}_q$*.*

Proof. Let $\mathscr{V}$ be a rank n vector bundle on Σ. Then $\operatorname{Spec}\mathbb{F}_q(\Sigma)\otimes_\Sigma \mathscr{V}$ is an n-dimensional vector space over $\mathbb{F}_q(\Sigma)$, so it has a basis $v_1,\dots,v_n$. Now these v_i are rational sections of $\mathscr{V}$, so they have poles. Let $X\subset\Sigma_0$ be the set of singularities for all the v_i.

Next, for all x in X, write $\mathscr{V}_x$ for the completion of $\mathscr{V}$ at x. It is a locally free and hence free $\widehat{\mathscr{O}}_x$-module, so we can pick a basis $u_x^1,\dots,u_x^n$ of $\mathscr{V}_x$. After passing to K_x, we obtain a matrix passing from $v_1,\dots,v_n$ to $u_x^1,\dots,u_x^n$ in $\mathrm{GL}_n(K_x)$. Changing our choice of basis does not change the double coset associated to these $\mathrm{GL}_n(K_x)$, so we obtain an element of the right-hand side.

Conversely, we use any double coset to reverse-engineer the associated vector bundle by emulating the above construction, using the fact that $\mathrm{GL}_n(\widehat{\mathscr{O}}_x)\,\mathrm{GL}_n(\mathbb{F}_q(\Sigma)) = \mathrm{GL}_n(K_x)$.[1] These constructions can readily be checked to be inverse to one another, completing the proof. ☐

In order to cover the entirety of Σ with our charts formed by elements of the double coset, we used the fact that Σ is a curve. This is why we study geometric Langlands on curves rather than general varieties.

Let's now proceed to *Hecke operators*, which are a collection of commuting operators in the space of automorphic functions. Hecke only carried out this construction for $n=2$, but there's no reason not to perform the same process for general n. We begin with the following object in the geometric setting.

1.3 Definition. Let r be an integer between 0 to n, inclusive. The *r-th Hecke stack* is defined to be[2]

$$\mathscr{H}\mathrm{eck}^r := \{(\mathscr{V},\mathscr{V}',x)\in \mathrm{Bun}_n\times\mathrm{Bun}_n\times\Sigma \mid \mathscr{V}'\subseteq\mathscr{V} \text{ and } \mathscr{V}/\mathscr{V}' \text{ is a skyscraper sheaf of rank } r \text{ at } x\}.$$

Concretely, what does the Hecke stack look like? Let t be a local parameter at x, and write $\mathscr{V}(x)$ for the fiber of $\mathscr{V}$ at x. Fix an r-dimensional subspace E of $\mathscr{V}(x)$, and consider the short exact sequence

$$0\longrightarrow \mathscr{V}_x \xrightarrow{t} \mathscr{V}_x \longrightarrow \mathscr{V}(x)\longrightarrow 0.$$

Write $\widetilde{E}$ for the preimage of E in $\mathscr{V}_x$. Then the collection of possible $\mathscr{V}'$, given the rest of the data, is the collection of subsheaves of $\mathscr{V}$ consisting of sections s such that s_x lies in $\widetilde{E}$ (that is, s_x lies in E modulo t) for some choice of E.

By the theory of elementary divisors (where we're using the fact that $\widehat{\mathscr{O}}_x$ is a PID), we can choose a basis $u_x^1,\dots,u_x^n$ of $\mathscr{V}_x$ such that $\widetilde{E}$ has a basis given by $tu_x^1,\dots,tu_x^r,u_x^{r+1},\dots,u_x^n$. We can use this to explicate our description of $\mathscr{V}'$ further. In addition, because we understand submodules of free modules of PIDs quite well, we can be fancier and impose more complicated conditions on $\mathscr{V}'$ at x, like taking different powers of t on the different basis elements instead.

The Hecke stack comes with three projection maps

$$\begin{tikzcd} & \mathscr{H}\mathrm{eck}^r \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] \arrow[rr, "p"] & & \Sigma \\ \mathrm{Bun}_n & & \mathrm{Bun}_n. & \end{tikzcd}$$

[1] This fact can be proved using the Iwasawa decomposition $\mathrm{GL}_n(\widehat{\mathscr{O}}_x)B(K_x)=\mathrm{GL}_n(K_x)$, the fact that uniformizers t at x can be taken to lie in $\mathbb{F}_q(\Sigma)$, and some nice row operations to kill off the tails in the "t-power series" expansions of elements in K_x.

[2] When interpreting this for general S-valued points, we want $\mathscr{V}/\mathscr{V}'$ to be a vector bundle of rank r.

Furthermore, our above description of the collection of possible $\mathscr{V}'$ indicates that

$$(\mathrm{pr}_2 \times p)^{-1}(\mathscr{V}, x) = \mathrm{Gr}^r(\mathscr{V}(x))$$

is the Grassmannian of rank r subspaces of $\mathscr{V}(x)$. Let's now actually define Hecke operators.

1.4 Definition. The *Hecke operator* H_x^r sends automorphic functions $f : \mathrm{Bun}_n(\Sigma)(\mathbb{F}_q) \longrightarrow \mathbb{C}$ to

$$H_x^r(f) \coloneqq (\mathrm{pr}_{2,x})_*(\mathrm{pr}_{1,x})^*(f).$$

We can see how this matches with the arithmetic case by noting that the pushforward of functions here corresponds to a summation over the fiber. For this interpretation, it is crucial that our residue fields $\mathbb{F}_q$ are finite.

One can readily check that the H_x^r commute for different x. What is a bit less immediate is that they also commute for different r:

1.5 Theorem. *The operators H_x^r commute, for any r.*

The modern formulation of geometric Satake requires the *affine Grassmannian*, which I shall now explain. Let K be a nonarchimedean local field (e.g. K_x), write $\mathcal{O}$ for its ring of integers, and let t be a uniformizer of $\mathcal{O}$. Suppose that the residue field of $\mathcal{O}$ is $\mathbb{F}_q$.

1.6 Definition. A *lattice* is a finitely generated $\mathcal{O}$-submodule L of K^n such that $\bigcup_{m\geq 0} \frac{1}{t^m} L = K^n$.

The affine Grassmannian, roughly speaking, puts a geometric structure on the set of lattices in K^n.

1.7 Example. We can just take the standard lattice $L_0 \coloneqq \mathcal{O}^n \subset K^n$. This is like a base point for our further discussion of lattices.

To get a handle on lattices, I will state the following lemma, which you should immediately recognize.

1.8 Lemma. *Let $L' \subseteq L$ be a pair of lattices. Then there exists an $\mathcal{O}$-basis $e_1, \dots, e_n$ of L and an unordered n-tuple $m_1 \geq \cdots \geq m_n \geq 0$ of non-negative integers such that $L' = \mathcal{O}t^{m_1}e_1 + \cdots + \mathcal{O}t^{m_n}e_n$.*

This follows from the structure theory of finitely generated modules over PIDs, once again. In general, we can find a matrix g in $\mathrm{M}_n(\mathcal{O})$ such that $L' = g(L)$, and Lemma 1.8 just gives us an optimal way of choosing such a g.

For our upcoming corollary, note that $\mathrm{GL}_n(K)$ acts on the set of lattices in K^n by left translation, with $\mathrm{GL}_n(\mathcal{O})$ being the stabilizer of L_0.

1.9 Corollary. *Let L and L' be sublattices of L_0. Then L lies in $\mathrm{GL}_n(\mathcal{O})L'$ if and only if L_0/L is isomorphic to L_0/L'.*

Proof. The $\Longrightarrow$ direction is immediate, so let us tackle the $\Longleftarrow$ direction. Use Lemma 1.8 to find a basis $e_1, \dots, e_n$ of L_0 adapted to L and a basis $e_1', \dots, e_n'$ of L_0 adapted to L'. Then the linear map defined by sending $e_i \mapsto e_i'$ is an automorphism φ of L_0, and because L_0/L is isomorphic to L_0/L', the sequences of integers corresponding to L and L' are equal. Therefore φ also sends L to L'. $\square$

1.10 Corollary. *Let L be a sublattice of L_0, and suppose that $L = g(L_0)$ for some g in $\mathrm{M}_n(\mathcal{O})$. Then $\mathrm{length}_{\mathcal{O}} L_0/L = m_1 + \cdots + m_n$ for the integers provided by Lemma 1.8, and $\det g = ct^{\mathrm{length}_{\mathcal{O}} L_0/L}$, where c lies in $\mathcal{O}^\times$.*

Enough with lattices for a moment. Their discussion already allows us to reinterpret the Hecke stack as

$$\mathscr{H}\mathrm{eck}^r = \left\{ (\mathscr{V}', \mathscr{V}, x, \phi) \;\middle|\; \begin{array}{c} \phi : \mathscr{V}'|_{\Sigma\smallsetminus x} \xrightarrow{\sim} \mathscr{V}|_{\Sigma\smallsetminus x} \text{ such that } \mathscr{V}'_x \text{ and } \mathscr{V}_x \\ \text{are in relative position } (1,\dots,1,0,\dots,0). \end{array} \right\}$$

where there are r copies of 1 in $(1,\dots,1,0,\dots,0)$.

Next, observe that for any two lattices L and L' in K^n, we have $t^m L' \subseteq L \subseteq \frac{1}{t^m}L'$ for sufficiently large m.[3] This allows us to extend Lemma 1.8 to the case where L and L' don't necessarily have inclusion relations, by allowing the m_i to potentially be negative.

Now suppose K has positive characteristic. We now have a provisional definition of $\mathbb{F}_q$-points on the affine Grassmannian as[4]

$$\mathrm{Gr}(\mathbb{F}_q) := \{\text{lattices in } K^n\}.$$

By noticing that $\mathrm{Gr}(\mathbb{F}_q)$ is a homogeneous space for $\mathrm{GL}_n(K)$ and choosing picking L_0 for the base point, we can identify $\mathrm{Gr}(\mathbb{F}_q)$ with $\mathrm{GL}_n(K)/\mathrm{GL}_n(\mathcal{O})$.

Our extended version of Lemma 1.8 tells us what the left $\mathrm{GL}_n(\mathcal{O})$-orbits are on $\mathrm{Gr}(\mathbb{F}_q)$. Namely,

$$\mathrm{Gr}(\mathbb{F}_q) = \coprod_{m_1\geq\cdots\geq m_n} \mathrm{GL}_n(\mathcal{O})\,\mathrm{diag}(t^{m_1},\dots,t^{m_n})\,\mathrm{GL}_n(\mathcal{O})/\mathrm{GL}_n(\mathcal{O}),$$

which is also known as the *Cartan decomposition*. The Cartan decomposition also holds for general reductive groups, though its proof is not nearly as nice as this case of GL_n. We have to replace the data of $m_1 \geq \cdots \geq m_n$ with dominant coweights in general.

I will now skip everything else and just explain why Drinfeld looked at this content. I'll make up for it next time with more details. Returning to our discussion of Hecke operators, note that we can obtain way more operators by replacing

$$\underbrace{1 \geq \cdots \geq 1}_{r \text{ times}} \geq 0 \geq \cdots \geq 0$$

with arbitrary $m_1 \geq \cdots \geq m_n$. The analog of Theorem 1.5 continues to hold in this context. Write H for the algebra they generate in the endomorphism ring of automorphic functions.

Returning to the setting of arbitrary nonarchimedean local fields K, denote the spherical Hecke algebra $C_c(\mathrm{GL}_n(\mathcal{O})\backslash\mathrm{GL}_n(K)/\mathrm{GL}_n(\mathcal{O}))$ by H as well. Now what is the point of Satake? Write $X_\bullet$ for $\mathbb{Z}^n$ (which will be the coweight lattice in general).[5] The Weyl group $W = \mathfrak{S}_n$ of GL_n acts on $X_\bullet$ and hence $\mathbb{C}[X_\bullet]$.

1.11 Theorem (Satake)**.** *We have an isomorphism* $H \xrightarrow{\sim} \mathbb{C}[X_\bullet]^W$.

1.12 Remark.

- We see that $\mathbb{C}[X_\bullet]^W = \mathbb{C}[T^\vee]^W$, where here T is the usual maximal torus of GL_n. In turn, we have $\mathbb{C}[T^\vee]^W = \mathbb{C}[G^\vee]^{G^\vee}$ by the Chevalley restriction theorem, where $G = \mathrm{GL}_n$. This is where the Langlands dual appears in some formulations of Satake.
- The Weyl character formula tells us that $\mathbb{C}[G^\vee]^{G^\vee}$ has a $\mathbb{C}$-basis given by characters of irreducible finite-dimensional representations of $G^\vee(\mathbb{C}) = \mathrm{GL}_n(\mathbb{C})$. In other words, the functions $\chi_{(m_1,\dots,m_n)}$ (as $(m_1,\dots,m_n)$ ranges through dominant weights of $G^\vee$ and hence dominant coweights of G) form a $\mathbb{C}$-basis of $\mathbb{C}[G^\vee]^{G^\vee}$.

[3]We immediately have $L \subseteq \frac{1}{t^m}L'$ for sufficiently large m, and applying Lemma 1.8 to this inclusion tells us that we can take m (enlarging if necessary) such that $t^m L' \subseteq L$ as well.

[4]Our use of Gr here for the affine Grassmannian is not to be confused with our earlier use of Gr for the usual Grassmannian.

[5]This notation also has the advantage of avoiding the use of $\mathbb{Z}[\mathbb{Z}^n]$ for group algebras.

- While the usual Satake isomorphism involves q, now that we have expressed both sides in terms of dominant coweights of G (which do not involve q), one can ask whether the isomorphism can be re-expressed as not to involve q. The answer is yes: Lusztig did this, and the answer involves *Kazhdan–Lusztig polynomials*. Somehow, they can cancel out the q terms perfectly.
- The constructions in the Satake isomorphism only depend on q as a Laurent polynomial.

Geometric Satake instead relates our group-theoretic data (analogous to $\mathbb{C}[G^\vee]^{G^\vee}$) directly to our geometrized version of H. Recently, the story of geometric Satake has also been recasted to the case when K is a p-adic field in the work of Scholze, whose work is very formal. I don't mean formal in the sense of logic, where one can prove certain statements in characteristic zero by turning them into a computer program that only needs input from positive characteristic to run—given that aspects of number field Langlands show up in situations as complicated as Wiles's proof of Fermat's Last Theorem, it seems unreasonable to do *too* much number theory in this logic-theoretic way.

I will try to explain more about Drinfeld's ideas next time.

2 March 29, 2018

The note-taker missed class today—he thanks Boming Jia as well as Xiao Griffin Wang for letting him consult their notes.

Let's work entirely in the geometric setup. Let k be any field, let our group G be GL_n, and let $\mathfrak{S}_n$ act on the $\mathbb{Z}$-module $\Lambda := \mathbb{Z}^n$ via permuting entries. Write K for the field of formal Laurent series $k(\!(t)\!)$, and write $\mathcal{O}$ for the subring of formal power series $k[\![t]\!]$. Carrying out our discussion from last time in this geometric setting, we have the k-points of the affine Grassmannian:

$$\mathrm{Gr}(k) := \{\text{lattices in } K^n\}.$$

The group $G(K)$ acts transitively on $\mathrm{Gr}(k)$, and the stabilizer of the standard lattice $L_0 := \mathcal{O}^n$ equals $G(\mathcal{O})$. Therefore we have

$$\mathrm{Gr}(k) = G(K)/G(\mathcal{O}).$$

Note that this shows there is a left action of $G(K)$ on $\mathrm{Gr}(k)$, which is given by operations on the opposite side as the one used in our transitive action.

2.1 Definition. Let L and L' be lattices in K^n. A *basis adapted to* (L, L') is an $\mathcal{O}$-basis $v_1, \dots, v_n$ of L such that $L' = t^{m_1}\mathcal{O} + \dots + t^{m_n}\mathcal{O}$ for some $\lambda := (m_1, \dots, m_n)$ in Λ. We say that this basis has *type* λ.

The following lemma is in line with our discussion from last time.

2.2 Lemma.

1) *Any pair of lattices (L, L') has an adapted basis, and its type λ is unique up to permutation. In this case, we say that L and L' are in* relative position $\lambda \in \Lambda/\mathfrak{S}_n$.

2) *Any pairs of lattices (L_1, L_2) and (L_1', L_2') have the same relative position if and only if they belong to the same orbit of the diagonal action of $G(K)$ on $\mathrm{Gr}(k) \times \mathrm{Gr}(k)$.*

2.3 Remark. For any abstract group A and subgroup B, the map

$$\begin{aligned} A\backslash(A/B \times A/B) &\longrightarrow B\backslash A/B \\ (a, a') &\longmapsto a^{-1}a' \end{aligned}$$

is a bijection, where the inverse is given by $a' \mapsto (1, a')$. Therefore we get

$$G(K)\backslash(\mathrm{Gr}(k) \times \mathrm{Gr}(k)) = G(\mathcal{O})\backslash G(K)/G(\mathcal{O}) = G(\mathcal{O})\backslash \mathrm{Gr}(k),$$

which helps us interpret Lemma 2.2.2).

Proof. For 1), observe that $t^r L'$ lies in L for sufficiently large r. Applying the same proof as that of Theorem 1.8 yields the desired result. As for 2), we just run the same argument as in Corollary 1.9. $\square$

The stack Gr is not going to be of finite type over k, so we want to stratify it into objects that are. Fix a positive integer r, and set

$$\mathrm{Gr}_r(k) := \{L \in \mathrm{Gr}(k) \mid t^r L_0 \subseteq L \subseteq t^{-r} L_0\}$$

on the level of k-points. We then see that

$$\mathrm{Gr}(k) = \varinjlim_r \mathrm{Gr}_r(k),$$

and quotienting by $t^r L_0$ tells us that

$$\mathrm{Gr}_r(k) = \left\{ t\text{-stable } k\text{-subspaces } \frac{L}{t^r L_0} \text{ of } \frac{t^{-r} L_0}{t^r L_0} = (k^n)^{2r} \right\},$$

where t acts nilpotently on $(k^n)^{2r}$. This will soon let us identify Gr_r a closed subvariety of a union of (the usual) Grassmannians.

2.4 Remark. Let's consider a few example fields first:

- When $k = \mathbb{F}_q$, the set $\mathrm{Gr}_r(k)$ is finite.
- When $k = \mathbb{C}$, the set $\mathrm{Gr}_r(k)$ forms a complex projective variety.
- We can explicate the action of t on $(k^n)^{2r}$. If $e_1, \dots, e_n$ is the standard $\mathcal{O}$-basis of L_0, then $t^j e_i$ forms a k-basis for $t^{-r} L_0 / t^r L_0$, where $1 \leq i \leq n$ and $-r \leq j \leq r-1$ are integers. If we first vary the j from $r-1, \dots, -r$ while keeping the i fixed, the matrix for t becomes

$$\begin{bmatrix} J_0 & & \\ & \ddots & \\ & & J_0 \end{bmatrix},$$

  where J_0 is the full $(2r \times 2r)$-Jordan block of eigenvalue zero.

Write T for the subgroup of diagonal matrices in G, and write N for the subgroup of unipotent upper-triangular matrices in G.

2.5 Lemma.

1) We have a bijection given by

$$\begin{aligned} \Lambda &\xrightarrow{\sim} \mathrm{Gr}(k)^{T(k)} \\ \lambda &\longmapsto L^\lambda := t^{m_1}\mathcal{O} + \cdots + t^{m_n}\mathcal{O}. \end{aligned}$$

2) We have a decomposition as sets

$$\mathrm{Gr}(k) = \coprod_{\lambda \in \Lambda} N(K) L^{\lambda}.$$

Equivalently, we have the following decomposition for $G(K)$:

$$G(F) = \coprod_{\lambda \in \Lambda} N(K)\,\mathrm{diag}(t^{m_1}, \ldots, t^{m_n}) G(\mathcal{O}).$$

This is the Iwasawa decomposition.

Proof. For part 1), injectivity is immediate, so let us move to surjectivity. Begin by noting that the $T(k)$-action on $\mathrm{Gr}(k)$ yields a $T(k)$-action on $t^{-r}L_0/t^r L_0$ that commutes with the action of t. We then see that any $T(k)$-stable k-subspace of $t^{-r}L_0/t^r L_0$ is of the form

$$E = E_1 \oplus \cdots \oplus E_n,$$

where the E_i is a subspace of $\bigoplus_{j=-r}^{r-1} k \cdot t^j e_i$. If E is also t-stable, then we can reduce the $t^j e_i$ to a smallest value of j, which indicates that $E_i = \bigoplus_{j=m_i}^{r-1} k \cdot t^j e_i$ for some m_i. Finally, taking the union over all r yields the desired result.

As for part 2), use Gaussian elimination to turn any lattice L in K^n into one of the form L^{λ} via row operations in $N(K)$. $\square$

Return now to a global setup: Let Σ be a smooth projective curve over k, let x be a k-point of Σ, set $\mathcal{O} = \widehat{\mathcal{O}}_x$, and write $\mathcal{O}_{\mathrm{out}}$ for the global sections of $\Sigma \smallsetminus x$.

2.6 Proposition. *We have an identification*

$$\mathrm{Gr}(k) = \left\{ (\mathscr{V}, \phi) \mid \mathscr{V} \text{ is a rank-}n\text{ vector bundle on } \Sigma \text{ and } \psi : \mathscr{V}|_{\Sigma \smallsetminus x} \xrightarrow{\sim} \mathscr{O}^n_{\Sigma \smallsetminus x} \right\} / \sim .$$

Proof. In the proof of Proposition 1.2 (which works over any field k in place of $\mathbb{F}_q$), we see that fixing a trivialization of $\mathscr{V}$ away from x corresponds to forcing all components of our adelic double quotient to lie in $G(\widehat{\mathcal{O}}_y)$ for closed points $y \neq x$. The last remaining place x yields the identification with $G(K)/G(\mathcal{O})$ and hence $\mathrm{Gr}(k)$. $\square$

As remarked last time, we now extend the Hecke stack to arbitrary positions λ in $\Lambda/\mathfrak{S}_n$. Set

$$\mathscr{H}\mathrm{eck}_x^{\lambda} := \left\{ (\mathscr{V}, \mathscr{V}', \phi) \;\middle|\; \begin{array}{c} \phi : \mathscr{V}|_{\Sigma \smallsetminus x} \xrightarrow{\sim} \mathscr{V}'|_{\Sigma \smallsetminus x} \text{ such that } \mathscr{V}_x \\ \text{and } \mathscr{V}'_x \text{ are in relative position } \lambda. \end{array} \right\}$$

As before, we have a pair of projections

$$\begin{tikzcd} & \mathscr{H}\mathrm{eck}_x^{\lambda} \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\ \mathrm{Bun}_n & & \mathrm{Bun}_n, \end{tikzcd}$$

where we're working only at the local level at x (rather than taking the Hecke stack over all of Σ). Write Bun_n^0 for the substack of rank-n vector bundles $\mathscr{V}$ such that $\mathscr{V}|_{\Sigma \smallsetminus x}$ is trivial, and write $(\mathrm{Gr}(k) \times \mathrm{Gr}(k))^{\lambda}$ for the pairs of lattices in relative position λ. Proposition 2.6 implies that

$$\mathrm{pr}_1^{-1}(\mathrm{Bun}_n^0(k)) = G(\mathcal{O}_{\mathrm{out}}) \backslash (\mathrm{Gr}(k) \times \mathrm{Gr}(k))^{\lambda} = G(\mathcal{O}_{\mathrm{out}}) \backslash \mathrm{Gr}(k) / \mathrm{Stab}(L_0, L^{\lambda}).$$

Note that $L^0 = L_0$.

2.7 Example. When $\lambda = 0$, this recovers a variant of Proposition 1.2 for the affine curve $\Sigma \smallsetminus x$.

Let us now switch gears to a fairly general setting for our discussion of Hecke algebras. Let M be a locally compact topological group, suppose it has a maximal compact subgroup K, and choose a left Haar measure μ on M with $\mu(K) = 1$. Write

$$C_c(K\backslash M/K) \coloneqq \{\mathbb{C}\text{-valued } K\text{-biinvariant continuous functions on } M \text{ with compact support}\}.$$

Now $C_c(K\backslash M/K)$ forms a $\mathbb{C}$-algebra under convolution, and in a manner similar to Remark 2.3, we can show that it is isomorphic to $C_{\text{rel-comp-supp}}(M\backslash(M/K \times M/K))$. This latter space also has a natural convolution product.

2.8 Remark. For any space X and sufficiently good class of functions on X (where we are intentionally being very vague), the convolution of two said functions f and g on $X \times X$ can be given by

$$(p_{13})_*(p_{12}^* f \cdot p_{23}^* g),$$

where the p_{ij} are the projection maps $X \times X \times X \longrightarrow X \times X$.

Now suppose Y is a topological space with a continuous action of M. Then we get a natural action

$$\begin{aligned} C_c(Y/K) \otimes_{\mathbb{C}} C_c(K\backslash M/K) &\longrightarrow C_c(Y/K) \\ f \otimes g &\longmapsto \left(y \mapsto \int_M \mathrm{d}m\, f(ym) g(m^{-1}) \right), \end{aligned}$$

which we denote as $f * g$.

Finally, let us specialize to the situation $M = G(K)$, $K = G(K)$, and

$$Y = \left\{ (\mathscr{V}, \mathcal{B}) \mid \mathscr{V} \in \mathrm{Bun}_n(k) \text{ and } \mathcal{B} \text{ is a basis of } \mathscr{V}_x \right\}.$$

We topologize Y using the adelic description, which identifies it with

$$\prod_{\substack{y \in \Sigma_0 \\ y \neq x}} \mathrm{GL}_n(\widehat{\mathcal{O}}_y) \Big\backslash \prod_{y \in \Sigma_0}{}' \mathrm{GL}_n(K_y) \Big/ \mathrm{GL}_n(k(\Sigma)).$$

2.9 Theorem. *The algebra $C_c(G(\mathcal{O})\backslash G(K)/G(\mathcal{O}))$ is commutative.*

The following proof is due to Gelfand.

Proof. This proof shall work for general M and K, as long as we have a continuous anti-involution $\tau : M \longrightarrow M$ such that $\tau(KmK) = KmK$ for all m in M. To see this, define $\tau : C_c(M) \longrightarrow C_c(M)$ via

$$(\tau f)(m) \coloneqq f(\tau(m)).$$

One readily checks that $\tau(f * g) = \tau(g) * \tau(f)$, where we use the fact that τ is an involution to show that its modulus under μ is trivial. Our assumption on τ indicates that $\tau(f) = f$ for all f in $C_c(K\backslash M/K)$, so

$$f * g = \tau(f * g) = \tau(g) * \tau(f) = g * f$$

in this setting, as desired. Finally, in our case, the Cartan decomposition implies that matrix transposition yields a satisfactory candidate for τ. $\square$

3 April 3, 2018

I will now explain generalizations of various things for arbitary reductive groups.[6] This makes some statements cleaner—when we focused on GL_n, we got a combinatorial mess. I shall switch to using z for the uniformizer of $K = k(\!(z)\!)$ instead of our previous usage of t, because we want to reserve t for elements of tori. Let G be a split reductive group over k.

3.1 Definition. We define the k-points of the *affine Grassmannian of* G as $\mathrm{Gr}_G(k) \coloneqq G(K)/G(\mathcal{O})$.

We will also be interested in the (reduced) variety as well as scheme structure of Gr_G, but we shall stick to its k-points for now. To geometrize our quotient description, we present the following ideas of Lusztig. Write $\mathfrak{g}$ for the Lie algebra of G over k. In the case of GL_n, we studied lattices in K^n, but for more general G, we shall study lattices in $\mathfrak{g}(K)$ itself.

Fix a non-degenerate *invariant* symmetric bilinear form $(\cdot,\cdot) : \mathfrak{g}\times\mathfrak{g} \longrightarrow k$, where invariant means that

$$([x,y],z) + (y,[x,z]) = 0$$

for all x, y, and z in $\mathfrak{g}$. This condition stems from deriving the condition of G-invariance. From here, we can consider the extension of $(\cdot,\cdot)$ to a bilinear form $\mathfrak{g}(K)\times\mathfrak{g}(K) \longrightarrow K$, which we also denote by $(\cdot,\cdot)$.

3.2 Example. When $\mathfrak{g}$ is $\mathfrak{gl}_n$ or $\mathfrak{sl}_n$, we can take $(x,y) \coloneqq \mathrm{tr}(xy)$. In the $\mathfrak{sl}_n$ case, we can also take the Killing form $(\cdot,\cdot)_{\text{Killing}}$, which differs from the above by an integer depending on n.

At this point, assume that G is in fact simply connected (and in particular semisimple, as tori always contribute nontrivial finite coverings).

3.3 Definition. A *lattice (in $\mathfrak{g}(K)$)* is a finitely generated $\mathcal{O}$-module L of $\mathfrak{g}(K)$ such that $K\otimes_{\mathcal{O}} L = \mathfrak{g}(K)$.

Given a lattice L in $\mathfrak{g}(K)$, we can take its *dual lattice*

$$L^\vee \coloneqq \{x \in \mathfrak{g}(K) \mid (x,y)\in\mathcal{O} \quad \forall y \in L\}.$$

Our finite generation condition on L bounds the denominators in $L^\vee$, so we see that $L^\vee$ is indeed a lattice. We give the following alternative description of the k-points of the affine Grassmannian:

$$\mathrm{Gr}'_G(k) \coloneqq \{\text{lattices } L \text{ in } \mathfrak{g}(K) \text{ such that } [L,L]\subseteq L \text{ and } L = L^\vee\}.$$

3.4 Example. Write L_0 for the lattice $\mathfrak{g}(\mathcal{O})$, where we use the fact that $\mathcal{O}$ is a k-algebra. We immediately see that L_0 is closed under brackets and also self-dual.

Note that $G(K)$ acts on $\mathrm{Gr}'_G(k)$ via the adjoint action of $G(K)$, since the adjoint action commutes with brackets (and our bilinear form is invariant). Here are some facts about this action:

- the stabilizer of L_0 in $G(K)$ is $G(\mathcal{O})$,
- this action of $G(K)$ is transitive.

Combining these two facts yields an identification

$$\mathrm{Gr}'_G(k) = G(K)/G(\mathcal{O}) = \mathrm{Gr}_G(k),$$

as promised. Therefore, from now on we will use $\mathrm{Gr}_G(k)$ to denote $\mathrm{Gr}'_G(k)$. To motivate this new definition of $\mathrm{Gr}(k)$, perhaps the following remark will be enlightening.

[6] Even though we shall soon restrict to the case of simply connected groups.

3.5 Remark. Let B be a Borel subgroup of G, and consider the maximal flag variety G/B. Nowadays, we can view G/B as the variety of Borel subgroups of G. Thus it is also the variety of all Borel subalgebras of $\mathfrak{g}$. Our definition of $\mathrm{Gr}'_G(k)$ attempts to mimic this reinterpretation of G/B, by cooking up a choice-free way of considering $G(K)/G(\mathcal{O})$.

The next lemma is immediate from our knowledge of lattices in finite-dimensional K-vector spaces.

3.6 Lemma. *Let L be a lattice in $\mathfrak{g}(K)$. Then we have $z^n L_0 \subseteq L \subseteq z^{-n} L_0$ for sufficiently large n.*

Therefore $L/z^n L_0$ is a k-subspace of $V_n := z^{-n}L_0/z^n L_0$. In our description of the GL_n situation, we only had to also be z-stable in addition to being a k-subspace of V_n. However, here we're working additionally with lattices that are closed under $[\cdot,\cdot]$ and self-dual. To deal with this, first define a symmetric k-bilinear form

$$\begin{aligned} \beta_n : V_n \times V_n &\longrightarrow k \\ (x,y) &\longmapsto \mathrm{Res}_{z=0}(x,y), \end{aligned}$$

where the first parentheses denote an ordered pair, while the second parentheses denotes our non-degenerate symmetric bilinear pairing.

3.7 Proposition. *The form β_n is non-degenerate.*

Proof. We immediately see this from the non-degeneracy of $(\cdot,\cdot)$ and the fact that we can multiply by an appropriate power of z to move nonzero entries of formal Laurent series into the z^{-1} coefficient. $\square$

Let us continue with Lusztig's method of equipping the affine Grassmannian with a variety structure. Before doing so, note that we can obtain a non-degenerate alternating trilinear form on V_n via sending

$$(x,y,z) \mapsto \beta_n([x,y],z).$$

This assignment is manifestly alternating in x and y. While the alternation in z is initially unclear, it follows from using the Killing form for $(\cdot,\cdot)$ and the fact that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. This can be used to construct invariant 3-forms on G, by translating β_n to other tangent spaces, and this was first noticed by Cartan.

With this in mind, define

$$\mathrm{Gr}_n(k) := \left\{ \begin{array}{c} z\text{-stable } k\text{-subspaces } E \text{ of } V_n \text{ that are maximal isotropic with} \\ \text{respect to } \beta_n \text{ and satisfy } \beta_n([x,y],z) = 0 \text{ for all } x,y,z \text{ in } E \end{array} \right\}.$$

3.8 Proposition. *The map sending L to $L/z^n L_0$ yields a bijection*

$$\{L \in \mathrm{Gr}_G(k) \mid z^n L_0 \subseteq L \subseteq z^{-n} L_0\} \xrightarrow{\sim} \mathrm{Gr}_n(k).$$

Sketch of the proof. Under the self-duality hypothesis, one can show that being closed under $[\cdot,\cdot]$ is equivalent to having $\beta_n([E,E],E) = 0$. And being self-dual itself is equivalent to the maximal isotropy condition on E. $\square$

Note that $\mathrm{Gr}_n(k)$ naturally has the structure of a projective variety. Because we have

$$\mathrm{Gr}_G(k) = \varinjlim_n \mathrm{Gr}_n(k),$$

we see that this allows us to equip Gr_G with the structure of an ind-variety. However, we did not consider possible reducedness in our discussion, so we do not know what the scheme structure on Gr_G is yet.

The conjugation action of $G(\mathcal{O})$ on $\mathrm{Gr}_G(k)$ descends to a $G(\mathcal{O})$-action on V_n, and this leads to the following computation. This is essentially the only computation in the business, and for this, we shall need the following Lie-theoretic notation. Let T be a maximal split torus of G over k, and write $X_\bullet := X_\bullet(T)$ for the lattice of cocharacters $\mathbb{G}_m \longrightarrow T$, which is abstractly isomorphic to $\mathbb{Z}^{\mathrm{rk}\, G}$.

For any λ in $X_\bullet$, write t^λ for the element $\lambda(z)$ of $T(K)$. We think of t^λ as a loop in $T(k)$. Define

$$L^\lambda := (\mathrm{ad}\, t^\lambda)(L_0) = (\mathrm{ad}\, t^\lambda)(\mathfrak{g}(\mathcal{O})),$$

where we note that $L^0 = L_0$ as in our GL_n situation.

Write $\mathfrak{t}$ for the Lie algebra of $\mathfrak{t}$, which a Cartan subalgebra of $\mathfrak{g}$. Write $R \subset \mathfrak{t}^*$ for the root system associated to $(\mathfrak{g}, \mathfrak{t})$, and form the associated root decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} k \cdot e_\alpha \implies \mathfrak{g}(\mathcal{O}) = \mathfrak{t}(\mathcal{O}) \oplus \bigoplus_{\alpha \in R} \mathcal{O} \cdot e_\alpha.$$

We use this to compute

$$(\mathrm{ad}\, t^\lambda)(\mathfrak{g}(\mathcal{O})) = \mathfrak{t}(\mathcal{O}) \oplus \bigoplus_{\alpha \in R} \mathcal{O} \cdot z^{\langle \lambda, \alpha \rangle} e_\alpha.$$

At this point, assume that λ is dominant, which is equivalent to asking that $\langle \lambda, \alpha \rangle \geq 0$ for all positive α. The above calculation indicates that $z^n L_0 \subseteq L^\lambda$ if and only if $\langle \lambda, \alpha \rangle \leq n$ for all positive α. This is equivalent to having $\langle \lambda, \alpha \rangle \geq -n$ for all negative α, which in turn is equivalent to saying that $L^\lambda \subseteq z^{-n} L_0$. Altogether, these conditions are all equivalent to L^λ lying in $\mathrm{Gr}_n(k)$.

Suppose this is the case, and write $\mathrm{Gr}^\lambda(k)$ for the $G(\mathcal{O})$-orbit of L^λ. Because $G(\mathcal{O})$ preserves $\mathrm{Gr}_n(k)$, we see that $\mathrm{Gr}^\lambda(k)$ lies in $\mathrm{Gr}_n(k)$. We identify

$$\mathrm{Gr}^\lambda(k) = G(\mathcal{O}) / \mathrm{Stab}_{G(\mathcal{O})}(L^\lambda)$$

thusly. We are interested in computing the dimension of $\mathrm{Gr}^\lambda(k)$ as a variety over k, which we can reduce to a question about Lie algebras. Namely,

$$\dim_k \mathrm{Gr}^\lambda(k) = \dim_k \frac{G(\mathcal{O})}{\mathrm{Stab}_{G(\mathcal{O})}(L^\lambda)} = \dim_k \frac{\mathfrak{g}(\mathcal{O})}{\mathrm{Lie}_{\mathcal{O}} \mathrm{Stab}_{G(\mathcal{O})}(L^\lambda)}.$$

This leads us to the question: what *is* $\mathrm{Stab}_{G(\mathcal{O})}(L^\lambda)$? We first see that

$$\mathrm{Stab}_{G(K)}(L^\lambda) = \mathrm{Stab}_{G(K)}((\mathrm{ad}\, t^\lambda)(\mathfrak{g}(\mathcal{O})) = (\mathrm{ad}\, t^\lambda)(\mathrm{Stab}_{G(K)}\, \mathfrak{g}(\mathcal{O})) = (\mathrm{ad}\, t^\lambda)(G(\mathcal{O})),$$

so we must have $\mathrm{Stab}_{G(\mathcal{O})}(L^\lambda) = G(\mathcal{O}) \cap (\mathrm{ad}\, t^\lambda)(G(\mathcal{O}))$. To compute the Lie algebra of this stabilizer, we use the fact that taking Lie algebras here commutes with taking intersections:

$$\mathrm{Lie}_{\mathcal{O}}(\mathrm{Stab}_{G(\mathcal{O})}(L^\lambda)) = \mathfrak{g}(\mathcal{O}) \cap (\mathrm{ad}\, t^\lambda)(\mathfrak{g}(\mathcal{O})) = \mathfrak{t}(\mathcal{O}) \oplus \bigoplus_{\substack{\alpha \in R \\ \alpha < 0}} \mathcal{O} \cdot e_\alpha \oplus \bigoplus_{\substack{\alpha \in R \\ \alpha > 0}} \mathcal{O} \cdot z^{\langle \lambda, \alpha \rangle} e_\alpha$$

The non-positive terms in our dimension calculation cancel, so we're left with

$$\dim_k \frac{\mathfrak{g}(\mathcal{O})}{\mathrm{Lie}_{\mathcal{O}} \mathrm{Stab}_{G(\mathcal{O})}(L^\lambda)} = \sum_{\alpha > 0} \langle \lambda, \alpha \rangle = 2\langle \lambda, \rho \rangle,$$

where ρ is the usual half-sum of all positive roots.

Write $Q^\vee \subseteq X_\bullet$ for the sublattice generated by the coroots. A basic fact is that $\langle \lambda, \rho \rangle$ is an integer for any λ in $Q^\vee$, though in general it is only a half-integer for λ in $X_\bullet$. In particular, $2\langle \lambda, \rho \rangle$ is even if λ lies in $Q^\vee$. Since we assumed that G is simply connected, we see that $X_\bullet = Q^\vee$, which nets us the following corollary.

3.9 Corollary. *The integer $2\langle\lambda,\rho\rangle$ is even.*

To more vividly see the simple connectedness in action, let me now pretend that I am a topologist or differential geometer rather than algebraic geometer. Let $k=\mathbb{C}$, and fix a maximal compact subgroup K of $G(\mathbb{C})$.[7] Because I still sort of want to be an algebraic geometer, write

$$\Omega(K) := \{\text{polynomial maps } f : S^1 \subset \mathbb{C}^\times \longrightarrow K \text{ such that } f(1)=1\}.$$

Then we can consider $\Omega(K) \hookrightarrow G(\mathbb{C}(\!(z)\!))$, since $K\subseteq G(\mathbb{C})$ and the maps involved are algebraic.[8] Of course, $G(\mathbb{C}(\!(z)\!))$ in turn maps to $\mathrm{Gr}_G(\mathbb{C})$. We have the following link to this classical geometric setup.

3.10 Theorem. *The composed map $\Omega(K)\longrightarrow \mathrm{Gr}_G(\mathbb{C})$ is an isomorphism of ind-complex manifolds.*

- The fact that we constrain the target K in $\Omega(K)$ (where K has half the real dimension of $G(\mathbb{C})$) corresponds to the fact that we take a quotient of $G(\mathbb{C}(\!(z)\!))$ when forming $\mathrm{Gr}_G(\mathbb{C})$.
- For GL_n, Theorem 3.10 is proven precisely using Gram–Schmidt orthonormalization.
- Topologists would actually care about all based C^∞ loops to K, but $\Omega(K)$ is dense in this space of smooth loops anyways, which in turn is dense in the space of all continuous loops. Furthermore, these two spaces of loops are homotopic anyways, so they're the same when it comes to homology and homotopy groups.

In this topological setting, we have a covering homomorphism

$$p : K^{\mathrm{sc}} \longrightarrow K^{\mathrm{ad}},$$

where K^{sc} and K^{ad} denote the simply connected and adjoint forms of K, respectively. The adjoint form K^{ad} has trivial center, and we have $\ker p = \pi_1(K^{\mathrm{ad}},1) = Z(K^{\mathrm{sc}})$. The fact that the loop group functor shifts homotopy up by a degree yields

$$\pi_0(\Omega(K^{\mathrm{sc}}),1) = \pi_1(K^{\mathrm{sc}},1) = 1.$$

In light of Theorem 3.10, this says that $\mathrm{Gr}_{G^{\mathrm{sc}}}(\mathbb{C})$ is connected. On the other hand, we similarly get

$$\pi_0(\Omega(K^{\mathrm{ad}}),1) = \pi_1(K^{\mathrm{ad}},1) = Z(K^{\mathrm{sc}}),$$

so Theorem 3.10 now tells us that $\mathrm{Gr}_{G^{\mathrm{ad}}}(\mathbb{C})$ consists of connected components labeled by $Z(K^{\mathrm{sc}})$. One can additionally show that the base component can be identified with $\mathrm{Gr}_{G^{\mathrm{sc}}}(\mathbb{C})$.

Return to our algebraic setup. Next time, we shall prove that the Gr^λ cover the entirety of the affine Grassmannian. These Gr^λ will end up forming a stratification of Gr_G, so the Gr^λ of smallest dimension must be closed. Said pieces correspond to λ that are *minuscule*, and this can serve as a definition of minusculity. Under the geometric Satake correspondence, they shall end up being matched with the highest weight representation of weight λ of $G^\vee$. I'll explain this more next time.

[7] Not to be confused with our loop field $K=k(\!(z)\!)$.

[8] People also write $G(\!(z)\!)$ and $G[\![z]\!]$ for the loop and positive loop group functors, respectively.

4 April 5, 2018

Today I'll discuss various aspects of affine Grassmannians for more general groups. My first comments will be on the scheme structure of Gr_G. My entire discussion will be conducted over $k = \mathbb{C}$ (with a brief interlude on number-theoretic settings of k in the middle).

Let G be a linear algebraic group, and consider the functor

$$\begin{aligned} L^+G : (k\text{-Alg}) &\longrightarrow (\text{Grp}) \\ R &\mapsto G(R[\![z]\!]). \end{aligned}$$

We call L^+G the *positive loop group of* G.

4.1 Proposition. *The positive loop group L^+G is representable by an affine scheme (though usually of infinite type) over k, which we denote by $G(\mathcal{O})$.*

Proof. One can see this by realizing LG as the inverse limit of the n-th jet groups of G. □

Define yet another functor

$$\begin{aligned} LG : (k\text{-Alg}) &\longrightarrow (\text{Grp}) \\ R &\mapsto G(R(\!(z)\!)), \end{aligned}$$

which we call the *loop group of* G. The following result is a bit more complicated.

4.2 Proposition. *The loop group LG is representable by an ind-scheme over k, which we denote by $G(K)$.*

Now that we have schemified $G(\mathcal{O})$ and $G(K)$, let us form their quotient.

4.3 Definition. The *affine Grassmannian* of G defined to be the quotient

$$\mathrm{Gr}_G \coloneqq G(K)/G(\mathcal{O}),$$

considered as a sheaf of sets. It is an ind-scheme.

4.4 Remark.

1) These objects are generally highly non-reduced. For instance, this occurs even for $\mathbb{G}_m(K)$.

2) The quotient morphism $G(K) \longrightarrow G(K)/G(\mathcal{O})$ is a $G(\mathcal{O})$-torsor, though we need to specify the topology in which we take our local trivializations. We choose the finest topology (and hence weakest trivialization condition) in the business, which is the fpqc topology.

When $G = \mathrm{GL}_n$, we want to give a lattice-theoretic description of the functor Gr_G in a similar vein as our initial discussion for k-points. For this, we shall need a notion of lattices for more general rings.

4.5 Definition. Let R be a ring. A *lattice (in $R(\!(z)\!)^n$)* is a finitely generated projective $R[\![z]\!]$-submodule L of $R(\!(z)\!)^n$ such that $R(\!(z)\!) \otimes_{R[\![z]\!]} L = R(\!(z)\!)^n$.

Write L_0 for the standard lattice $R[\![z]\!]^n$. As before, for any lattice L, we have $z^m L_0 \subseteq L \subseteq z^{-m} L_0$ for sufficiently large m. In this setting, we need some more ingredients (which were automatic in our previous case of $R = k$) to establish basic results, like connections with the usual Grassmannian. For example, the following proposition requires a half-page argument using short exact sequences.

4.6 Proposition. *The quotient module $z^{-m} L_0 / L$ is a projective R-module.*

Next, I shall give a brief overview of the mixed-characteristic case of for affine Grassmannians. Now take k to be the finite field $\mathbb{F}_p$, let $\mathcal{O}$ be $\mathbb{Z}_p$, and write K for $\mathbb{Q}_p$. Let's focus on the case of GL_n, which already presents much difficulty. We would like to define an ind-scheme Gr over k such that

$$\mathrm{Gr}(k) = \left\{\mathbb{Z}_p\text{-lattices in } \mathbb{Q}_p^n\right\}.$$

To accomplish this, let us recall how we pass from k to $\mathcal{O}$ and K. The way of doing this is with Witt vectors:

(1) We have $\mathbb{Z}_p = W(k)$ and $\mathbb{Q}_p = W(k)[\frac{1}{p}]$, so for any k-algebra R, we want our functor of points to be given by

$$\mathrm{Gr}(R) \coloneqq \left\{W(R)\text{-lattices in } W(R)[\tfrac{1}{p}]^n\right\}.$$

However, for non-reduced R, the Witt vectors sometimes are not an integral domain, and sometimes p can even be a zerodivisor! So this is not a meaningful formulation for general R, which is terrible.

(2) What's the fix? Well, an examination of the construction of Witt vectors reveals that we should only be interested in perfect algebras R:

4.7 Definition. Let R be a k-algebra. We say that R is *perfect* if the Frobenius endomorphism $\mathrm{Fr} : R \longrightarrow R$ given by $r \mapsto r^p$ is a bijection.

Why should we only care about perfect algebras? Well, because of the following fact.

4.8 Lemma. *Let R be a perfect k-algebra. Then p is not a zerodivisor in $W(R)$.*

Thus the above formulation of $\mathrm{Gr}(R)$ is pretty meaningful if we restrict to the category of perfect k-algebras. Similarly, we can also define a functor Gr_i (which consists of lattices bounded by $p^i W(R)^n$ and $p^{-i}W(R)^n$), and as before we have

$$\mathrm{Gr} = \varinjlim_i \mathrm{Gr}_i\,.$$

The following theorem is true, albeit very hard in this Witt setting.

4.9 Proposition. *The functor Gr is representable by an ind-scheme $\varinjlim_i X_i$, where the X_i are perfect schemes over k that correspond to the Gr_i, and the transition maps $X_i \longrightarrow X_{i+1}$ are closed embeddings.*

(3) Now these X_i are very much of infinite type in general, because perfect schemes with indeterminates require the presence of arbitrary p-th power roots of said indeterminates. However, the following recent result of Bhatt–Scholze indicates that this is the only problem:

4.10 Theorem (Bhatt–Scholze)**.** *The X_i are perfections of projective varieties over k.*

A few years prior, Zhu had proved an intermediate weaker result, which said that the X_i were at least perfections of algebraic spaces over k.

This concludes our brief overview of the considerations involved in the number-theoretic case. Return to our world of $k = \mathbb{C}$, and let us first study the case when $G = T = \mathbb{G}_m^n$ is a torus. The character and cocharacter lattices

$$X_\bullet \coloneqq X_\bullet(T) \coloneqq \mathrm{Hom}(\mathbb{G}_m, T) \text{ and } X^\bullet \coloneqq X^\bullet \coloneqq \mathrm{Hom}(T, \mathbb{G}_m)$$

have a perfect pairing $X^\bullet \times X_\bullet \longrightarrow \mathbb{Z}$ given by sending $(\chi, \gamma) \mapsto (\chi \circ \gamma)$, because we can naturally identify $\operatorname{Hom}(\mathbb{G}_m, \mathbb{G}_m)$ with $\mathbb{Z}$. We also have a canonical identification

$$\begin{aligned} \mathbb{G}_m \otimes_{\underline{\mathbb{Z}}} \underline{X_\bullet} &\xrightarrow{\sim} T \\ (x, \lambda) &\longmapsto \lambda(x) \end{aligned}$$

as group schemes, which inspires us to make the following definition.

4.11 Definition. The *dual torus* $T^\vee$ is $T^\vee := \mathbb{G}_m \otimes_{\underline{\mathbb{Z}}} \underline{X^\bullet}$.

This duality is a baby case of the Langlands business. Our $T^\vee$ switches around characters and cocharacters as follows:

$$X_\bullet(T^\vee) = X^\bullet \text{ and } X^\bullet(T^\vee) = X_\bullet.$$

Note that we can identify $X_\bullet$ with $\pi_1(T(\mathbb{C}), 1)$, because loops in $T(\mathbb{C})$ can readily be given by the polynomial functions that comprise $X_\bullet$ (restricted to S^1). Therefore

$$T^\vee(\mathbb{C}) = \mathbb{G}_m(\mathbb{C}) \otimes_{\mathbb{Z}} X^\bullet = \mathbb{G}_m(\mathbb{C}) \otimes_{\mathbb{Z}} \operatorname{Hom}(X_\bullet, \mathbb{Z}) = \operatorname{Hom}(\pi_1(T(\mathbb{C}), 1), \mathbb{G}_m(\mathbb{C})).$$

In other words, we can canonically view $T^\vee$ as parameterizing isomorphism classes of (1-dimensional) local systems on T. This is just a typical day in the Langlands correspondence—we switch between a group and its dual, all the while moving to local systems.

From now on, we ignore issues of non-reducedness and only work at the level of varieties. Recall that we have a canonical identification $K^\times/\mathcal{O}^\times = \mathbb{Z}$ given by taking valuations, so we see that

$$X_\bullet \xrightarrow{\sim} T(\!(z)\!)/T[\![z]\!] = \mathrm{Gr}_T$$

via sending $\lambda \mapsto \lambda(z)$.

Now that we have examined the case of a torus, let us move to $G = \mathrm{SL}_n$. Because SL_n lies in GL_n, we also expect that $\mathrm{Gr}_{\mathrm{SL}_n} \hookrightarrow \mathrm{Gr}_{\mathrm{GL}_n}$. To describe $\mathrm{Gr}_{\mathrm{SL}_n}$ in terms of $\mathrm{Gr}_{\mathrm{GL}_n}$, we shall need the notion of *relative dimension* for lattices. For any pair of lattices L and L' in $k(\!(z)\!)^n$, we see that L and L' contain $z^m L_0$ for sufficiently large m. Therefore $\dim_k L/(L \cap L')$ is finite.

4.12 Definition. The *relative dimension of* (L, L') is

$$\dim(L, L') := \dim_k L/(L \cap L') - \dim_k L'/(L \cap L').$$

This minus sign ensures that relative dimension is independent of the choice of m.

4.13 Proposition. *We have* $\mathrm{Gr}_{\mathrm{SL}_n}(k) = \{L \in \mathrm{Gr}_{\mathrm{GL}_n}(k) \mid \dim(L, L_0) = 0\}$.

Note that $\dim(-, L_0)$ varies continuously. Therefore Proposition 4.13 implies that $\mathrm{Gr}_{\mathrm{SL}_n}(k)$ contains the pointed component of $\mathrm{Gr}_{\mathrm{GL}_n}(k)$. Combining this with the consequences of Theorem 3.10 mentioned last time, we see that $\mathrm{Gr}_{\mathrm{SL}_n}(k)$ is precisely the pointed component of $\mathrm{Gr}_{\mathrm{GL}_n}(k)$.

We should also note that we have a nice line bundle $\det$ on $\mathrm{Gr}_{\mathrm{SL}_n}(k)$, which is roughly given as follows. The fiber of $\det$ over a lattice L is given by the line

$$\det_k(L/L \cap L_0) \otimes \left(\det_k(L_0/(L \cap L_0))\right)^*,$$

which analogizes the difference definition used for $\dim(L, L')$. The additivity of $\det$ also indicates that this is independent of m.

4.14 Proposition. *The line bundle* $\det$ *is ample, and it induces an (ind-)projective embedding of* $\mathrm{Gr}_{\mathrm{SL}_n}$.

The line bundle $\det$ is analogous to the line bundle related to the Plücker embedding of usual Grassmannians into projective space. We can construct an analog of $\det$ on Gr_G for arbitrary reductive G as well. Fix a symmetric non-degenerate invariant bilinear form $\beta : \mathfrak{g} \times \mathfrak{g} \longrightarrow k$, and let us define an extension

$$0 \longrightarrow k \longrightarrow \widehat{\mathfrak{g}} \longrightarrow \mathfrak{g}(\!(z)\!) \longrightarrow 0$$

of Lie algebras over k as follows. We set $\widehat{\mathfrak{g}} := k \oplus \mathfrak{g}(\!(z)\!)$ as a k-vector space, and form the extension corresponding to the form $c_\beta : \mathfrak{g}(\!(z)\!) \times \mathfrak{g}(\!(z)\!) \longrightarrow k$ given by $(x, y) \mapsto \mathrm{Res}_{z=0}\, \beta(x, y)$. Now this form is trivial when restricted to $\mathfrak{g}[\![z]\!]$, so we see that there is a splitting $\mathfrak{g}[\![z]\!] \longrightarrow \widehat{\mathfrak{g}}$. On the group-theoretic side, this corresponds to a diagram

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{G}_m \arrow[r] & \widehat{G} \arrow[r] & G(\!(z)\!) \arrow[r] & 1 \\ & & & G[\![z]\!] \arrow[u, hook] \arrow[ul, dotted] & \end{tikzcd}$$

Thus we can form the quotient $\widehat{G}/G[\![z]\!]$, and we get a $\mathbb{G}_m$-torsor

$$\widehat{G}/G[\![z]\!] \longrightarrow G(\!(z)\!)/G[\![z]\!] = \mathrm{Gr}_G .$$

This $\mathbb{G}_m$-torsor gives us the desired line bundle.

4.15 Example. For $G = \mathrm{SL}_n$, we recover the determinant bundle $\det$ by choosing $\beta(x, y) = \mathrm{tr}(xy)$.

Let's also study an adjoint group $G = \mathrm{PGL}_n$. Recall that PGL_n fits into a short exact sequence

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathrm{GL}_n \longrightarrow \mathrm{PGL}_n \longrightarrow 1,$$

so we expect $\mathrm{Gr}_{\mathrm{PGL}_n}$ to be some sort of quotient of $\mathrm{Gr}_{\mathrm{GL}_n}$. Let me tell you what this quotient is:

4.16 Proposition. *We have* $\mathrm{Gr}_{\mathrm{PGL}_n}(k) = \{L \in \mathrm{Gr}_{\mathrm{GL}_n}(k)\}/(L \sim zL)$.

Based on our discussion of consequences of Theorem 3.10, we expect $\mathrm{Gr}_{\mathrm{PGL}_n}(k)$ to consist of n copies of $\mathrm{Gr}_{\mathrm{SL}_n}(k)$. Furthermore, we can find the following subvarieties of these copies: for each integer $0 \leq m \leq n-1$, write

$$\mathrm{Gr}^m(k) := \left\{ z^{\mathbb{Z}} \widetilde{E} \in \mathrm{Gr}_{\mathrm{PGL}_n}(k) \;\middle|\; \widetilde{E} \text{ is the preimage of an } m\text{-dimensional subspace } E \text{ of } L_0/zL_0 \right\}.$$

We immediately see that $\mathrm{Gr}^m(k)$ is identified with the usual Grassmannian of m-dimensional subspaces of k^n, which is a projective variety. This $\mathrm{Gr}^m(k)$ is also a $G(\mathcal{O})$-orbit in $\mathrm{Gr}_{\mathrm{PGL}_n}(k)$, and its projectivity indicates that it is closed. It turns out that each $\mathrm{Gr}^m(k)$ lies in a different connected component of $\mathrm{Gr}_{\mathrm{PGL}_n}(k)$, and it is the unique closed $G(\mathcal{O})$-orbit in its connected component.

Let us return to the tale of $G(\mathcal{O})$-orbits we began to tell last time. Let T be a maximal torus of G, let B be a Borel subgroup of G containing T, and write N for the unipotent radical of B.

4.17 Theorem. *We can cover* $\mathrm{Gr}_G(k)$ *as follows:*

- *We have*

$$\mathrm{Gr}_G(k) = \bigcup_{\lambda \in X_\bullet(T)} G(\mathcal{O}) \cdot t^\lambda G(\mathcal{O})/G(\mathcal{O}),$$

which we call the Cartan decomposition.

- *We have a decomposition of sets*

$$\mathrm{Gr}_G(k) = \bigcup_{\lambda \in X_\bullet(T)} N(K) \cdot t^\lambda G(\mathcal{O})/G(\mathcal{O}),$$

  which we call the Iwasawa decomposition*.*

To prove Theorem 4.17, we need the following lemma.

4.18 Lemma. *The fixed point set* $\mathrm{Gr}_G(k)^{T(k)}$ *equals* $\{t^\lambda G(\mathcal{O}) \mid \lambda \in X_\bullet(T)\}$.

Next time I'll discuss how to get disjoint union versions of Theorem 4.17 by using dominant coweights, and how it relates to the Weyl action.

5 April 10, 2018

Let's recall our running notation. We still work over $k = \mathbb{C}$, with the additional rings $K = k(\!(z)\!)$ and $\mathcal{O} = k[\![z]\!]$. We use G to denote a connected reductive group, we choose a maximal torus T of G, and we write $X_\bullet = X_\bullet(T)$ for its cocharacter lattice. To any λ in $X_\bullet(T)$, we can form $t^\lambda := \lambda(z)$ in $T(k(\!(z)\!))$. Write $X_\bullet^{\mathrm{dom}}$ for the subset of dominant coweights in $X_\bullet(T)$. One has the following refinement of Theorem 4.17:

5.1 Theorem. *We can cover* $\mathrm{Gr}_G(k)$ *with disjoint subsets as follows:*

1) We have the Cartan decomposition

$$\mathrm{Gr}_G(k) = \coprod_{\lambda \in X_\bullet^{\mathrm{dom}}} G(\mathcal{O}) \cdot t^\lambda G(\mathcal{O})/G(\mathcal{O})$$

as sets.

2) We have the Iwasawa decomposition

$$\mathrm{Gr}_G(k) = \coprod_{\lambda \in X_\bullet} N(K) \cdot t^\lambda G(\mathcal{O})/G(\mathcal{O})$$

as sets.

Here are some remarks on how to prove Theorem 5.1:

- These decompositions of the affine Grassmannian are similar to the use of the Bruhat decomposition to study flag varieties. Their proofs also share similar ideas...
- ...However, a technical issue that arises is that, despite $\mathrm{Gr}_G = G(\!(z)\!)/G[\![z]\!]$ being like a homogeneous space, there are many senses in which it is not smooth. For instance, the stratified pieces

$$G[\![z]\!] \cdot t^\lambda G[\![z]\!]/G[\![z]\!]$$

  covering Gr_G are *not* smooth. This problems stems from the infinite-dimensionality of the objects $G[\![z]\!]$ and $G(\!(z)\!)$ that we are studying.
- To recover something like a tangent spaces and nice local behavior in this infinite-dimensional setting, we can proceed as follows. It turns out that $G(k[\frac{1}{z}]) \cdot t^\lambda G(\mathcal{O})/G(\mathcal{O})$ is an open neighborhood of $t^\lambda G(\mathcal{O})/G(\mathcal{O})$ in $\mathrm{Gr}_G(k)$. Reducing to the level of Lie algebras around this neighborhood, one can somehow use the fact that $\mathfrak{g}(\!(z)\!) = \mathfrak{g}[\![z]\!] + \mathfrak{g}[\frac{1}{z}]$ to make leeway.

- Here is an algebro-geometric way of proving 2). Start by noticing that

$$T(K) = \coprod_{\lambda \in X_\bullet} t^\lambda \cdot T(\mathcal{O}).$$

Now we know that $B = N \rtimes T$, so combining these facts tells us that the Iwasawa decomposition as given in 2) is equivalent to the statement

$$G(K) = B(K) \cdot G(\mathcal{O}).$$

In turn, this is equivalent to asking for the map

$$B(\mathcal{O})\backslash G(\mathcal{O}) \longrightarrow B(K)\backslash G(K)$$

to be surjective. We want to pass to points of $B\backslash G$, since then we can just apply the valuative criterion of properness to the projective variety $B\backslash G$. In other words, we want to have

$$B(\mathcal{O})\backslash G(\mathcal{O}) = (B\backslash G)(\mathcal{O}) \text{ and } B(K)\backslash G(K) = (B\backslash G)(K).$$

Now $G \longrightarrow G/B$ is a B-torsor, so this amounts to checking that the nonabelian étale cohomology $H^1_{\text{ét}}(\operatorname{Spec} R, B)$ is trivial for our rings R of interest. This is indeed true for $R = K$ and $R = \mathcal{O}$[9], so we can proceed as stated.

As a digression, we can use the Cartan decomposition to prove (a slightly weak version of) the *Hilbert–Mumford theorem*[10], which is a useful result in geometric invariant theory. Suppose that G acts on a projective variety X. Let x be in $X(k)$, and write Y for the G-orbit of x in X. We also write $\partial Y := \overline{Y} \smallsetminus Y$. In this setup, we have the following famous theorem in geometric invariant theory.

5.2 Theorem (Hilbert–Mumford)**.** *If the boundary ∂Y is nonempty, then there exists a 1-parameter subgroup $\gamma : \mathbb{G}_m \longrightarrow G$ and a k-point x' of Y such that $\lim_{z\to 0} \gamma(z)x'$ lies in the boundary $(\partial Y)(k)$.*

Proof. Write $f : G \longrightarrow Y$ for the action map $g \mapsto gx$. We have a k-point $\mathcal{Y}_k$ in $(\partial Y)(k)$ because it's nonempty, and we can perturb it to an $\mathcal{O}$-point $\mathcal{Y}$ in $\overline{Y}(\mathcal{O})$ such that $\mathcal{Y}_K$ lies in $Y(K)$. That is, there exists a K-point g_K of G such that $f(g_K) = \mathcal{Y}_K$. The Cartan decomposition for g_K gives us

$$g_1 t^\lambda g_2 x = \mathcal{Y}_K \implies \lim_{z\to 0} g_1(z)t^\lambda(z)g_2(z)x = \mathcal{Y}_k \implies \lim_{z\to 0} t^\lambda(z)g_2(0)x = g_1(0)^{-1}\mathcal{Y}_k \in (\partial Y)(k)$$

for some g_1 and g_2 in $G(\mathcal{O})$, since we can actually evaluate g_1 and g_2 at $z = 0$. Taking $x' = g_2(0)x$ concludes the proof. □

Let's return to the number-theoretic setting and present the classical Satake correspondence. Suppose now that $K = \mathbb{Q}_p$, write $\mathcal{O} = \mathbb{Z}_p$, and let $M := C_c(N(K)\backslash G(K)/G(\mathcal{O}))$. The (number-theoretic analog of) Iwasawa decomposition indicates that elements m of M can uniquely written as

$$m = \sum_{\lambda \in X_\bullet} m_\lambda \cdot \mathbf{1}_\lambda,$$

where $\mathbf{1}_\lambda$ is the indicator function of $N(K)t^\lambda G(\mathcal{O})$, and the m_λ are complex numbers that are zero for cofinitely many λ. We also have a $\mathbb{C}$-algebra

$$R := C_c(B(K)/T(\mathcal{O})N(K)),$$

[9] For K, this follows from breaking B up into copies of $\mathbb{G}_m$ and $\mathbb{G}_a$, where the $\mathbb{G}_m$ case is given by Hilbert's theorem 90 and the $\mathbb{G}_a$ case is given by the normal basis theorem. For $\mathcal{O}$, this follows from splitness over $\mathcal{O}$ and the K-case.

[10] In the full Hilbert–Mumford theorem, we may take $x' = x$.

which we can identify with $\mathbb{C}[X_\bullet]$ via the Iwasawa decomposition. We see that R acts on M from the left via convolution. Similarly, we see that the $\mathbb{C}$-algebra

$$H \coloneqq C_c(G(\mathcal{O})\backslash G(K)/G(\mathcal{O}))$$

acts on M from the right via convolution. Now M is a rank 1 free left R-module generated by $\mathbf{1}_0$, which allows us to define the following map.

5.3 Definition. The *Satake map* is the map $S : H \longrightarrow R$ defined by the property $\mathbf{1}_0 \cdot h = S(h) \cdot \mathbf{1}_0$ for all h in H.

Taking $S(h)S(h')\mathbf{1}_0 = S(h)\mathbf{1}_0 h' = \mathbf{1}_0 hh'$ shows that S is an algebra morphism. The classical Satake correspondence describes the image and kernel of S. Write W for the Weyl group of G.

5.4 Theorem (Satake)**.** *The Satake map induces an isomorphism $S : H \xrightarrow{\sim} R^W$, where R^W denotes the Weyl group invariants in $R = \mathbb{C}[X_\bullet]$.*

This result resembles the Harish-Chandra isomorphism, and I'm sure Satake was thinking about said isomorphism when he did this work.

The Cartan decomposition implies that H has a $\mathbb{C}$-basis given by elements of the form

$$h^\lambda \coloneqq \mathbf{1}_{G(\mathcal{O})t^\lambda G(\mathcal{O})},$$

where $\mathbf{1}$ denotes indicator functions once again, and λ ranges over $X_\bullet^{\mathrm{dom}}$. Our first instinct then is to ask where S sends the h^λ. The answer is given in the following complicated formula, due to Macdonald. It's a truly interesting formula that shows up in other places, and the unity of its presence is explained by the geometric Satake correspondence.

As we can carry out the Satake isomorphism over any finite extension of $\mathbb{Q}_p$, we use the variable q in our expressions, which denotes the size of the residue field. So in our current setup, $q = p$. For any element w of W, we write $\ell(w)$ for its length. Fix a Borel subgroup B of G containing T, which determines a set of positive roots.

5.5 Theorem (Macdonald)**.** *For any dominant cocharacter λ, we have the following formula:*

$$S(h^\lambda) = \frac{q^{\langle\rho,\lambda\rangle}}{W_\lambda(q^{-1})} \sum_{w\in W} \left(\prod_{\alpha>0} \frac{1 - q^{-1}\mathbf{1}_{-w(\alpha^\vee)}}{1 - \mathbf{1}_{w(\alpha^\vee)}} \right) \mathbf{1}_{w(\lambda)} \in \mathbb{C}[X_\bullet],$$

where W_λ denotes the stabilizer of λ in W, and $W_\lambda(q^{-1})$ denotes the polynomial $\sum_{w\in W_\lambda} q^{-\ell(w)}$ in q^{-1}.

- For $G = \mathrm{GL}_n$, the expression in the product over positive α is known as the *Hall polynomial*. It was discovered by Hall in the 40s, and it arises in the finite-dimensional representation theory of $\mathrm{GL}_n(\mathbb{F}_q)$ as well. This should not be surprising, as Langlands tells us that the Hecke algebras of GL_n should contain the representation theory of $(\mathrm{GL}_n)^\vee$ (and here this is just GL_n once again). Indeed, one can reprove this connection with $\mathrm{GL}_n(\mathbb{F}_q)$ using geometric Satake, but this is ill-advised.

- We would like we would like to mimic the setup of classical Satake when trying to prove and develop geometric Satake. Classical Satake forms a useful basis, since it provides inspiration on how to tackle our infinite-dimensional affine Grassmannians.

Next time, I will try to hint on geometric Satake and explain its relation to the Langlands program.

6 April 12, 2018

Today will essentially a digression on why our discussion matters to representation theory. We'll explain the global Langlands conjecture for GL_n (at least in the everywhere unramified case), which has been proven over function fields by Drinfeld and Lafforgue. This discussion shall illustrate the fact that the Satake correspondence already gives you half of this result, in some sense.

In this lecture, take $k = \mathbb{F}_q$ to be a finite field, and let G be a split reductive group over $\mathbb{Z}$. Then $G(K)$ is a locally compact totally disconnected group, and $G(\mathcal{O})$ is a maximal compact subgroup of $G(K)$.

6.1 Definition. A *spherical representation of* $G(K)$ is a homomorphism $\rho : G(K) \longrightarrow \mathrm{GL}(M)$, where M is a vector space over $\mathbb{C}$, that satisfies the following properties:

1) *Smoothness*: for all m in M, its stabilizer $\mathrm{Stab}_{G(K)}(m)$ contains an open subgroup of $G(K)$,

2) *Sphericity*: the subspace $M^{G(\mathcal{O})}$ is nonzero,

3) M is irreducible.

We often denote M by M_ρ.

In the world of smooth representations of $G(K)$, the $\mathbb{C}$-algebra

$$C_c^\infty(G(K)) \coloneqq \left\{ \begin{array}{c} \text{convolution algebra of locally constant} \\ \text{functions } f : G(K) \longrightarrow \mathbb{C} \text{ with compact support} \end{array} \right\}.$$

is of utmost importance. This algebra acts on any smooth representation of $G(K)$, so in particular it acts on any spherical representation M. Furthermore, under this action, the subalgebra $H \coloneqq C_c(G(\mathcal{O})\backslash G(K)/G(\mathcal{O}))$ acts on $M^{G(\mathcal{O})}$, as we have $H \cdot M^{G(\mathcal{O})} = M^{G(\mathcal{O})}$.

We have the following basic result:

6.2 Proposition. *For all spherical representations M of $G(K)$, the dimension of $M^{G(\mathcal{O})}$ equals* 1.

Proof. Note that $M^{G(\mathcal{O})}$ is a module over H. Therefore it suffices to show that $M^{G(\mathcal{O})}$ is simple, because Theorem 2.9 indicates that H is commutative. If W is a proper submodule of $M^{G(\mathcal{O})}$, then I claim that $(C_c^\infty(G(K)) \cdot W) \cap V^{G(\mathcal{O})} = W$. This would yield a proper subrepresentation $(C_c^\infty(G(K)) \cdot W$ of M, which is impossible since M is irreducible.

As for $(C_c^\infty(G(K)) \cdot W) \cap V^{G(\mathcal{O})} = W$, one inclusion is immediate. Conversely, suppose we have

$$v = \sum_{i=1}^{r} \varphi_i \cdot w_i \in V^{G(\mathcal{O})}$$

for some φ_i in $C_c^\infty(G(K))$ and w_i in W. Writing $e \coloneqq \mathbf{1}_{G(\mathcal{O})}$, we see that $ew_i = w_i$ and $ev = v$. Therefore

$$v = \sum_{i=1}^{r} (e * \varphi_i * e) \cdot w_i \in W,$$

as desired, which shows that $(C_c^\infty(G(K)) \cdot W) \cap V^{G(\mathcal{O})}$ is contained in W. $\square$

6.3 Corollary. *For any spherical representation V of $G(K)$, there exists a unique character $\chi_V : H \longrightarrow \mathbb{C}$ such that $h \cdot v = \chi_V(h)v$ for all h in H and v in V.*

The following lemma shall also be useful, and it follows immediately from the finitude of $\mathbb{P}^n(\mathbb{F}_q)$.

6.4 Lemma. *Let X be a projective variety over $\mathbb{F}_q$, Then $X(\mathbb{F}_q)$ is finite.*

It turns out that the geometric structure on Gr_G (which we defined in Lecture 4) coincides with the direct limit

$$\varinjlim_m \mathrm{Gr}_m$$

when the Gr_m are considered as schemes, where the Gr_m are as given in Lecture 3. Now Lemma 6.4 ensures that the $\mathrm{Gr}_m(\mathbb{F}_q)$ are finite sets, and because any $G(\mathcal{O})$-orbit in $\mathrm{Gr}_G(\mathbb{F}_q)$ lies in some $\mathrm{Gr}_m(\mathbb{F}_q)$, we see that these $G(\mathcal{O})$-orbits are finite as well.

Recall the construction of dual tori from Lecture 4. In this context, we use the notation

$$\begin{aligned} X_\bullet(T) &\xrightarrow{\sim} X^\bullet(T^\vee) \\ \lambda &\longmapsto e^\lambda, \end{aligned}$$

which is both traditional as well as suggestive. Combining this with the Iwasawa decomposition allows us to identify

$$R = \mathbb{C}[X_\bullet(T)] = \mathbb{C}[X^\bullet(T^\vee)] = \mathbb{C}[T^\vee].$$

In this light, the Satake isomorphism is an isomorphism

$$S : H \xrightarrow{\sim} R^W = \mathbb{C}[T^\vee]^W = \mathbb{C}[T^\vee/W].$$

Interpreting this isomorphism geometrically, Corollary 6.3 indicates that, for any spherical representation M, we obtain a unique $\mathbb{C}$-point t_M in $(T^\vee/W)(\mathbb{C})$ satisfying $h \cdot m = S(h)(t_M)$.

Let's now consider the global (unramified everywhere) Langlands conjectures for $G = \mathrm{GL}_n$ over function fields, since treating the case of a general group G introduces many complications. For GL_n, the Weyl group is $W = \mathfrak{S}_n$, and $\mathbb{C}[T^\vee/W] = \mathbb{C}[z_1^\pm, \dots, z_n^\pm]^{\mathfrak{S}_n}$ is the ring of symmetric Laurent polynomials in n variables. Write

$$p \coloneqq \prod_{i=1}^n (s - z_i) = \sum_{i=0}^n (-1)^i \sigma_i(z_1, \dots, z_n) s^{n-i} \in \mathbb{C}[T^\vee/W][s]$$

for the polynomial in s with coefficients in $\mathbb{C}[z_1^\pm, \dots, z_n^\pm]^{\mathfrak{S}_n}$, where $\sigma_i(z_1, \dots, z_n)$ is the i-th elementary symmetric polynomial.

6.5 Definition. Let M be a spherical representation of $G(K)$. The *L-function of M* is the rational function $L(M, s)$ in $\mathbb{C}(s)$ obtained from $1/p$ by plugging in t_M for $(z_1, \dots, z_n)$.

Anyways, we are in a global geometric situation, so let X be a smooth projective geometrically connected curve over $\mathbb{F}_q$. Write F for the function field $\mathbb{F}_q(X)$ of X, and write X_0 for the set of closed points of X.

6.6 Example. When $X = \mathbb{P}^1_{\mathbb{F}_q}$, our set X_0 equals

$$X_0 = \{\text{irreducible monic polynomials in } \mathbb{F}_q[s]\} \cup \{\infty\}.$$

For any x in X_0, we can form a local situation $K_x \supset \mathcal{O}_x \supset \mathfrak{m}_x$ by taking $\mathcal{O}_x$ to be the completion of the stalk $\mathscr{O}_{X,x}$, $\mathfrak{m}_x$ to be its maximal ideal, and K_x to be its fraction field. The residue field $k_x \coloneqq \mathcal{O}/\mathfrak{m}_x$ is a finite extension of k, and we write $\deg x$ for its degree over k.

To state the Langlands conjecture, we shall need to introduce the adeles. Write

$$\mathbb{A} := \left\{ (a_x)_x \in \prod_{x \in X_0} K_x \,\middle|\, a_x \in \mathcal{O}_x \text{ for cofinitely many } x \right\} \text{ and } \mathbb{O} := \prod_{x \in X_0} \mathcal{O}_x$$

for the rings of adeles and integral adeles, respectively. The diagonal map $f \mapsto (f)_x$ sends F to $\mathbb{A}$, since functions only have poles at finitely many points. For general split reductive groups G over $\mathbb{Z}$, we have

$$G(F) \subseteq G(\mathbb{A}) \supseteq G(\mathbb{O}) = \prod_{x \in X_0} G(\mathcal{O}_x).$$

When $G = \mathrm{GL}_n$, the group G has a 1-dimensional nontrivial center of diagonal matrices, which we deal with as follows. Fix an element $a = (a_x)_x$ of $\mathbb{A}^\times$ satisfying

$$\deg(a) := \sum_{x \in X_0} \deg(x) \cdot v_x(a_x) = 1.$$

If we convert a into an element of $\mathbb{O}^\times \backslash \mathbb{A}^\times / F^\times = \mathrm{Pic}(X)$ via Proposition 1.2, we see that it corresponds to a line bundle with nontrivial degree. Next, form the double quotient space

$$X_a := a^{\mathbb{Z}} \backslash G(\mathbb{A}) / G(F),$$

where we view a as a diagonal matrix in $G = \mathrm{GL}_n$. Because a lies in the center of G, we see that $G(\mathbb{A})$ acts on X_a via left translation. At this point, I will now state a bunch of motivating theorems and definitions which play no role in the future.

6.7 Definition.

1) An *(unramified) automorphic function* is a function $f : X_a \longrightarrow \mathbb{C}$ which is $G(\mathbb{O})$-invariant.

2) Such an f is called *cuspidal* if, for any proper parabolic subgroup P with Levi decomposition $L \ltimes U$ of G, the integral

$$\int_{U(\mathbb{A})/U(F)} \mathrm{d}u \, f(gu) = 0$$

for all g in $G(\mathbb{A})$.

Cuspidal functions satisfy the following miraculous statement.

6.8 Proposition. *For any of our general G, any cuspidal f has compact support in X_a.*

Write $C_{\mathrm{cusp}}(X_a)$ for the space of cuspidal functions. We see that $G(\mathbb{A})$ acts on $C_{\mathrm{cusp}}(X_a)$ and $C_c^\infty(X_a)$ by sending $f : X_a \longrightarrow \mathbb{C}$ to the function $g \cdot f$ given by

$$(g \cdot f)(x) := f(g^{-1}x)$$

for all g in $G(\mathbb{A})$ and x in X_a. Then Proposition 6.8 indicates that we have an inclusion of representations

$$C_c^\infty(X_a) \supseteq C_{\mathrm{cusp}}(X_a).$$

Note that $C_c^\infty(X_a)$ is a pre-Hilbert space, where the inner product is given by

$$\int_{X_a} \mathrm{d}x \, f_1(x)\overline{f_2(x)}.$$

Here, we get $\mathrm{d}x$ via the quotient measure on $X_a = a^{\mathbb{Z}} \backslash G(\mathbb{A}) / G(F)$ arising from the Haar measure on $G(\mathbb{A})$ and the discrete measures on $a^{\mathbb{Z}}$ and $G(F)$.

6.9 Theorem. *The space of cuspidal functions decomposes as*

$$C_{\mathrm{cusp}}(X_a) = \bigoplus_{\rho} M_\rho,$$

where the M_ρ are irreducible representations of $G(\mathbb{A})$. We say that such an M_ρ is an (unramified) cuspidal representation *of $G(\mathbb{A})$.*

We have presented much of our notation and results as if they were for general groups G, because it indeed works more generally. However, the following truly difficult result of Piatetsky-Shapiro (who was a single person) is for $G = \mathrm{GL}_n$ (and is not true in general!), and it's the whole reason why the GL_n case is easier.

6.10 Theorem (Multiplicity one for GL_n)**.** *In the decomposition of Theorem 6.9, each irreducible representation of $G(\mathbb{A})$ occurs at most once.*

To connect our unramified global situation to our spherical local situation, we have the following decomposition theorem of Flath.

6.11 Theorem (Flath)**.** *For any M_ρ as appearing in Theorem 6.9, we have*

$$M_\rho = {\bigotimes_{x\in X_0}}' M_{\rho,x},$$

for some spherical representations $M_{\rho,x}$ of $G(K_x)$.

Recall that the restricted tensor product is constructed as follows: for every x in X_0, choose a nonzero m_x in $M_{\rho,x}^{G(\mathcal{O}_x)}$. Then the restricted tensor product is defined to be

$${\bigotimes_{x\in X_0}}' M_{\rho,x} \coloneqq \varinjlim_{S} \left(\bigotimes_{s\in X_0\smallsetminus S} m_x \otimes \bigotimes_{s\in S} M_{\rho,x} \right),$$

where S ranges over all finite subsets of X_0.

In conclusion, for any cuspidal representation $\rho : \mathrm{GL}_n(\mathbb{A}) \longrightarrow \mathrm{GL}(M_\rho)$, Flath's theorem yields a bunch of spherical representations $M_{\rho,x}$ of $\mathrm{GL}_n(K_x)$, where x ranges over X_0. These $M_{\rho,x}$ are equipped with L-functions $L(M_{\rho,x}, s)$. By taking the infinite product of these $L(M_{\rho,x}, s)$, one obtains a *global L-function*, and number theory is interested in the analytic properties of these resulting functions. But we won't discuss the topic further.

The Langlands conjectures say that the entirety of our above discussion has analogs for Galois representations. Let $\overline{K}_x^{\mathrm{sep}}$ be a separable closure of K_x, and write Γ_x for $\mathrm{Gal}(\overline{K}_x^{\mathrm{sep}}/K_x)$. We have a short exact sequence

$$1 \longrightarrow I_x \longrightarrow \Gamma_x \longrightarrow \widehat{\mathbb{Z}} \longrightarrow 1,$$

where I_x denotes the inertia subgroup of Γ_x, and we identified $\mathrm{Gal}(\overline{k}_x^{\mathrm{sep}}/k_x)$ with $\widehat{\mathbb{Z}}$. The $\widehat{\mathbb{Z}}$ term contains $\mathrm{Fr}_x^{\mathbb{Z}}$ inside, where $\mathrm{Fr}_x : \overline{k}_x^{\mathrm{sep}} \longrightarrow \overline{k}_x^{\mathrm{sep}}$ is the Frobenius automorphism acting via $x \mapsto x^{1/\#k_x}$.

In the global setting, we have a Galois group $\Gamma_F \coloneqq \mathrm{Gal}(\overline{F}^{\mathrm{sep}}/F)$. For each x in X_0, this group contains an associated decomposition subgroup (which is well-defined up to conjugacy), and this group is isomorphic to Γ_x. Fix a prime number ℓ not equal to $\mathrm{char}\, k$.

6.12 Definition. An *(everywhere unramified) Galois representation* is a continuous homomorphism $\sigma : \Gamma_F \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ such that

1) There exists a finite extension E of $\mathbb{Q}_\ell$ such that $\sigma(\Gamma_F)$ lands in $\mathrm{GL}_n(E)$,[11]

2) For all x in X_0, the restriction of σ to I_x is trivial.

For any Galois representation $\sigma : \Gamma_F \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$, it makes sense to evaluate $\sigma(\mathrm{Fr}_x)$ because I_x acts trivially. This allows us to construct the following analog of L-functions.

6.13 Definition. Let $\sigma : \Gamma_F \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ be a Galois representation, and let x be in X_0. The *local L-function at x* is

$$L_x(\sigma, s) \coloneqq \det(s\,\mathrm{id} - \sigma(\mathrm{Fr}_x))^{-1} \in \overline{\mathbb{Q}}_\ell(s).$$

At this point, fix a field isomorphism $\mathbb{C} \approx \overline{\mathbb{Q}}_\ell$, which we use to identify these two fields. With all these pieces in our hands, we can finally state the (unramified) Langlands conjecture for GL_n. This was proven by Drinfeld for $n = 2$ and by Laurent Lafforgue for all n. The $n = 1$ case is classical.

6.14 Theorem (Unramified Langlands correspondence for GL_n)**.** *There exists a unique bijection*

$$\left\{ \begin{array}{c} \textit{isomorphism classes of cuspidal} \\ \textit{representations of } \mathrm{GL}_n(\mathbb{A}) \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \textit{isomorphism classes of irreducible Galois} \\ \textit{representations } \Gamma_F \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell) \end{array} \right\}$$
$$\rho \longmapsto \sigma$$

such that, for all x in X_0, we have an equality

$$L(M_{\rho,x}, s) = L_x(\sigma, s).$$

I had some other things to say, but I don't have time for that anymore. A crucial step in proofs of the Langland correspondence is to reformulate the Galois side in a geometric manner, which is important for geometric Langlands as well. Let's start talking about that today. This reformulation is merely a linguistic change, so it has no actual content in it, but it does have *intuitive* content.

Geometrically, everywhere unramified continuous representations $\sigma : \Gamma_F \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ correspond to rank n local systems $\mathscr{E}_\sigma$ on X. For every x in X_0, the image $\sigma(\mathrm{Fr}_x)$ of Frobenius acts on the stalk $\mathscr{E}_\sigma|_x$ of $\mathscr{E}_\sigma$ at x, and in this optic we have

$$L_x(\sigma, s)^{-1} = \det(s\,\mathrm{id} - \sigma(\mathrm{Fr}_x)) = \sum_{i=0}^{n} (-1)^i \operatorname{tr}\left(\sigma(\mathrm{Fr}_x) \,\middle|\, \bigwedge^i \mathscr{E}_\sigma|_x\right) s^i,$$

where we have used the linear-algebraic fact that

$$\sigma_i(x_1, \dots, x_n) = \operatorname{tr}\left(\operatorname{diag}(x_1, \dots, x_n) \,\middle|\, \bigwedge^i K^n\right)$$

for any field K. This inspires us to geometrize the automorphic side as well: recall that we had the spherical representation $M_{\rho,x}$ yields a point $t_{M,x} = (z_{1,x}, \dots, z_{n,x})$ in $(\mathbb{C}^\times)^n/\mathfrak{S}_n$. We see that Theorem 6.14's condition on L-functions is equivalent to asking that

$$\operatorname{tr}\left(\operatorname{diag}(z_{1,x}, \dots, z_{n,x}) \,\middle|\, \bigwedge^i \mathbb{C}^n\right) = \operatorname{tr}\left(\sigma(\mathrm{Fr}_x) \,\middle|\, \bigwedge^i \mathscr{E}_\sigma|_x\right)$$

for all x in X_0.

[11] One can use the Baire category theorem to show that this condition is superfluous.

- Some conjectures of Deligne say that this correspondence doesn't depend on our choice of isomorphism between $\mathbb{C}$ and $\overline{\mathbb{Q}}_\ell$ (and that these traces are even algebraic numbers).
- Langlands was aware of this reformulation of his conjectures, but it's generally hard to see what ought to replace $\bigwedge^i$ for general groups G. After all, it's not true if you just replace $\bigwedge^i$ with arbitrary algebraic representations of GL_n.
- At some point, somebody (perhaps Langlands himself, maybe inspired by the theory of Shimura varieties) conjectured that the correct generalization was that one should use *minuscule* representations. But why are these the right answer? We'll try putting this on a conceptual standing next time.

7 April 17, 2018

The plan of today is to first explain a wishful-thinking categorification of last time's discussion (which is a beautiful dream that unfortunately seems out of reach), and then to explain the Tannakian formalism, which is the first step in actually carrying out a categorical upgrade of Satake.

How should we proceed to categorify the Langlands conjectures? Recall that our local correspondence

- Considered $G(\mathcal{O})$-invariants of spherical representations M, which were 1-dimensional,
- Viewed them as modules over H to obtained a character from $M^{G(\mathcal{O})}$,
- Used the Satake isomorphism to identify these characters with $\mathbb{C}$-points of $T^\vee/W$.

Therefore we must first categorify Hecke algebras, which we accomplish using the *Hecke category*. First, note that the $G(K)/G(\mathcal{O})$ in

$$H = C_c[G(\mathcal{O})\backslash G(K)/G(\mathcal{O})]$$

is just the set of k-points of Gr_G. Next, recall that it is Grothendieck's philosophy that interesting functions on a variety over k arise from traces of Frobenius acting on stalks at geometric points. To incorporate the left $G(\mathcal{O})$-action on $G(K)/G(\mathcal{O})$, we would like only to consider $G(\mathcal{O})$-invariant sheaves. And finally, H has an algebra structure (and Theorem 2.9 indicates that it is commutative!), so we want a *monoidal* structure that is also *symmetric*. Altogether, we want to replace H with

$$\underline{\mathrm{Sat}} = \{\text{some symmetric monoidal category of “}G(\mathcal{O})\text{-invariant sheaves” on } \mathrm{Gr}_G\},$$

roughly speaking. It turns out that, whatever $\underline{\mathrm{Sat}}$ is, it will end up being a Tannakian category, and the geometric Satake correspondence shall amount to an equivalence of Tannakian categories between $\underline{\mathrm{Sat}}$ and some sort of representation category.

In the global setting, recall that Proposition 1.2 already allows us to categorify $C(G(F)\backslash G(\mathbb{A})/G(\mathbb{O}))$ into a category of sheaves on Bun_G in the case of $G = \mathrm{GL}_n$. This is the picture for general G as well. But now here come the pipe dreams: recall that, via the local Hecke operators for all x in X_0, the local Hecke algebra

$$H_x \coloneqq C_c[G(\mathcal{O}_x)\backslash G(K_x)/G(\mathcal{O}_x)]$$

acts on automorphic functions. To categorify this, we would want $\underline{\mathrm{Sat}}$, as a symmetric monoidal category, acts on our category of sheaves on Bun_G.

7.1 Remark. As a motivational example of a module category $\mathscr{M}$ over a monoidal category $\mathscr{C}$, let G be a topological group, and let $\mathscr{C}$ be the category of finite-dimensional continuous real representations of G. Let X be a topological space with a continuous action of G, and let $\mathscr{M}$ be the category of G-equivariant vector bundles on X. Then the tensor product $\mathscr{C} \times \mathscr{M} \longrightarrow \mathscr{M}$ functor yields a module category structure on $\mathscr{M}$ over $\mathscr{C}$.

Recall the definition of $\mathscr{H}\mathrm{eck}^i$ from Lecture 1, which was defined for integers $0 \leq i \leq n$. Note that this choice of $\mathscr{H}\mathrm{eck}^i$ corresponded to the minuscule cocharacters

$$(\underbrace{1, \dots, 1}_{i \text{ times}}, 0, \dots, 0),$$

which are precisely the coweights that correspond to the wedge power representations of GL_n.

In the classical situation, a nonzero vector v in a cuspidal representation is given as a tensor product $v = \bigotimes_{x \in X_0} v_x$. Furthermore, for each x in X_0, there exists a character $\chi_x : H_x \longrightarrow \mathbb{C}$ such that v_x satisfies $h \cdot v_x = \chi_x(h) v_x$ for all h in H_x. The vector v is a function in $C(G(F)\backslash G(\mathbb{A})/G(\mathbb{O}))$, which corresponds to some sort of sheaf $\mathcal{F}$ on Bun_G. With the diagram

$$\begin{tikzcd}
& \mathscr{H}\mathrm{eck}^i \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] \arrow[rr, "p"] & & X \\
\mathrm{Bun}_G & & \mathrm{Bun}_G. &
\end{tikzcd}$$

in mind, we might guess that categorical analog of the Hecke eigenvector equation is then

$$(\mathrm{pr}_2 \times p)_* \, \mathrm{pr}_1^* \, \mathcal{F} = \mathcal{F} \boxtimes \mathcal{E}_i$$

for all i, where the $\mathcal{E}_i$ are some local systems on X. Note that this pipe dream asks that we package all the local characters χ_x into the global objects $\mathcal{E}_i$. The choice of i here corresponds to choosing the representations $\bigwedge^i \mathbb{C}^n$—how can we extend this to other representations V of $\mathrm{GL}_n(\mathbb{C})$?

We carry out this extension by refining the above guess as follows. Returning to the local situation, we begin by stating the geometric Satake correspondence (even though we do not know what all the terms involved mean).

7.2 Theorem (Geometric Satake)**.** *There is a natural equivalence of symmetric monoidal categories*

$$S : \mathrm{Rep}_{\mathbb{C}} G^\vee \xrightarrow{\sim} \underline{\mathrm{Sat}},$$

where $G^\vee$ denotes the Langlands dual[12]*, and $\mathrm{Rep}_{\mathbb{C}} G^\vee$ denotes the category of finite-dimensional continuous representations of $G^\vee(\mathbb{C})$ over $\mathbb{C}$.*[13]

7.3 Example. For $G = \mathrm{GL}_n$, the dual group $G^\vee$ is isomorphic to GL_n again. Furthermore, the category $\mathrm{Rep}_{\mathbb{C}} G^\vee$ is generated by the $\bigwedge^i \mathbb{C}^n$ (for integers $1 \leq i \leq n-1$) as a Tannakian category.

With the geometric Satake correspondence in hand, it would then seem natural to require that for all V in $\mathrm{Rep}_{\mathbb{C}} G^\vee$, there exists a local system $\mathcal{E}_V$ on X such that

$$S(V) * \mathcal{F} = \mathcal{F} \boxtimes \mathcal{E}_V$$

[12] See Definition 13.8.

[13] In other words, the category of algebraic representations of $G^\vee$.

for all of our sheaves $\mathcal{F}$ on Bun_G, where $*$ denotes a hypothetical global upgrade of our hypothetical module category structure (where we recover the local setting by taking a fiber at a point of X). Because we want this equality to be highly compatible, we also hope that

$$S(V_1 \otimes V_2) * \mathcal{F} = \mathcal{F} \boxtimes (\mathcal{E}_{V_1} \otimes \mathcal{E}_{V_2})$$

for all V_1 and V_2 in $\mathrm{Rep}_{\mathbb{C}}\, G^\vee$. But then the assignment $V \mapsto \mathcal{E}_V$ would yield a monoidal functor

$$\mathrm{Rep}_{\mathbb{C}}\, G^\vee \longrightarrow \mathrm{Rep}\, \pi_1(X),$$

where $\pi_1(X)$ is whatever fundamental group we need to use in order to identify our sheaves on X with representations of $\pi_1(X)$. Tannakian formalism indicates[14] that the above monoidal functor comes from a homomorphism $\pi_1(X) \longrightarrow G^\vee(\mathbb{C})$. But this is precisely what the Langlands conjecture is trying to do!

Sadly, this conceptual cookbook for constructing the Langlands conjecture is only a pipe dream. And even many parts of our discussion that *are* known have not yet been defined in lecture—one really needs to put bones on our recipes. We'll stop here, and next time will entirely be on monoidal categories.

8 April 19, 2018

Today, I'll talk about all sorts of monoidal categories, and in the middle I'll hopefully make rigorous some of last time's discussion. Let k be a field, and let $\mathscr{C}$ be an *essentially small* (I don't know what this means. Just kidding, it means equivalent to a small) abelian k-linear category.

8.1 Definition. A *monoidal structure on* $\mathscr{C}$ is a k-linear bifunctor $\otimes : \mathscr{C} \times \mathscr{C} \longrightarrow \mathscr{C}$ equipped with

- a natural associativity constraint $\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$,
- an object 1 equipped with an isomorphism $1 \otimes 1 \xrightarrow{\sim} 1$,

that satisfies the following axioms:

- the *unit axiom*: the functors $X \mapsto 1 \otimes X$ and $X \mapsto X \otimes 1$ are self-equivalences of categories $\mathscr{C} \longrightarrow \mathscr{C}$,
- the *pentagon axiom*: for all X, Y, Z, and W, the diagram

$$\begin{tikzcd}[column sep=tiny] & ((X \otimes Y) \otimes Z) \otimes W \arrow[dl] \arrow[dr] & \\ (X \otimes (Y \otimes Z)) \otimes W \arrow[d] & & (X \otimes Y) \otimes (Z \otimes W) \arrow[d] \\ X \otimes ((Y \otimes Z) \otimes W) \arrow[rr] & & X \otimes (Y \otimes (Z \otimes W)) \end{tikzcd}$$

  commutes, where the morphisms are the only possible ones obtained from α.

The monoidal categorical analog of uniqueness of identity is the following readily proven proposition.

8.2 Proposition. *The object* 1 *is unique up to a unique isomorphism.*

8.3 Definition. Let $(\mathscr{C}, \otimes)$ be a monoidal category. We say it is *symmetric* if there exists a commutativity constraint $s_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$ that satisfies the following axioms:

[14] See Proposition 8.12.

1) $s_{Y,X} \circ s_{X,Y} = \mathrm{id}_{X \otimes Y}$,

2) the *first hexagon axiom*: for all U, V, and W, the diagram

$$\begin{tikzcd} U \otimes (V \otimes W) \arrow[r] \arrow[d] & (V \otimes W) \otimes U \arrow[d] \\ (U \otimes V) \otimes W \arrow[d] & V \otimes (W \otimes U) \arrow[d] \\ (V \otimes U) \otimes W \arrow[r] & V \otimes (U \otimes W) \end{tikzcd}$$

commutes, where the morphisms are sometimes the commutativity maps and sometimes the associativity maps,

3) the *second hexagon axiom*: it's the same as the first hexagon axiom, except use the maps $s_{Y,X}$ instead of $s_{X,Y}$ and hence reverse all instances of commutativity arrows.

8.4 Examples.

1) Let R be a commutative k-algebra. Then the category $\mathscr{C} = (R\text{-mod})$ is symmetric monoidal, where the operation is given by tensor products and we use the usual identifications for associativity and commutativity.

2) Here is a non-example: if you are a perverse person and decide to define $s_{X,Y}$ via sending $x \otimes y \mapsto -y \otimes x$ in the above example, then you do not get a symmetric structure, since the hexagon axiom contains three instances of the commutativity map (and hence the sign you introduced doesn't cancel).

What are the morphisms of monoidal categories?

8.5 Definition. Let $\mathscr{C}$ and $\mathscr{C}'$ be monoidal categories. A *monoidal functor* is a functor $F : \mathscr{C} \longrightarrow \mathscr{C}'$ equipped with natural isomorphisms $\beta_{F,X,Y} : F(X) \otimes F(Y) \xrightarrow{\sim} F(X \otimes Y)$ such that the hexagon

$$\begin{tikzcd} (F(U) \otimes F(V)) \otimes F(W) \arrow[r] \arrow[d] & F(U) \otimes (F(V) \otimes F(W)) \arrow[d] \\ F(U \otimes V) \otimes F(W) \arrow[d] & F(U) \otimes F(V \otimes W) \arrow[d] \\ F((U \otimes V) \otimes W) \arrow[r] & F(U \otimes (V \otimes W)) \end{tikzcd}$$

commutes for all U, V, and W, where the morphisms are obtained from $\beta_{F,X,Y}$ and the associativity maps.

8.6 Remark (Mac Lane coherence). The pentagon axiom implies that any string of tensor products is well-defined, in the sense that there is a canonical isomorphism between any choice of parentheses. Furthermore, the pentagon and hexagon axioms imply that, when the monoidal category has a symmetric structure, any string of tensor products is also well-defined independently of the order of the factors.

Someone notes that this is related to the contractibility of the permutohedron, which apparently is hard to prove. In any case, Mac Lane has already done our work for us.

8.7 Definition. Let $\mathscr{C}$ be a symmetric monoidal category. We say it is *rigid* if for all X in $\mathscr{C}$, there exists an $X^\vee$ in $\mathscr{C}$ and morphisms $\mathrm{coev}_X : 1 \longrightarrow X \otimes X^\vee$ and $\mathrm{ev}_X : X^\vee \otimes X \longrightarrow 1$ such that

- the morphism $X \longrightarrow (X \otimes X^\vee) \otimes X \longrightarrow X \otimes (X^\vee \otimes X) \longrightarrow X \otimes 1 \longrightarrow X$ is the identity,
- the morphism $X^\vee \longrightarrow X^\vee \otimes (X \otimes X^\vee) \longrightarrow (X^\vee \otimes X) \otimes X^\vee \longrightarrow 1 \otimes X^\vee \longrightarrow X^\vee$ is the identity.

A *tensor category* is a rigid symmetric monoidal category.

A priori, these associated $X^\vee$ and coev_X and ev_X are just objects and morphisms, but it turns out one can upgrade this functorially as follows.

8.8 Proposition. *The assignment $X \mapsto X^\vee$ extends to a functor, and for this functor structure, the ev_X and coev_X are natural transformations.*

- It is known that 1 is a simple object in $\mathscr{C}$ if and only if $\mathrm{End}(1)$ is a field. From now on, we shall assume that $\mathrm{End}(1)$ is just the field k itself. This is just as well, since the ring $\mathrm{End}(1)$ acts $\mathbb{Z}$-linearly on every homset in $\mathscr{C}$ in a way that is compatible with composition of morphisms.
- With this new assumption, we see that
$$1 \longrightarrow X \otimes X^\vee \stackrel{f \otimes \mathrm{id}_{X^\vee}}{\longrightarrow} X \otimes X^\vee \longrightarrow 1$$
is just a multiple $\lambda_{f,X} \in k$ of the identity morphism. We define the *trace* of f to be $\mathrm{tr}(f) \coloneqq \lambda_{f,X}$ and we define the *dimension* of X to be $\dim X \coloneqq \mathrm{tr}(\mathrm{id}_X)$.

8.9 Examples.

1) Let (Vect) be the category of finite-dimensional vector spaces over k. Then it is evidently a tensor category, and the above discussion is immediately seen to be linear algebra in the usual sense.
2) The category (R-Mod) is not rigid, because dual objects $X^\vee$ do not always exist.
3) The category (R-Proj) of finitely generated projective modules over R does satisfy the rigidity conditions, but it is no longer abelian.
4) Generalizing 1), let G be an abstract group. Then the category $\mathrm{Rep}_k\, G$ of finite-dimensional representations of G over k is a tensor category. Similarly, if G is instead an algebraic group over k, then the category $\mathrm{Rep}_k\, G$ of finite-dimensional algebraic representations of G over k is also a tensor category.

In 1) and 4), we have a forgetful functor to (Vect), which we axiomatize as follows.

8.10 Definition. Let $\mathscr{C}$ and $\mathscr{D}$ be monoidal categories. A *fiber functor of $\mathscr{C}$* is an exact faithful monoidal functor $\omega : \mathscr{C} \longrightarrow \mathscr{D}$.

8.11 Example. As mentioned in Examples 8.9, for any algebraic group G over k, we have a fiber functor $\omega_G : \mathrm{Rep}_k\, G \longrightarrow$(Vect) given by the forgetful functor.

Some variant of the following proposition was mentioned last time as a method of creating morphisms of groups:

8.12 Proposition. *Let H and G be algebraic groups over k, and let $F : \mathrm{Rep}_k\, H \longrightarrow \mathrm{Rep}_k\, G$ be a monoidal functor such that $\omega_G \circ F$ is isomorphic to ω_H. Then F is induced from a unique homomorphism $f : G \longrightarrow H$ via precomposition.*

We can use the language of monoidal categories to consider module categories over monoidal categories. Let $\mathscr{M}$ be an abelian category. Then the category $\mathrm{Fun}(\mathscr{M}, \mathscr{M})$ of additive self-functors has a natural monoidal structure given by composition of functors.

8.13 Definition. A *module category over* $\mathscr{C}$ is an abelian category $\mathscr{M}$ equipped with a monoidal functor $F : \mathscr{C} \longrightarrow \mathrm{Fun}(\mathscr{M}, \mathscr{M})$. More explicitly (via currying), this is the data of a bifunctor $* : \mathscr{C} \times \mathscr{M} \longrightarrow \mathscr{M}$ given by $X * M \coloneqq F(X)(M)$, along with an associativity constraint

$$\beta_{F,X,Y,M} : (F(X) \circ F(Y))(M) \xrightarrow{\sim} F(X \otimes Y)(M)$$

that satisfies a version of the hexagon axiom as in Definition 8.5.

Let us now return to our discussion from last time. Recall that X is our smooth geometrically irreducible curve over $k = \mathbb{F}_q$, and write $\Delta : X \longrightarrow X \times X$ for the diagonal morphism. Informally, for any space $\mathfrak{X}$, we shall write $\underline{S}(\mathfrak{X})$ for some category of sheaves on $\mathfrak{X}$. Write $*$ for the monoidal structure on $\underline{\mathrm{Sat}}$. Let's make the following assumptions:

- For all h in $\underline{\mathrm{Sat}}$, suppose that we have a functor $\mathscr{H}_h : \underline{S}(\mathrm{Bun}_G) \longrightarrow S(X \times \mathrm{Bun}_G)$,
- For all h_1 and h_2 in $\underline{\mathrm{Sat}}$, there exists a natural isomorphism μ_{h_1,h_2} from the composed functor

$$\underline{S}(\mathrm{Bun}_G) \xrightarrow{\mathscr{H}_{h_1}} \underline{S}(X \times \mathrm{Bun}_G) \xrightarrow{\mathscr{H}_{h_2}} \underline{S}(X \times X \times \mathrm{Bun}_G) \xrightarrow{\Delta^*} \underline{S}(X \times \mathrm{Bun}_G)$$

  to the functor $\mathscr{H}_{h_1 * h_2} : \underline{S}(\mathrm{Bun}_G) \longrightarrow \underline{S}(X \times \mathrm{Bun}_G)$ (where we interpret the $\mathscr{H}_{h_2}$ via some sort of structure that allows us to ignore the extra factor of X, which geometrically corresponds to taking the product of our correspondence $X \times \mathrm{Bun}_G \dashrightarrow X$ with X),
- The natural isomorphism μ satisfies appropriate pentagon axioms.

8.14 Definition. Let $\mathcal{E}$ be an object of $\underline{S}(\mathrm{Bun}_G)$. We say $\mathcal{E}$ is an *eigenobject* if for all h in $\underline{\mathrm{Sat}}$, there exists a $\chi(h)$ in $\underline{S}(X)$ and an isomorphism $\gamma_h : \mathscr{H}_h(\mathcal{E}) \xrightarrow{\sim} \chi(h) \boxtimes \mathcal{E}$ that satisfies certain associativity axioms.

8.15 Proposition. *Let $\mathcal{E}$ be an eigenobject, and suppose that there exists a nonempty open subspace U of Bun_G such that $\mathcal{E}|_U$ is a constant sheaf. Then χ defines a monoidal functor $\underline{\mathrm{Sat}} \longrightarrow \underline{S}(X)$.*

Proof. Fix some b in U. Liberally and haphazardly applying various natural isomorphisms, we have

$$\begin{aligned}\chi(h_1 * h_2) \boxtimes \mathcal{E} &= \mathscr{H}_{h_1 * h_2}(\mathcal{E}) = \Delta^*(\mathscr{H}_{h_2}(\mathscr{H}_{h_1}(\mathcal{E}))) = \Delta^*(\mathscr{H}_{h_2}(\chi(h_1) \boxtimes \mathcal{E})) = \Delta^*(\chi(h_1) \boxtimes \mathscr{H}_{h_2}(\mathcal{E})) \\ &= \Delta^*(\chi(h_1) \boxtimes \chi(h_2) \boxtimes \mathcal{E}) = \big(\chi(h_1) \otimes \chi(h_2)\big) \boxtimes \mathcal{E}.\end{aligned}$$

Pulling back to $X \times \{b\}$ and using the triviality of $\mathcal{E}|_U$ (and hence $\mathcal{E}_b$) yields the desired result. □

The geometric Satake equivalence yields an equivalence of symmetric monoidal categories

$$\mathrm{Rep}_{\mathbb{C}} G^\vee \xrightarrow{\sim} \underline{\mathrm{Sat}}.$$

Let $\mathcal{E}$ be an eigenobject as in Proposition 8.15. Then composing the above equivalence with $\chi : \underline{\mathrm{Sat}} \longrightarrow \underline{S}(X)$ and applying Proposition 8.12 yields a morphism $\pi_1(X) \longrightarrow G^\vee(\mathbb{C})$, where once again $\pi_1(X)$ is whatever fundamental group needed to make $\underline{S}(X)$ equivalent to $\mathrm{Rep}_k \pi(X)$.

Here's the technical difficulty in realizing the above recipe: even for GL_n, the functors $\mathscr{H}$ arise from a pullback-pushforward on sheaves, and for this to be well-behaved, we need to work in the derived category. But the derived category is very much not an abelian category! And in general, defining $\mathscr{H}_h$ for arbitrary h in $\underline{\mathrm{Sat}}$ (or equivalently for objects in $\mathrm{Rep}_{\mathbb{C}} G^\vee$, by geometric Satake) is hard, though we remark that we have visibly constructed such $\mathscr{H}_h$ when h corresponds to the wedge power representations.

Let's now turn to bialgebras. For now, all algebras shall be commutative.

8.16 Definition. A *bialgebra over k* is a k-algebra A equipped with morphisms

$$\Delta : A \longrightarrow A \otimes_k A,\ S : A \longrightarrow A, \text{ and } \varepsilon : A \longrightarrow k$$

of k-algebras that turn $\operatorname{Spec} A$ into an affine group scheme over k upon applying Spec. Alternatively, we can take the axioms for being a group object, reverse all the arrows, and require that (Δ, S, ε) satisfy the new, reversed axioms.

The anti-equivalence between affine schemes over k and k-algebras shows that a bialgebra over k is precisely the data of an affine group scheme over k.

8.17 Definition. Let G be an affine group scheme over k, and let V be a k-vector space. A *representation of G in V* is a k-linear $k[G]$-comodule structure on V, i.e. a k-linear map $r : V \longrightarrow V \otimes_k k[G]$ such that

$$(\mathrm{id}_V \otimes_k \Delta) \circ r = (r \otimes_k \mathrm{id}_{k[G]}) \circ r \text{ and } (\mathrm{id}_V \otimes \varepsilon) \circ r = \mathrm{id}_V .$$

The first requirement is the dual of $g(g'm) = (gg')m$, and the second is the dual of $(1)m = m$.

View V as a scheme over k (via taking a direct limit of finite-dimensional subspaces of V, viewed as copies of $\mathbb{A}_k^N$[15]). For any homomorphism $\rho : G \longrightarrow \mathrm{GL}(V)$ of group schemes, we obtain a representation of G in V as follows. The homomorphism $\rho(A)$ takes the identity map in $G(A)$ to an A-linear automorphism of $V \otimes_k A$, which is uniquely determined by its restriction $r : V \longrightarrow V \otimes_k A$ to V. This process is reversible, which shows that our notion of representations of G are equivalent to the usual sense.

The following highly useful proposition allows us to reduce questions about representations to those about finite-dimensional ones.

8.18 Proposition. *Let $r : V \longrightarrow V \otimes_k k[G]$ be a representation of G in V. Then V is the union of finite-dimensional subrepresentations.*

This is essentially the only idea in the theory, and it follows readily from tensor finitude.

Proof of Proposition 8.18. Let v lie in V, and choose a basis $\{e_i\}_i$ of $k[G]$ over k. Write

$$r(v) = \sum_{i \in I} v_i \otimes e_i$$

for some v_i in V, where I is a finite set. We shall prove that the span of v with the v_i is a subrepresentation of V. For all i in I, write

$$\Delta(e_i) = \sum_{j,k} c_{i,j,k}(e_j \otimes e_k),$$

for $c_{i,j,k}$ in k such that cofinitely many are zero. Note that

$$\sum_{i,j,k} c_{i,j,k}(v_i \otimes e_j \otimes e_k) = (\mathrm{id}_V \otimes \Delta) \left(\sum_i v_i \otimes e_i \right) = (\mathrm{id}_V \otimes_k \Delta)(r(v)) = (r \otimes_k \mathrm{id}_{k[G]})(r(v))$$
$$= (r \otimes_k \mathrm{id}_{k[G]}) \left(\sum_i v_i \otimes e_i \right) = \sum_i r(v_i) \otimes e_i.$$

Comparing e_i-components shows that $\sum_{i,j} c_{i,j,k}(v_i \otimes e_j) = r(v_k)$. Therefore we see that r preserves the span of v and the v_k, as desired. As the sum of finite-dimensional subrepresentations remains finite, this finishes the proof. □

[15] Note that for infinite-dimensional V, this construction does not agree with $\operatorname{Spec} \operatorname{Sym}^\bullet V^*$ nor $\operatorname{Spec} \operatorname{Sym}^\bullet V$.

9 April 24, 2018

Hopefully, today we'll prove the main theorem on tensor categories. We'll start with a simple and highly general construction due to Deligne. Let $\mathscr{A}$ be an essentially small k-linear abelian category, and let X be an object of $\mathscr{A}$. Recall that (Vect) denotes the category of finite-dimensional k-vector spaces. Given any V in (Vect), our goal will be to define $V \otimes X$. Suppose that $\operatorname{Hom}(X, Y)$ is finite-dimensional for all X and Y in $\mathscr{A}$.

9.1 Lemma. *Let V be an object of (Vect).*

1) *The functor $\mathscr{A} \longrightarrow$(Vect) given by $Y \mapsto \operatorname{Hom}(V, \operatorname{Hom}(X, Y))$ is corepresentable, and we denote its corepresenting object by $V \otimes X$.*

2) *The functor $\mathscr{A} \longrightarrow$(Vect) given by $Y \mapsto \operatorname{Hom}(V \otimes Y, X)$ is representable, and we denote its representing object by $\mathscr{H}\mathrm{om}(V, X)$.*

I shall only prove part 1), as part 2) is very similar.[16] Note that these representablility results are precisely the kinds of equations we expect to hold for $V \otimes X$ and $\mathscr{H}\mathrm{om}(V, X)$.

Proof of part 1). Write n for the dimension of V. Note that any automorphism $k^n \xrightarrow{\sim} k^n$ induces an automorphism $X^n \xrightarrow{\sim} X^n$. For any two isomorphisms $\alpha : k^n \xrightarrow{\sim} V$ and $\beta : k^n \xrightarrow{\sim} V$, the composite $\alpha \circ \beta^{-1} : k^n \xrightarrow{\sim} k^n$ therefore induces an automorphism $\varphi_{\alpha,\beta} : X^n \xrightarrow{\sim} X^n$. For any such α, β, and γ, we have $\varphi_{\alpha,\beta} \circ \varphi_{\beta,\gamma} = \varphi_{\alpha,\gamma}$, and we define $V \otimes X$ to be the colimit over all these automorphisms $\varphi_{\alpha,\beta}$. $\square$

Let $\mathscr{C}$ be a tensor category, which we recall means a rigid symmetric monoidal k-linear abelian category. We are always assuming that 1 is simple and that $\operatorname{End}(1) = k$.

9.2 Remark. Recall that we have dual objects $X^\vee$.

1) We see that $(X^\vee)^\vee \cong X$, since dual objects are unique up to isomorphism, and X satisfies the conditions for being the dual of $X^\vee$.

2) For any X and Y in $\mathscr{C}$, set $\mathscr{H}\mathrm{om}(X, Y) \coloneqq X^\vee \otimes Y$. It is a fact that

$$\operatorname{Hom}(1, \mathscr{H}\mathrm{om}(X, Y)) = \operatorname{Hom}(1, X^\vee \otimes Y) = \operatorname{Hom}(X, Y).$$

9.3 Corollary. *If every object in $\mathscr{C}$ has finite length, then $\operatorname{Hom}(X, Y)$ is finite-dimensional for all X and Y in $\mathscr{C}$.*

Proof. Let $f_1, \dots$ be a k-linearly independent sequence in $\operatorname{Hom}(X, Y)$, and let $F_i : 1 \longrightarrow \mathscr{H}\mathrm{om}(X, Y)$ be the corresponding morphisms in $\mathscr{C}$. For any n, we can form the direct sum

$$\bigoplus_{i=1}^{n} F_i : 1^{\oplus n} \longrightarrow \mathscr{H}\mathrm{om}(X, Y),$$

and the the linear independence of the f_i along with the simplicity of 1 imply that $\bigoplus_{i=1}^n F_i$ is a monomorphism, as $1^{\oplus n}$ is semisimple. This would then indicate that $\mathscr{H}\mathrm{om}(X, Y)$ has infinite length. $\square$

Let's now return to bialgebras over k. Let G be an affine group scheme over k, and write $A = k[G]$ for the corresponding bialgebra. We can break up A in the following manner, similarly to Proposition 8.18.

[16] One considers a similar construction that encapsulates all bases of V, except we now take a limit rather than a colimit.

9.4 Lemma. *The algebra A is the union $A = \bigcup_i A_i$ of sub-bialgebras A_i such that A_i is finitely generated as a k-algebra.*

Using the anti-equivalence between affine schemes over k and k-algebras, we see that this means any affine group scheme G over k is of the form

$$G = \varprojlim_i G_i$$

for some algebraic groups G_i over k, where an *algebraic group* is an affine group scheme of finite type.

Proof of Lemma 9.4. We apply a trick much like the one used to prove Proposition 8.18. Let v lie in $k[G]$, and choose a basis $\{e_i\}_i$ of $k[G]$ over k. Write

$$\Delta(v) = \sum_{i \in I} v_i \otimes e_i,$$

for some v_i in $k[G]$, where I is a finite set. One can use the axioms of bialgebras to show that the k-subalgebra generated by

$$\{v\} \cup \{v_i, S(v_i)\}_{i \in I}$$

is closed under Δ and S, and we conclude as in Proposition 8.18. □

The next lemma is an important equivalent characterization of the algebraic groups.

9.5 Lemma. *An affine group scheme G over k is algebraic if and only if it has a faithful representation, i.e. if it is isomorphic to a closed subgroup of GL_n for some n.*

Proof. Closed subgroups of GL_n are of finite type, so this direction follows. Conversely, note that $k[G]$ forms a representation ρ of G via right translation. Then Proposition 8.18 indicates that $k[G]$ is the union of finite-dimensional subrepresentations $\rho_i : G \longrightarrow \mathrm{GL}(V_i)$ (i.e. $k[G] = \bigcup_i V_i$). Now regular representation $k[G]$ is faithful, and we have

$$\bigcap_i \ker \rho_i = \ker \rho = \{1\} \implies I\left(\bigcap_i \ker \rho_i\right) = \sum_i I(\ker \rho_i) = I(\{1\}).$$

The finite type hypothesis implies that $k[G]$ is noetherian, so some finite subsum $\sum_{i=1}^n I(\ker \rho_i)$ already equals $I(\{1\})$, by the ascending chain condition. Taking the union $\bigcup_{i=1}^n V_i$ yields the desired faithful representation. □

From now on, assume that k is algebraically closed.

9.6 Definition. Let G be an algebraic group over k. We say that G is *reductive* if G contains no nontrivial normal unipotent subgroups.

The goal of the next proposition is to translate properties of G into properties of its representations.

9.7 Proposition. *Let G be an affine group scheme over k, and suppose $\operatorname{char} k = 0$.*

1) *The group G is algebraic if and only if there exists an object X in $\operatorname{Rep}_k G$ such that $\operatorname{Rep}_k G$ is generated from X via taking direct sums, tensor products, and subquotients.*

2) *Suppose additionally that G is algebraic. Then G is reductive if and only if $\operatorname{Rep}_k G$ is semisimple.*

Proof. I originally thought I wouldn't do the proof, but I was wrong:

1) If G is algebraic, Lemma 9.5 indicates that G is a closed subgroup of GL_n. Here, we can take X to be the standard representation k^n, since we can obtain any finite-dimensional subrepresentation of $k[G]$ from k^n using our allowed constructions.

   Conversely, suppose we have such an object X. Then any g in $G(k)$ acting trivially on X acts trivially on every representation of G, which indicates that $g = 1$. Thus X is a faithful finite-dimensional representation of G, and Lemma 9.5 tells us that G is algebraic.

2) It's a fact that any algebraic group G has a maximal normal unipotent subgroup $R_u(G)$. We can form the quotient $G/R_u(G)$, which then is reductive. Assuming that $\mathrm{Rep}_k G$ is semisimple, it suffices to show that $R_u(G)$ is trivial. Since $R_u(G)$ is unipotent, any finite-dimensional representation V of G has a nonzero $R_u(G)$-stable vector. As $R_u(G)$ is normal in G, we see that $V^{R_u(G)}$ is G-stable. Applying these two facts to irreducible representations V yields $V = V^{R_u(G)}$, and our semisimplicity assumption extends this for all finite-dimensional V. The same argument as above then shows that $R_u(G)$ is trivial.

   Conversely, if G is reductive, then the theory of reductive groups tells us that $\mathrm{Rep}_k G$ is semisimple. □

We're almost getting to the final part! Let G be an affine group scheme over k, and let $\omega : \mathrm{Rep}_k G \longrightarrow (\mathrm{Vect})$ be a fiber functor. Recall this means that ω is an exact faithful monoidal functor. With ω in hand, we can make the following definition.

9.8 Definition. Write $\mathrm{Aut}^{\otimes} \omega$ for the functor $(k\text{-Alg}) \longrightarrow (\mathrm{Grp})$ that sends any k-algebra R to the set of collections $\lambda = \{\lambda_X \in \mathrm{Aut}_R(R \otimes \omega(X))\}_X$, where X ranges over (isomorphism classes of) objects in $\mathrm{Rep}_k G$, such that

- $\lambda_k = \mathrm{id}_k$,
- for all isomorphism classes X and Y in $\mathrm{Rep}_k G$, we have $\lambda_X \otimes_R \lambda_Y = \lambda_{X \otimes Y}$ after identifying
$$(R \otimes \omega(X)) \otimes_R (R \otimes \omega(Y)) = R \otimes (\omega(X) \otimes_k \omega(Y)) = R \otimes \omega(X \otimes Y)$$
  using the compatibility isomorphisms given by the monoidal functor ω,
- for all isomorphism classes of morphisms $\alpha : X \longrightarrow Y$ in $\mathrm{Rep}_k G$, the diagram
$$\begin{tikzcd} R \otimes \omega(X) \arrow[r] \arrow[d] & R \otimes \omega(Y) \arrow[d] \\ R \otimes \omega(X) \arrow[r] & R \otimes \omega(Y) \end{tikzcd}$$
  commutes.

The group operation is defined via $(\lambda \cdot \lambda')_X := \lambda_X \circ \lambda'_X$.

9.9 Proposition. *The morphism of functors $G \longrightarrow \mathrm{Aut}^{\otimes} \omega$ given by sending g in $G(R)$ to the action of g on $R \otimes \omega(X)$ for all X in $\mathrm{Rep}_k G$ is an isomorphism.*

In other words, we can recover an affine group scheme G from $\mathrm{Rep}_k G$, by taking $\mathrm{Aut}^{\otimes} \omega$. Proposition 9.9 only takes place in a situation where we already know that our tensor category is isomorphic to $\mathrm{Rep}_k G$ for some affine group scheme G, but we can also perform this for abstract tensor categories too. We'll explain this last point next time.

10 April 26, 2018

Let's start by just assuming k is an algebraically closed field. Let G be an affine group scheme over k, and recall that we have the usual forgetful functor $\omega_G : \mathrm{Rep}_k\, G \longrightarrow$(Vect). Proposition 9.9 claims to give an isomorphism $G \longrightarrow \mathrm{Aut}^{\otimes}\,\omega_G$ of functors. Before we consider Proposition 9.9, we first use it to prove Proposition 8.12.

Proof of Proposition 8.12. Define a morphism $F^* : \mathrm{Aut}^{\otimes}\,\omega_G \longrightarrow \mathrm{Aut}^{\otimes}\,\omega_H$ as follows: for any k-algebra R, send λ in $(\mathrm{Aut}^{\otimes}\,\omega_G)(R)$ to the element $F^*\lambda$ of $(\mathrm{Aut}^{\otimes}\,\omega_H)(R)$ defined by $(F^*\lambda)_Y := \lambda_{F(Y)}$. Then Proposition 9.9 converts F^* into a morphism $f : G \longrightarrow H$ of algebraic groups over k with the desired property. $\square$

For the rest of today, assume that k also has characteristic zero, and let $\mathscr{C}$ be a tensor category for which $\mathrm{End}(1) = k$. The following statement is the main theorem of the Tannakian formalism, which amounts to a version of Proposition 9.9 for abstract tensor categories.

10.1 Theorem. *Let $\omega : \mathscr{C} \longrightarrow$(Vect) be a fiber functor. Then $\mathrm{Aut}^{\otimes}\,\omega$ is represented by an affine group scheme G over k such that there exists an equivalence $\overline{\omega} : \mathscr{C} \xrightarrow{\sim} \mathrm{Rep}_k\, G$ of tensor categories for which $\omega_G \circ \overline{\omega}$ is isomorphic to ω.*

However, we also have the following variant of Theorem 10.1 for categories with less structure.

10.2 Theorem. *Let $\mathscr{C}$ be an abelian k-linear symmetric monoidal category, except it need not satisfy the pentagon or hexagon axioms. Furthermore, let $\omega : \mathscr{C} \longrightarrow$(Vect) be an exact faithful k-linear functor, suppose we have an isomorphism $\nu : k \xrightarrow{\sim} \omega(1)$ as well as natural isomorphisms $\tau_{X,Y} : \omega(X) \otimes \omega(Y) \xrightarrow{\sim} \omega(X \otimes Y)$. Finally, suppose that for all X in $\mathscr{C}$ such that $\dim_k \omega(X) = 1$, there exists an $X^\vee$ in $\mathscr{C}$ such that $X \otimes X^\vee$ is naturally isomorphic to 1. Then the conclusion of Theorem 10.1 holds.*

10.3 Examples.

1) Write $(\mathrm{Vect}^{\mathbb{Z}})$ for the category of finite-dimensional $\mathbb{Z}$-graded k-vector spaces, and let ω be the usual forgetful functor. This satisfies all the hypotheses in Theorem 10.1, so we can get an affine group scheme out of this. And we can show that $(\mathrm{Vect}^{\mathbb{Z}})$ is equivalent to $\mathrm{Rep}_k\, \mathbb{G}_m$.

2) Let Γ be an abstract group, and let $\mathrm{Rep}\,\Gamma$ be the category of finite-dimensional representations of Γ over k. This also satisfies the hypotheses of Theorem 10.1, so there exists an affine group scheme $G(\Gamma)$ over k such that $\mathrm{Rep}\,\Gamma$ is equivalent to $\mathrm{Rep}_k\, G(\Gamma)$.

   For example, if $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, then $G(\Gamma)$ will end up being SL_2. In general, if you take a Zariski-dense subgroup Γ of an algebraic group G, then $G(\Gamma)$ will just be G.

3) Let X be a path-connected, locally path-connected, semilocally simply connected topological space (thus it has a universal cover). Write $\mathscr{C}$ for the category of finite-dimensional k-local systems on X. For any x in X, the functor given by $\mathscr{L} \mapsto \mathscr{L}|_x$ is a fiber functor (this is the origin of the terminology "fiber functor"), and because $\mathscr{C}$ is equivalent to the category of finite-dimensional representations of $\pi_1(X, x)$ over k, part 2) shows that $\mathscr{C}$ is equivalent to $\mathrm{Rep}_k\, G(\pi_1(X, x))$.

For a non-example, consider the category of $\mathbb{Z}/2\mathbb{Z}$-graded finite-dimensional vector spaces over k, where the commutativity constraint is given by the Koszul rule for signs. Then some dimensions might be negative, which can never happen for $\mathrm{Rep}_k\, G$.

10.4 Theorem. *Suppose the hypotheses of Theorem 10.1 are satisfied. Then* *a) For all X in $\mathscr{C}$, we have $\dim_{\mathscr{C}} X = \dim_k \omega(X)$,*

b) Any object X in $\mathscr{C}$ has finite length,

c) For any X and Y in $\mathscr{C}$, the k-vector space $\operatorname{Hom}_{\mathscr{C}}(X, Y)$ is finite-dimensional.

Proof. This all follows from using $\overline{\omega}$ to identify $\mathscr{C}$ with $\operatorname{Rep}_k G$. □

For the next ten minutes, forget about tensor categories and focus on abelian categories. Let A be a (not necessarily commutative) finite-dimensional k-algebra, and write (A-mod) for the category of finite-dimensional (left) A-modules. Consider the following motivational lemma.

10.5 Lemma. *Let X be an object of (A-mod), and let $\alpha : X \longrightarrow X$ be a k-linear map. Suppose that, for any non-negative integer n and any A-submodule of Y of $X^{\oplus n}$, the map $\alpha^{\oplus n}$ preserves Y. Then there exists a in A such that α equals the map given via multiplication by a.*

Proof. Let $x_1, \dots, x_n$ be a k-basis of X, and let Y be the A-submodule of $X^{\oplus n}$ generated by $x_1 \oplus \cdots \oplus x_n$. As $\alpha^{\oplus n}$ preserves Y, we see that $\alpha^{\oplus n}(x_1 \oplus \cdots \oplus x_n) = a(x_1 \oplus \cdots \oplus x_n) = ax_1 \oplus \cdots \oplus ax_n$ for some a in A. But $\alpha^{\oplus n}(x_1 \oplus \cdots \oplus x_n) = \alpha(x_1) \oplus \cdots \oplus \alpha(x_n)$. Since $x_1, \dots, x_n$ is a k-basis of X, we therefore see that α acts via multiplication by a on all of X. □

What follows shall be similar to Lemma 10.5 in flavor. Let $\mathscr{C}$ be a k-linear abelian category, and let X be an object of $\mathscr{C}$. Write $\langle X \rangle$ for the full subcategory of $\mathscr{C}$ whose objects are the subquotients of $X^{\oplus n}$, as n ranges through non-negative integers. Suppose we have an exact faithful k-linear functor $\omega : \mathscr{C} \longrightarrow$(Vect), and define

$$A_X := \left\{ \alpha \in \operatorname{End}(\omega(X)) \mid \text{for all } n \geq 0 \text{ and } Y \subseteq X^{\oplus n},\ \alpha^{\oplus n}(Y) \subseteq Y \right\}.$$

Note that A_X is a k-subalgebra of $\operatorname{End}(\omega(X))$, which is finite-dimensional over k.

10.6 Proposition. *There exists an equivalence $\overline{\omega}_X : \langle X \rangle \xrightarrow{\sim} (A_X\text{-mod})$ such that $f \circ \overline{\omega}_X$ is isomorphic to $\omega|_{\langle X \rangle}$, where $f : (A_X\text{-mod}) \longrightarrow$(Vect) is the forgetful functor.*

Proof. Let a be in A_X. By construction, a acts on $\omega(Y)/\omega(Y') = \omega(Y/Y')$ for any subobjects $Y' \subseteq Y \subseteq X^{\oplus n}$ for any n, so therefore a acts on $\omega(Z)$ for any Z in $\langle X \rangle$. This allows us to view objects in $\langle X \rangle$ as A_X-modules, netting us a functor $\overline{\omega}_X : \langle X \rangle \longrightarrow (A_X\text{-mod})$.

To show that $\overline{\omega}_X$ is an equivalence, we will produce a quasi-inverse. This is a complicated construction of Deligne—while it does the trick, I do not understand it. First, for any inclusions $W \subseteq V$ in (Vect) and $Z \subseteq Y$ in $\mathscr{C}$, define

$$(Z : W) := \ker\left(\mathscr{H}\!om(V, Y) \longrightarrow \mathscr{H}\!om(W, Y/Z)\right),$$

which is an object of $\mathscr{C}$. Next, use this notation to define

$$P_X := \bigcap_{\substack{n \geq 0 \\ Y \subseteq X^{\oplus n}}} \mathscr{H}\!om(\omega(X), X) \cap (Y : \omega(Y)),$$

where the inclusions used to form $(Y : \omega(Y))$ are $\omega(Y) \subseteq \omega(X)^{\oplus n}$ and $Y \subseteq X^{\oplus n}$, and the intersection is taken via embedding $\mathscr{H}\!om(\omega(X), X)$ diagonally into $\mathscr{H}\!om(\omega(X)^{\oplus n}, X^{\oplus n})$. Note that P_X lies in $\langle X \rangle$.

We remark that the constructions in Lemma 9.1 are functorial, and this gives us the bottom arrow of the following commutative diagram:

$$\begin{tikzcd} A_X \arrow[rr] \arrow[d, hook] & & \operatorname{End} P_X \arrow[d, hook] \\ \operatorname{End}\omega(X) \arrow[r] & \operatorname{End}(\mathscr{H}\mathrm{om}(\omega(X), X)), & \end{tikzcd}$$

where the vertical arrows are inclusion maps, and the top-most arrow is given by our A_X-actions.

This diagram shows that we have a version of an "A_X-action" on P_X. This allows us to define a functor $(A_X\text{-mod}) \longrightarrow \langle X\rangle$ via the formula

$$M \mapsto P_X \otimes_{A_X} M.$$

How is this tensor product defined? Well, our map $A_X \longrightarrow \operatorname{End} P_X$ yields a map $P_X \otimes A_X \longrightarrow P_X$ by the universal property from Lemma 9.1. For any A_X-module M, we get an action map $A_X \otimes M \longrightarrow M$, and tensoring with P_X yields map $\alpha_1 : P_X \otimes (A_X \otimes M) \longrightarrow P_X \otimes M$. On the other hand, tensoring $P_X \otimes A_X \longrightarrow P_X$ with M yields another map $\alpha_2 : (P_X \otimes A_X) \times M \longrightarrow P_X \otimes M$.

Morally speaking, these maps give the actions of A_X on $P_X \otimes M$ via M and P_X, respectively. Therefore, to form $P_X \otimes_{A_X} M$, we want them to coincide. So we just define

$$P_X \otimes_{A_X} M \coloneqq (P \otimes M)/\operatorname{im}(\alpha_1 - \alpha_2).$$

One can check that all this indeed gives a quasi-inverse for $\overline{\omega}_X$, concluding the proof. $\square$

We want to prove Theorem 10.1 by taking the limit of Proposition 10.6, but one has to put in some effort to make it work. We'll carry this out next time.

11 May 1, 2018

Today, we'll finish discussing the proof of Theorem 10.1, and we'll also introduce an extension of Theorem 10.1 due to Deligne (though we won't use said extension in the future). Recall that k is an algebraically closed field of characteristic zero. Let $\mathscr{C}$ now be a tensor category for which $\operatorname{End}(1) = k$, and let X be an object of $\mathscr{C}$.

Write $B_X \coloneqq A_X^\vee$ for the k-linear dual of A_X. Now B_X has the structure of a k-coalgebra by contravariance, and in the language of coalgebras and comodules, Proposition 10.6 says that we have an equivalence of categories

$$\omega_X^\vee : \langle X\rangle \xrightarrow{\sim} (B_X\text{-comod}),$$

where $(B_X\text{-comod})$ denotes the category of finite-dimensional B_X-comodules.

We'll take the limit of this situation as follows. For any two objects X and Y of $\mathscr{C}$, write $X \leq Y$ if X is a direct summand of Y. This forms a poset structure on the set of (isomorphism classes of) objects of $\mathscr{C}$, and for $X \leq Y$, we see that $\langle X\rangle \subseteq \langle Y\rangle$. Now elements of A_Y preserve the direct summand X of Y, so we obtain an algebra homomorphism $A_Y \longrightarrow A_X$. Dualizing gives us a coalgebra homomorphism $B_X \longrightarrow B_Y$, and we define $B \coloneqq \varinjlim_X B_X$ to be the direct limit over these maps.

11.1 Proposition. *There exists an equivalence $\overline{\omega} : \mathscr{C} \xrightarrow{\sim} (B\text{-comod})$ such that $f \circ \overline{\omega}$ is isomorphic to ω, where $f : (B\text{-comod}) \longrightarrow (\mathrm{Vect})$ is the forgetful functor.*

By taking $G = \operatorname{Spec} B$, Proposition 11.1 finishes our proof of Theorem 10.1.

Proof. Since ω is a tensor functor, we have natural isomorphisms $\omega(X \otimes X') \xrightarrow{\sim} \omega(X) \otimes \omega(X')$. This induces an isomorphism of algebras

$$\operatorname{End}(\omega(X \otimes X')) \xrightarrow{\sim} \operatorname{End}(\omega(X) \otimes \omega(X')) \xrightarrow{\sim} (\operatorname{End}\omega(X)) \otimes (\operatorname{End}(\omega(X')),$$

where the second isomorphism is due to the rigidity of $\mathscr{C}$. Under this morphism, $A_{X\otimes X'}$ is sent to $A_X \otimes A_{X'}$, so this nets us a k-algebra morphism

$$A_{X\otimes X'} \longrightarrow A_X \otimes A_{X'}.$$

Taking k-linear duals yields a k-coalgebra morphism $B_X \otimes B_{X'} \longrightarrow B_{X\otimes X'}$, and since tensors commute with direct limits, taking the limit over all X and X' gives us a coalgebra morphism $\mu : B \otimes B \longrightarrow B$. This map μ will equip B with the structure of a commutative algebra (where the commutativity of μ is due to the commutativity of $\otimes$).

Well, we want to construct the unit for our prospective algebra structure on B. You can imagine how we'll get it—set $X = 1$. Then $A_1 = \operatorname{End}_k(k) = k$, so $B_1 = k$, and this copy of k naturally maps into the direct limit B. One can verify that these maps give B the structure of a bialgebra over k (in the sense of Definition 8.16), and taking the limit of Proposition 10.6 gives the desired result. $\square$

Tannakian duality has a complicated history. It's most commonly named after Tannaka, but it turns out Krein had proven it contemporaneously in Ukraine. Krein didn't initially get credit because he was a functional analyst and nobody knew what he did. Later, a gap was discovered in Tannaka's proof, which Deligne–Milne fixed. Then in the 2000s, Deligne extended this work by offering the following characterization of these representation categories:

11.2 Theorem. *Let $\mathscr{C}$ be a tensor category over k. The following are equivalent:*

1) $\mathscr{C}$ is equivalent to $\operatorname{Rep}_k G$ for some affine group scheme G over k,

2) for all X in $\mathscr{C}$, the dimension $\dim X$ of X is a non-negative integer,

3) for all X in $\mathscr{C}$, we have $\bigwedge^n X = 0$ for sufficiently large n,

4) for all X in $\mathscr{C}$, there exists a non-negative integer $\ell(x)$ such that $\operatorname{length}(X^{\otimes n}) \leq \ell(x)^n$ for all positive integers n.

There are some preliminaries to discuss—for instance, we must define wedge products in $\mathscr{C}$.

11.3 Definition. Let Γ be a finite group. An *action* of Γ on Y in $\mathscr{C}$ is a group homomorphism $\Gamma \longrightarrow \operatorname{Aut}(Y)$.

11.4 Example. For all X in $\mathscr{C}$, the symmetric group $\mathfrak{S}_n$ on n letters acts on $X^{\otimes n}$ by permutation.

By k-linearity, we obtain ring homomorphisms $k[\Gamma] \longrightarrow \operatorname{End}(Y)$. In the setting of Example 11.4, note that $e := \frac{1}{n!}\sum_{s\in\mathfrak{S}_n} \operatorname{sgn}(s)s$ is an idempotent in the group algebra $k[\mathfrak{S}_n]$.

11.5 Definition. For all X in $\mathscr{C}$, write $\bigwedge^n X$ for the image $\operatorname{im}(e : X^{\otimes n} \longrightarrow X^{\otimes n})$.

As with finite-dimensional vector spaces, one has the following result for dimensions of wedge products.

11.6 Proposition. *We have* $\dim(\bigwedge^n X) = \binom{\dim X}{n} := \dfrac{(\dim X)\cdot(\dim X - 1)\cdots(\dim X - n + 1)}{n\cdot(n-1)\cdots 2\cdot 1}$.

This discussion works for other Schur functors (and Theorem 11.2 can be expressed in terms of Schur functors), but we shall not discuss Schur functors here. To prove Proposition 11.6, we start with the following lemma.

11.7 Lemma. *Suppose we have a cycle of morphisms*

$$\cdots \xrightarrow{u_n} X_1 \xrightarrow{u_1} X_2 \xrightarrow{u_2} \cdots$$

in $\mathscr{C}$. Then the trace of the endomorphism $\bigotimes_{i=1}^n u_i : \bigotimes_{i=1}^n X_i \longrightarrow \bigotimes_{i=1}^n X_i$ equals

$$\operatorname{tr}\left(\bigotimes_{i=1}^n u_i \,\middle|\, \bigotimes_{i=1}^n X_i\right) = \operatorname{tr}\left(u_n \circ \cdots \circ u_1 \mid X_1\right).$$

Of course, one can readily prove this when $\mathscr{C} = (\text{Vect})$ by using bases and writing traces as a sum over the diagonal entries. The proof of Lemma 11.7 in the abstract setting of $\mathscr{C}$ is an exercise in conducting linear algebra without explicitly saying that traces are equal to the sum of diagonal entries.

11.8 Corollary. *Let $\sigma = (1, \dots, n)$ be the standard n-cycle in $\mathfrak{S}_n$. Then $\operatorname{tr}(\sigma \mid X^{\otimes n}) = \dim X$.*

Proof. Apply Lemma 11.7 to $X_1 = \cdots = X_n = X$ and $u_1 = \cdots = u_n = \mathrm{id}_X$. ☐

By iterating Corollary 11.8, we obtain the following consequence.

11.9 Corollary. *Let σ in $\mathfrak{S}_n$ be a product of m disjoint cycles. Then $\operatorname{tr}(\sigma \mid X^{\otimes n}) = (\dim X)^m$.*

Proof. Apply Corollary 11.8 to each cycle in the cycle decomposition of σ, and use the fact that traces of tensor products are products of traces. ☐

With Corollary 11.9 in hand, we proceed to prove Proposition 11.6 using a universality argument.

Proof of Proposition 11.6. Because e is an idempotent, we have

$$\dim(\operatorname{im} e) = \operatorname{tr}(e \mid X^{\oplus n}) = \frac{1}{n!}\operatorname{tr}(a \mid X^{\otimes n}),$$

where a denotes the element $\sum_{s \in \mathfrak{S}_n} \operatorname{sgn}(s) s$ of the group algebra $k[\mathfrak{S}_n]$. Corollary 11.9 gives us a formula for computing $\operatorname{tr}(a \mid X^{\oplus n})$, and this formula implies that there exists a universal polynomial $p_n(t)$ in $\mathbb{Z}[T]$, independent of $\mathscr{C}$ and X, such that

$$\operatorname{tr}(a \mid X^{\otimes}) = p_n(\dim X).$$

When $\mathscr{C} = (\text{Vect})$ and X has dimension d, the usual calculation of wedge power dimensions tells us that

$$\frac{1}{n!} p_n(d) = \dim \bigwedge^n X = \binom{n}{d},$$

and since the value of $p_n(t)$ at all non-negative integers t determines the entries of $p_n(t)$polynomial uniquely, we see that this identity continues to hold for all $\mathscr{C}$ and X. ☐

In turn, Proposition 11.6 enables us to prove a piece of Theorem 11.2.

Proof of 2) $\iff$ 3) in Theorem 11.2. Assume that 2) holds, and let $\dim X = n$ for some non-negative integer n. Then Proposition 11.6 indicates that $\dim(\bigwedge^{n+1} X) = 0$. We claim that, under the hypothesis of 2), having dimension zero implies that you are the zero object. To see this, if Y in $\mathscr{C}$ is nonzero, then the map $\delta : 1 \longrightarrow \operatorname{End}(Y) = Y \otimes Y^\vee$ is nonzero and hence injective. We have

$$\dim(\operatorname{coker}\delta) = \dim(Y^\vee \otimes Y) - \dim 1 = (\dim Y)(\dim Y^\vee) - 1 \in \mathbb{Z}_{\geq 0}$$

by hypothesis, so we cannot have $\dim Y = 0$.

Conversely, suppose that 3) holds, and let n be a positive integer for which $\bigwedge^n X = 0$. Then $\dim(\bigwedge^n X) = 0$, so using the formula given by Proposition 11.6 indicates that $(\dim X) - k = 0$ for some integer $1 \leq k \leq n-1$. □

The proof that 2) $\iff$ 4) in Theorem 11.2 is similar.

We want to travel from the abstract conditions 2), 3), and 4) in Theorem 11.2 to a genuine fiber functor (which is necessary and sufficient for 1)). How on earth will we do this? Well, it involves a general trick that will also appear later in our discussion of geometric Satake.

Let $\mathscr{C}$ be an abelian category, and let $\operatorname{Ind}\mathscr{C}$ be the category of ind-objects of $\mathscr{C}$. If $\mathscr{C}$ is additionally a monoidal category, we can form the following definition.

11.10 Definition. Let R be an object of $\operatorname{Ind}\mathscr{C}$ (or of $\mathscr{C}$). We say R is a *ring object* if it is equipped with morphisms $m : R \otimes R \longrightarrow R$ and $u : 1 \longrightarrow R$ such that $(m \otimes \mathrm{id}_R) \circ m = (\mathrm{id}_R \otimes m) \circ m$.

We define a monoidal structure on $\operatorname{Ind}\mathscr{C}$ via taking direct limits of $\otimes$ in $\mathscr{C}$. Ring objects are often called *algebra objects* instead.

11.11 Example. Let G be an algebraic group over k, and let $\mathscr{C}$ be $\operatorname{Rep}_k G$. Then $k[G]$, considered as a representation of G via the regular representation, can be considered as an object of $\operatorname{Ind}\mathscr{C}$ by Proposition 8.18. Furthermore, $k[G]$ is a commutative ring object in $\operatorname{Ind}\mathscr{C}$ under (pointwise) multiplication of regular functions.

In the setting of Example 11.11, let X be in $\operatorname{Rep}_k G$. We have

$$\operatorname{Hom}_{\operatorname{Ind}(\operatorname{Rep}_k G)}(1, k[G] \otimes X) = (k[G] \otimes X)^G = \operatorname{Hom}_{(\mathsf{Var}/k)}(G \longrightarrow X)^G = X$$

as an object in (Vect), and this is precisely the trick that will allow us to obtain fiber functors. In particular, this calculation yields the following corollary.

11.12 Corollary. *We have $k[G] \otimes X \approx k[G]^{\dim X}$ in $\operatorname{Ind}(\operatorname{Rep}_k G)$.*

For any commutative ring object A in $\mathscr{C}$, one can define a notion of A-modules in $\mathscr{C}$ as morphisms $A \otimes M \longrightarrow M$ satisfying the usual module axioms. For $\mathscr{C}$ satisfying 2), one can define the rank $\operatorname{rk}_A M$ of an A-module M as follows: apparently we can develop enough commutative algebra to make sense of localizations[17] for commutative ring objects, and then we define ranks via considering ranks generically on "$\operatorname{Spec} A$." The assumption 2) ensures that $\operatorname{rk}_A M$ lies in $\mathbb{Z}_{\geq 0}$.

Locally, we can decompose A-modules M into free parts as follows.

11.13 Lemma. *Let A be a nonzero commutative ring object in $\mathscr{C}$, and let M be an A-module with $\operatorname{rk}_A M \geq 1$. Then A has a localization $A \longrightarrow B$ such that $M_{loc} = B \oplus N$ for some B-module N.*

We'll discuss how to use A-modules to finish the proof of Theorem 11.2 next time.

[17] By localization, we really mean extension of scalars to a nonzero commutative ring object. See Lecture 12.

12 May 3, 2018

I have decided to provide more details on the proof of Theorem 11.2. Recall that k is an algebraically closed field of characteristic zero and that $\mathscr{C}$ is a tensor category over k. At this point, for any commutative k-algebra R, write (R-mod) for the category of R-modules. This has a symmetric monoidal structure given by tensor products.

12.1 Theorem. *Assume that for all objects X in $\mathscr{C}$, its dimension $\dim X$ is a non-negative integer. Then there exists a nonzero commutative k-algebra R and an exact monoidal functor $\omega : \mathscr{C} \longrightarrow (R\text{-mod})$.*

The resulting commutative ring R will generally not be finitely generated over k, but of course we may write it as a direct limit of k-algebras that are.

And how do we prove Theorem 12.1? Let A be a commutative ring object in $\operatorname{Ind}\mathscr{C}$, and write ($A$-mod) for the category of A-module objects. We can also form tensor products over A as in the proof of Proposition 10.6:

12.2 Definition. Let M and N be A-modules. Their *tensor product* is

$$M \otimes_A N := \operatorname{coker}\left(M \otimes A \otimes N \rightrightarrows M \otimes N \right),$$

where the two arrows are obtained from the action morphisms on M and N, respectively.

The tensor product $\otimes_A$ equips (A-mod) with a symmetric monoidal structure (where the unit is A itself under left multiplication, which we denote using 1_A). The tensor products $\otimes_A$ over commutative ring objects in $\operatorname{Ind}\mathscr{C}$ satisfies the transitivity properties you'd want out of successive tensor products.

Given a morphism $A \longrightarrow B$ of commutative ring objects in $\operatorname{Ind}\mathscr{C}$, we can use tensor products to obtain an extension of scalars functor

$$\begin{aligned}(A\text{-mod}) &\longrightarrow (B\text{-mod})\\ M &\longmapsto M_B := B \otimes_A M,\end{aligned}$$

where the action map $B \otimes M_B \longrightarrow M_B$ arises from the multiplication morphism on B and the action map on M via

$$B \otimes M_B = B \otimes (B \otimes_A M) = B \otimes ((B \otimes_A A) \otimes M) = (B \otimes B) \otimes_A (A \otimes M) \longrightarrow B \otimes_A M = M_B.$$

12.3 Examples.

(a) The unit object 1 can be equipped with a commutative ring object structure via the isomorphism $1 \otimes 1 \xrightarrow{\sim} 1$. In this case, we see that (1-mod) is equivalent to $\operatorname{Ind}\mathscr{C}$ as symmetric monoidal categories. Furthermore, the identity morphism $1 \longrightarrow A$ for any commutative ring object A is an algebra morphism, so taking extension of scalars yields a functor

$$\begin{aligned}\operatorname{Ind}\mathscr{C} &\longrightarrow (A\text{-mod})\\ M &\longmapsto A \otimes M.\end{aligned}$$

(b) We recover the usual theory of commutative k-algebras by setting $\mathscr{C} = (\text{Vect})$. We use $\operatorname{Ind}\mathscr{C}$ instead of $\mathscr{C}$ since we don't require our k-algebras to be finite over k.

12.4 Definition. Let X and Y be A-modules. We say that X and Y are *locally isomorphic* if there exists a nonzero morphism $A \longrightarrow B$ of commutative ring objects such that X_B is isomorphic to Y_B.

12.5 Example. Let $\mathscr{C} = \operatorname{Rep}_k G$, where G is an affine group scheme over k. Then Proposition 8.18 indicates that $A \mapsto \operatorname{Spec} A$ yields an (anti-)equivalence between commutative ring objects in $\operatorname{Ind}\mathscr{C}$ and affine schemes over k equipped with a right G-action.

12.6 Lemma. *Let X and Y be objects of $\operatorname{Rep}_k G$, considered as objects of (1-mod) as in Example 12.3.(a). Then X and Y are locally isomorphic if and only if $\dim X = \dim Y$.*

Proof. Tensoring up to the ind-finite commutative ring object $k[G]$ yields $k[G] \otimes X \approx k[G]^{\dim X}$ by Corollary 11.12. Applying this to Y in place of X immediately gives the desired equivalence. $\square$

Since $(A\text{-mod})$ is a symmetric monoidal category, there is a notion of dualizable objects M of $(A\text{-mod})$. For any such M, we have morphisms

$$1_A \xrightarrow{\delta} M \otimes M^\vee \xrightarrow{\mathrm{ev}} 1_A$$

as usual, and we define $\dim_A M \coloneqq \mathrm{ev} \circ \delta$. This $\dim_A M$ is an element of $\operatorname{End}_A(1_A) = \operatorname{Hom}_A(A, A)$, which in turn is equal to $\operatorname{Hom}_{\operatorname{Ind}\mathscr{C}}(1, A)$ since any A-module morphism $A \longrightarrow A$ is uniquely determined by its action on $u : 1 \longrightarrow A$.

As with wedge products, we can form symmetric products in our abstract setting as well. The following analogous dimension count holds in the abstract setting:

12.7 Lemma. *Let M be a dualizable object in $(A\text{-mod})$ such that $d \coloneqq \dim_A M$ is a positive integer. Then*

$$\dim_A(\operatorname{Sym}^n_A M) = \frac{d \cdot (d+1) \cdots (d+n-1)}{n!}.$$

The proof of Lemma 12.7 is analogous to that of Proposition 11.6.

12.8 Proposition. *Let $0 \longrightarrow X' \longrightarrow X \xrightarrow{b} 1 \longrightarrow 0$ be a short exact sequence in $\mathscr{C}$. Then there exists a nonzero commutative ring object A in $\operatorname{Ind}\mathscr{C}$ such that $0 \longrightarrow X'_A \longrightarrow X_A \longrightarrow 1_A \longrightarrow 0$ splits in $(A\text{-mod})$.*

One can show that Proposition 12.8 generalizes as follows.

12.9 Corollary. *Let $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be a short exact sequence in $\mathscr{C}$. Then there exists a nonzero commutative ring object A in $\operatorname{Ind}\mathscr{C}$ such that $0 \longrightarrow M'_A \longrightarrow M_A \longrightarrow M''_A \longrightarrow 0$ splits in $(A\text{-mod})$.*

As for Proposition 12.8 itself, we shall illustrate the idea behind its proof using an example. Let X be an algebraic variety over $\mathbb{C}$, let $\mathscr{E}$ be a holomorphic vector bundle on X, and let $p : E \longrightarrow X$ be its total space. Then we have a canonical section $s : E \longrightarrow p^*E$ given by $e \mapsto (e, e)$. To ensure that s generates a sub-bundle, we have to avoid the locus where s vanishes. But this is precisely the zero section in E, so we want to cut out the zero section in the total space E.

Let's now carry out this idea in terms of algebra, rather than geometry.

Proof of Proposition 12.8. Set $B \coloneqq \operatorname{Sym}^\bullet X^\vee$, which is a commutative ring object in $\operatorname{Ind}\mathscr{C}$, and take the dual morphism $b^\vee : 1 \longrightarrow X^\vee$. By abuse of notation, write $1 : 1 \longrightarrow B$ for the unit morphism, and write $(1 - b^\vee)$ for the B-submodule of B generated by the image of $1 - b^\vee : 1 \longrightarrow B$.

Finally, set $A \coloneqq B/(1 - b^\vee)$. The multiplication morphism $X^\vee \otimes \operatorname{Sym}^\bullet X^\vee \longrightarrow \operatorname{Sym}^\bullet X^\vee$ yields a morphism $B \longrightarrow X_B$ by adjointness, and quotienting by $(1 - b^\vee)$ yields a morphism $A \longrightarrow X_A$. On the other hand, we have a morphism

$$X \otimes (\operatorname{Sym}^\bullet X^\vee)/(1 - b^\vee) = X_A \xrightarrow{b_A} 1_A,$$

and their composition is the identity of 1_A by construction. $\square$

In the proof of Proposition 12.8, instead of just punching out the zero section, we restricted to the "$\{b^\vee = 1\}$" locus (which is even stronger and works just as well).

12.10 Proposition. *Let M be a nonzero dualizable A-module. Then there exists a nonzero A-algebra B such that M_B is isomorphic to $1_B \oplus N$ for some B-module N.*

This is the Lemma 11.13 of last time, except I didn't know how to even state it properly back then. We shall use work from proving Proposition 12.10 to also prove the following result.

12.11 Corollary. *Let X be an object in $\mathscr{C}$. Then there exists a nonzero commutative ring object B in $\operatorname{Ind}\mathscr{C}$ such that X_B is isomorphic to $1_B^{\dim X}$.*

Proof of Proposition 12.10. We want to find an appropriate B as well as a commutative diagram

$$\begin{tikzcd} 1_B \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & M_B \arrow[r] & 1_B. \end{tikzcd}$$

Set $S \coloneqq \operatorname{Sym}_A^\bullet M^\vee$. Then we get a morphism $1_S \longrightarrow M_S$ as in the proof of Proposition 12.8, so now we just want a morphism $M_S \longrightarrow 1_S$. Briefly reverting to the notation of our motivational geometric setting, write $p^\vee : E^\vee \longrightarrow X$ for the total space of $\mathscr{E}^\vee$. Then we get a morphism of bundles $(p^\vee)^*E \longrightarrow \mathbb{A}^1_{E^\vee}$ via sending $(f, e) \mapsto (f, f(e))$, and we can further restrict to the incidence variety

$$Z \coloneqq \{(f, e) \in E^\vee \times_X E \mid f(e) = 1\} \subset E^\vee \times_X E = (p^\vee)^*E = p^*E^\vee$$

in order to obtain our desired commutative diagram.

To emulate this in the algebraic setting, write $S^\vee \coloneqq \operatorname{Sym}_A^\bullet M$, which we remark is generally not the dual of S. We obtain a morphism $M_{S^\vee} \longrightarrow 1_{S^\vee}$ of $S^\vee$-modules simply from the multiplication morphism $M \otimes \operatorname{Sym}_A^\bullet M \longrightarrow \operatorname{Sym}_A^\bullet M$. Next, note that we have a morphism $\delta : 1_A \longrightarrow M \otimes_A M^\vee$, form $C \coloneqq S \otimes_A S^\vee$, and set $B \coloneqq C/(1-\delta)$, where we construct $(1-\delta)$ as in the proof of Proposition 12.8. This B plays the role of our incidence variety.

The A-algebra B satisfies our section condition, but we have to check that it's nonzero. By tensoring $\delta : 1_A \longrightarrow M \otimes_A M^\vee$ with C and composing with the multiplication morphism, we obtain a morphism $\delta : C \longrightarrow C$ of C-modules. By observing that $B = C/(1-\delta)$ also equals the limit

$$\varinjlim \left(C \xrightarrow{\delta} C \xrightarrow{\delta} \cdots \right),$$

we see that it suffices to check that $\delta^n : C \longrightarrow C$ is nonzero for all n. Now $\delta : 1_A \longrightarrow M \otimes_A M^\vee$ and the morphisms $(\operatorname{Sym}_A^n M) \otimes_A (\operatorname{Sym}_A^n M^\vee) \longrightarrow (\operatorname{Sym}_A^{n+1} M) \otimes_A (\operatorname{Sym}_A^{n+1} M^\vee)$ obtained from δ are monomorphisms, so it suffices to check that the $(\operatorname{Sym}_A^n M) \otimes (\operatorname{Sym}_A^n M^\vee)$ are nonzero. But M is nonzero, so the argument used in proving 2) $\iff$ 3) in Theorem 11.2 shows that $\dim_A M$ is nonzero. Then Lemma 12.7 shows that

$$\dim_A \left((\operatorname{Sym}_A^n M) \otimes (\operatorname{Sym}_A^n M^\vee) \right) = \dim_A(\operatorname{Sym}_A^n M) \dim_A(\operatorname{Sym}_A^n M^\vee)$$

is nonzero, as desired. $\square$

Proof of Corollary 12.11. Begin with $A = 1$, and inductively apply Proposition 12.10 (using the transitivity of extension of scalars, the additivity of dimension, and the fact that $\dim_A 1_A = 1$) to obtain a nonzero commutative ring object B in $\operatorname{Ind}\mathscr{C}$ such that X_B is isomorphic to $1_B^{\dim X} \oplus N$, where $\dim_A N = 0$.

We now want to show that $N = 0$. Set $d := \dim X$. Checking dimensions and using Proposition 11.6 shows that $\bigwedge^{d+1} X = 0$. The usual formula for wedges of sums holds, and combining this with the compatibility of wedging with extension of scalars shows that

$$0 = \left(\bigwedge^{d+1} X\right)_B = \bigwedge\nolimits_B^{d+1}(X_B) = \bigwedge\nolimits_B^{d+1}(1_B^d \oplus N) = \bigoplus_{p+q=d+1} \bigwedge\nolimits_B^{p}(1_B^d) \otimes_B \bigwedge\nolimits_B^{q}(N).$$

The $(p, q) = (d, 1)$ term in the above sum is

$$\bigwedge\nolimits_B^{d}(1_B^d) \otimes_B N = 1_B \otimes_B N = N,$$

so we see that $N = 0$. □

We conclude by using our work to prove Theorem 12.1.

Proof of Theorem 12.1. Corollary 12.11 shows that every X in $\mathscr{C}$ is free after extending scalars to some nonzero commutative ring object A in $\operatorname{Ind}\mathscr{C}$, and Corollary 12.9 indicates that every short exact sequence splits after extending scalars in a similar fashion.

We can take these A in a direct system, allowing us to take $\mathcal{A} := \varinjlim A$. For any object X of $\mathscr{C}$, set $\omega(X) := \operatorname{Hom}_{\mathscr{A}}(1_{\mathscr{A}}, X_{\mathscr{A}})$, which is a module over $R := \operatorname{Hom}_{\mathscr{C}}(1, \mathscr{A}) = \operatorname{End}_{\mathscr{A}}(\mathscr{A})$. Then ω is exact and monoidal by construction. □

13 May 8, 2018

Maintain the notations of last time (so k is an algebraically closed field of characteristic zero). Let B be a commutative k-algebra, and let (B-mod) denote the category of B-modules with the symmetric monoidal structure given by $\otimes_B$.

13.1 Lemma. *Let M be a B-module. Then M is dualizable if and only if it is finite projective over B.*

Proof. Suppose that M is dualizable. For all B-modules N, we have $M^\vee \otimes_B N = \operatorname{Hom}_B(M, N)$. Taking $N = B$ shows that $M^\vee = \operatorname{Hom}_B(M, B)$, and taking $N = M$ yield $M^\vee \otimes_B M = \operatorname{Hom}_B(M, M)$. This allows us to identify id_M with $\sum_{i=1}^d m_i^\vee \otimes m_i$ for some $m_i^\vee$ in $M^\vee$ and m_i in M. The sum of the $m_i^\vee$ and m_i yield maps

$$M \longrightarrow B^d \longrightarrow M,$$

and the fact that $\sum_{i=1}^d m_i^\vee \otimes m_i$ corresponds to id_M means that the composite of the above two maps equals id_M as well. Thus M is a direct summand of B^d, as desired. Conversely, if M is finite projective over B, we already know how to form its dual (by, say, working on $\operatorname{Spec} B$). □

Next, we move towards a more geometric perspective. Write $X := \operatorname{Spec} B$, and let G be an affine algebraic group over k.

13.2 Definition. Let $\widetilde{X}$ be a faithfully flat scheme over X with a G_X-action. We say $\widetilde{X}$ is an *étale G-torsor* (or a *principal G-bundle*) if the map

$$\begin{aligned} \widetilde{X} \times_k G = \widetilde{X} \times_X G_X &\longrightarrow \widetilde{X} \times_X \widetilde{X} \\ (x, g) &\longmapsto (x, gx) \end{aligned}$$

is an isomorphism.

By fppf descent, Definition 13.2 coincides with the definition of étale G-torsors given in terms of local trivializations.

13.3 Proposition. *There exists a canonical equivalence of categories*

$$\{\textit{principal } G\textit{-bundles over } X\} \longleftrightarrow \left\{\textit{fiber functors } \operatorname{Rep}_k G \longrightarrow (B\textit{-mod})\right\}.$$

13.4 Remark. If we let X be an arbitrary k-scheme, then Proposition 13.3 remains true if we replace $(B$-mod$)$ with the category of quasi-coherent $\mathscr{O}_X$-modules, with the monoidal structure given by $\otimes_{\mathscr{O}_X}$.

Proof. In one direction, let $P \longrightarrow X$ be a principal G-bundle. Then we obtain a fiber functor

$$\begin{aligned} \omega_P : \operatorname{Rep}_k G &\longrightarrow (B\text{-mod}) \\ V &\longmapsto P \times_G V, \end{aligned}$$

where we view $P \times_G V$ as a geometric vector bundle over X and hence as a B-module via taking global sections.

In the other direction, let $\omega : \operatorname{Rep}_k G \longrightarrow (B\text{-mod})$ be a fiber functor. Write R for the regular representation $k[G]$ of G, and decompose it as $R = \varinjlim_i V_i$ for some V_i in $\operatorname{Rep}_k G$ as in Proposition 8.18. Extending ω to $\operatorname{Ind}(\operatorname{Rep}_k G)$,[18] the fact that ω preserves duals indicates that the $\omega(V_i)$ must be dualizable. Therefore Lemma 13.1 indicates that the $\omega(V_i)$ are finite projective and in particular flat over B, so we see that $\omega(R) = \varinjlim_i \omega(V_i)$ is also flat over over B. The (pointwise) multiplication map $R \otimes R \longrightarrow R$ yields a morphism $\omega(R) \otimes_B \omega(R) \longrightarrow \omega(R)$ upon applying ω, and this equips $\omega(R)$ with the structure of a commutative B-algebra. Take $P := \operatorname{Spec} \omega(R)$.

The map $p : P \longrightarrow X$ is flat, and furthermore I claim that it is faithfully flat. To see this, consider the short exact sequence

$$0 \longrightarrow k \longrightarrow R \longrightarrow R/k \longrightarrow 0$$

in $\operatorname{Ind}(\operatorname{Rep}_k G)$. The exactness of ω yields a short exact sequence

$$0 \longrightarrow B = \omega(k) \longrightarrow \omega(R) \longrightarrow \omega(R/k) \longrightarrow 0$$

of B-modules, and the same argument as the one for $\omega(R)$ shows that $\omega(R/k)$ is flat over B. Therefore tensoring with any B-module M yields an exact sequence

$$0 \longrightarrow M \longrightarrow \omega(R) \otimes_B M \longrightarrow \omega(R/k) \otimes_B M \longrightarrow 0.$$

In particular, we see that the vanishing of $\omega(R) \otimes_B M$ implies the vanishing of M.

We only need to verify one more condition. For any V in $\operatorname{Rep}_k G$, Corollary 11.12 indicates that $V \otimes R = R \otimes \underline{V}$ as objects of $\operatorname{Ind}(\operatorname{Rep}_k G)$, where $\underline{V}$ denotes the underlying vector space of V, and $R \otimes \underline{V}$ is as in Lemma 9.1. Taking limits shows that $R \otimes R = R \otimes \underline{R}$, and applying ω yields

$$\omega(R) \otimes_B \omega(R) = \omega(R) \otimes_k k[G].$$

Finally, taking Spec shows that $\operatorname{Spec} \omega(R) = P$ satisfies the isomorphism condition in Definition 13.2, as desired. $\square$

There's one more Tannakian proposition that we shall later find useful.

[18] In fancy words, this is the left Kan extension of ω to $\operatorname{Ind}(\operatorname{Rep}_k G)$.

13.5 Proposition. *Let $\omega : \operatorname{Rep}_k G \longrightarrow (B\text{-mod})$ be a fiber functor of the form ω_P for some principal G-bundle $P \longrightarrow X$. Let e be an element of $\operatorname{End}\omega$, and suppose that $e_{V_1\otimes V_2} = e_{V_1} \otimes \mathrm{id}_{V_2} + \mathrm{id}_{V_1} \otimes e_{V_2}$ for all V_1 and V_2 in $\operatorname{Rep}_k G$. Then there exists a unique section $e_{\mathfrak{g}}$ in $\Gamma(X, P \times_G \operatorname{Lie} G)$ such that e_V is induced by $e_{\mathfrak{g}}$ for all V in $\operatorname{Rep}_k G$.*

Let's gradually segue back to geometric Satake. From now on, let $k = \mathbb{C}$, and let T be an algebraic torus over k. Write $X^\bullet(T)$ for the weight lattice of T. Recall that $\operatorname{Rep}_k T$ is equivalent to the category of finite-dimensional $X^\bullet(T)$-graded vector spaces over k, where the grading is given by weight spaces.

13.6 Corollary. *Giving a fiber functor from $\operatorname{Rep}_k G$ to $X^\bullet(T)$-graded finite-dimensional vector spaces over k is the same as giving a morphism $T \longrightarrow G$ of algebraic groups.*

Proof. Apply Proposition 8.12 along with our above discussion on $\operatorname{Rep}_k T$. $\square$

Write K for $k(\!(z)\!)$, and write $\mathcal{O}$ for $k[\![z]\!]$. Say G is connected reductive, and write Gr_G for the affine Grassmannian of G over k. Write $G^\vee$ for the Langlands dual group of G. Recall that we want to construct a tensor category $\underline{\mathrm{Sat}}$ of certain sheaves on $\mathrm{Gr}_G(\mathbb{C})$, and we want $\underline{\mathrm{Sat}}$ to have a monoidal structure given by some sort of convolution. Finally, we want $\underline{\mathrm{Sat}}$ to be monoidally equivalent to $\operatorname{Rep}_k G^\vee$.

Let's first solve this problem in the case when $G = T$ is a (split) torus. In this situation, we have seen in Lecture 4 that

$$\mathrm{Gr}_T(k) = X_\bullet(T)$$

with the discrete topology, and this enables us to quickly solve our problem. Namely, let $\underline{\mathrm{Sat}}$ be the category of sheaves of finite-dimensional $\mathbb{C}$-vector spaces on $X_\bullet(T)$ (which then in particular must have finite support). Because $X_\bullet(T)$ is discrete, this is the data of a family $\mathcal{F} = \{\mathcal{F}_\lambda\}_{\lambda\in X_\bullet(T)}$, where the $\mathcal{F}_\lambda$ are objects of (Vect) that vanish for cofinitely many λ. In these terms, the global sections functor can be written as

$$\begin{aligned}\Gamma : \underline{\mathrm{Sat}} &\longrightarrow \operatorname{Rep}_k T^\vee\\ \mathcal{F} &\longmapsto \bigoplus_{\lambda\in X_\bullet(T)} \mathcal{F}_\lambda.\end{aligned}$$

We also want to define some sort of convolution product on $\underline{\mathrm{Sat}}$. For this, we can use the following fairly general construction: let H be a topological group, and write $m : H \times H \longrightarrow H$ for the multiplication map.

13.7 Definition. Let $\mathscr{F}$ and $\mathscr{F}'$ be sheaves of $\mathbb{C}$-vector spaces on H. Define their convolution to be

$$\mathscr{F} * \mathscr{F}' := m_*(\mathscr{F} \boxtimes \mathscr{F}'),$$

where we take all our operations to be derived.

Set $H = X_\bullet(T)$. Inspired by Definition 13.7, let $\mathcal{F}$ and $\mathcal{F}'$ be two objects of $\underline{\mathrm{Sat}}$, and write $\mathscr{F}$ and $\mathscr{F}'$ for the corresponding sheaves of finite-dimensional $\mathbb{C}$-vector spaces on $X_\bullet(T)$. We define their convolution via

$$(\mathcal{F} * \mathcal{F}')_\lambda := \Gamma(m^{-1}(\lambda), \mathscr{F} \boxtimes \mathscr{F}') = \bigoplus_{\substack{\mu,\nu\in X_\bullet(T)\\ \mu+\nu=\lambda}} \mathcal{F}_\mu \otimes \mathcal{F}'_\nu.$$

In other words, the sheaf of $\mathbb{C}$-vector spaces corresponding to $\mathcal{F} * \mathcal{F}'$ is $\mathscr{F} \boxtimes \mathscr{F}'$, so all we really needed to say was that $\mathscr{F} \boxtimes \mathscr{F}'$ remains a sheaf of finite-dimensional $\mathbb{C}$-vector spaces on $X_\bullet(T)$.[19] This convolution product immediately turns our functor $\Gamma : \underline{\mathrm{Sat}} \longrightarrow \operatorname{Rep}_k T^\vee$ into a monoidal equivalence, as desired.

[19] We can and will just work in the abelian rather than derived category in this case, since $X_\bullet(T)$ is discrete.

Return now to the case of an arbitrary connected reductive G, and choose a maximal torus T of G. Write $R \subset X^\bullet(T)$ for the root system of G corresponding to our choice of T, write $\mathfrak{t} := \operatorname{Lie} T$, and view $X^\bullet(T)$ as a subset of $\mathfrak{t}^*(\mathbb{C})$.

13.8 Definition. The *Langlands dual of* G, denoted by $G^\vee$, is the connected reductive group over k formed using the root datum $(R^\vee, X_\bullet(T), R, X^\bullet(T))$.

Note that $G^\vee$ comes with a maximal torus $T^\vee$ such that $X^\bullet(T^\vee) = X_\bullet(T)$ and $X_\bullet(T^\vee) = X^\bullet(T)$, though there is a choice involved in embedding $T^\vee \hookrightarrow G^\vee$.

By transporting along our hypothetical equivalence between $\underline{\mathrm{Sat}}$ and $\operatorname{Rep}_k G^\vee$, we would expect the Satake category $\underline{\mathrm{Sat}}$ to have the following properties and structures:

1) $\underline{\mathrm{Sat}}$ is semisimple, as Proposition 9.7 tells us that $\operatorname{Rep}_k G^\vee$ is semisimple,
2) $\underline{\mathrm{Sat}}$ is rigid symmetric monoidal, as $\operatorname{Rep}_k G^\vee$ is too,
3) we have a fiber functor $\underline{\mathrm{Sat}} \longrightarrow \operatorname{Rep}_k T^\vee$ (which comes from $T^\vee \hookrightarrow G^\vee$),
4) the isomorphism classes of simple objects of of $\underline{\mathrm{Sat}}$ are in bijection with dominant weights of $X^\bullet(T^\vee)$ (equivalently, dominant coweights of $X_\bullet(T)$).

Note that we can avoid the ambiguity of $T^\vee \hookrightarrow G^\vee$ by working with $X_\bullet(T)$-graded finite-dimensional vector spaces over k instead of $\operatorname{Rep}_k T^\vee$.

We begin by trying to tackle part 4). Recall that for any coweight λ of T, we obtain an element $z^\lambda := \lambda(z)$ of $G(K)$. If we write $\mathrm{Gr}_\lambda(k)$ for the $G(\mathcal{O})$-orbit of the image of z^λ in $\mathrm{Gr}_G(k) = G(K)/G(\mathcal{O})$, the Cartan decomposition yields a stratification

$$\mathrm{Gr}_G(k) = \coprod_{\lambda \in X_\bullet(T)^{\mathrm{dom}}} \mathrm{Gr}_\lambda(k).$$

Given that the pieces of this stratification biject with our desired classification of simple objects, it seems we should try to form said simple objects via constant sheaves supported on a given piece of the stratification. However, if you only use constructible sheaves, this is not well-behaved, as I shall explain below in Remark 13.10. Instead, one needs to use the category

$$\mathrm{Perv}_{G(\mathcal{O})\text{-constr}} := \left\{ \begin{array}{c} \text{perverse sheaves}^{20} \text{ on } \mathrm{Gr}_G(\mathbb{C}) \text{ that are } \mathbb{C}\text{-local systems on the} \\ \mathrm{Gr}_\lambda \text{ and are supported on finitely many strata} \end{array} \right\},$$

which lives inside the derived category $D^b_c(\mathrm{Gr}_G)$ of constructible sheaves over $\mathbb{C}$ on Gr_G that are supported on finitely many strata. The simple objects of $\mathrm{Perv}_{G(\mathcal{O})\text{-constr}}$ are the *intersection complexes* $\mathrm{IC}(\overline{\mathrm{Gr}_\lambda})$.

Let's now try to describe part 2), i.e. the convolution product. Fix a maximal compact subgroup K of $G(\mathbb{C})$, and recall our version of the loop group $\Omega(K)$ from Lecture 3. Theorem 3.10 indicates that $\Omega(K)$ is isomorphic to $\mathrm{Gr}_G(\mathbb{C})$ as ind-complex manifolds. Now $\Omega(K)$ has a group structure given by pointwise multiplication, so we can apply Definition 13.7 in this setting.

One of our ultimate goals is to prove the following result.

13.9 Theorem.

1) The category $\mathrm{Perv}_{G(\mathcal{O})\text{-constr}}$ *is semisimple,*
2) $\mathrm{Perv}_{G(\mathcal{O})\text{-constr}}$ *is stable under the convolution product defined above,*
3) The convolution product is symmetric,

4) *There exists a fiber functor from* $\mathrm{Perv}_{G(\mathcal{O})\text{-constr}}$ *to the category of* $X_\bullet(T)$*-graded finite-dimensional vector spaces over* k.

Note that this shows $\mathrm{Perv}_{G(\mathcal{O})\text{-constr}}$ has almost all the properties we want for $\underline{\mathrm{Sat}}$.

13.10 Remark.

- If you use constructible sheaves instead of perverse sheaves, you can't prove part 2) in Theorem 13.9. In order to get the associativity of the convolution product, one needs to derive the pushforward functor, in order to make it transitive. But once you derive, you lose your abelian category. To recover the abelian category (but while deriving), you need to turn to perverse sheaves. However, it still takes lots work to show that these perverse sheaves satisfy the properties we want.

- In order to obtain a fiber functor on $\mathrm{Perv}_{G(\mathcal{O})\text{-constr}}$, you could try to just take cohomology. This worked when we studied tori, and it even had a natural grading in that situation. However, in general we lose the grading because taking cohomology is sort of like integrating your sheaves on all of $G(\mathbb{C})$. On the other hand, recall from Lecture 5 that classical Satake involves integrating only on the maximal closed connected unipotent subgroup of G.

  So cohomology doesn't work. You could try to somehow finagle Gelfand's trick into this situation, but that ultimately also doesn't work (we will explore the reason why next time). The actual fiber functor is something entirely different and is due to Drinfeld.

14 May 10, 2018

I shall begin by explaining some simple things, before proceeding to the proof of part 1) in Theorem 13.9. Write $\mathscr{P}$ for the category $\mathrm{Perv}_{G(\mathcal{O})\text{-constr}}$ we introduced last time. Rather than giving you definitions and telling you what a perverse sheaf actually is, I will take an abstract approach and only tell you the properties of $\mathscr{P}$ necessary for proving our desired results. Let's start with some quick facts:

14.1 Proposition.

- *$\mathscr{P}$ is an abelian category,*
- *Any object in $\mathscr{P}$ has finite length,*
- *The simple objects of $\mathscr{P}$ are the* intersection complexes $\mathrm{IC}_\lambda := \mathrm{IC}(\overline{\mathrm{Gr}_\lambda})$.

This results from the theory of perverse sheaves.

- Recall from last time that we obtained a convolution product $*$ for sheaves of $\mathbb{C}$-vector spaces on $\mathrm{Gr}_G(\mathbb{C})$ after fixing a maximal compact subgroup K of $G(\mathbb{C})$ and identifying $\Omega(K)$ with $\mathrm{Gr}_G(\mathbb{C})$. While $*$ a priori also depends on the choice of K, we shall later give an intrinsic definition of convolution (which matches our first definition) that does not.[21]

- One can methodically check that our convolution product comes with associativity isomorphisms
$$(\mathscr{F} * \mathscr{F}') * \mathscr{F}'' \xrightarrow{\sim} \mathscr{F} * (\mathscr{F}' * \mathscr{F}''),$$
  and we offer an interesting method of doing so here. For any topological group K, we have a well-defined n-fold multiplication map $m_n : K^n \longrightarrow K$ (which uses the associativity of multiplication on K). Then one can define the convolution of the sheaves $\mathscr{F}_1, \dots, \mathscr{F}_n$ to be
$$\mathscr{F}_1 * \cdots * \mathscr{F}_n := (m_n)_*(\mathscr{F}_1 \boxtimes \cdots \boxtimes \mathscr{F}_n).$$
  From here, one deduces the associativity of $*$ from comparing with our original $n = 2$ case.

[21] See Lecture 17.

Let us introduce the functor

$$\begin{aligned}\mathscr{P} &\longrightarrow \{\mathbb{Z}\text{-graded finite-dimensional vector spaces over } \mathbb{C}\}\\ \mathscr{F} &\longmapsto H^\bullet(\mathscr{F}) := \bigoplus_{k=0}^{\infty} H^k(\mathrm{Gr}_G(\mathbb{C}), \mathscr{F}).\end{aligned}$$

Our $H^\bullet(\mathscr{F})$ is indeed finite-dimensional, since $\mathscr{F}$ is supported on finitely many strata. To see that $H^\bullet$ is monoidal, note that the Künneth formula yields

$$\begin{aligned}H^\bullet(\mathscr{F} * \mathscr{F}') = H^\bullet(m_*(\mathscr{F} \boxtimes \mathscr{F}')) &= \bigoplus_{k=0}^{\infty} H^k(\mathrm{Gr}_G(\mathbb{C}) \times \mathrm{Gr}_G(\mathbb{C}), \mathscr{F} \boxtimes \mathscr{F}')\\ &= \bigoplus_{p,q=0}^{\infty} H^p(\mathrm{Gr}_G(\mathbb{C}), \mathscr{F}) \otimes H^q(\mathrm{Gr}_G(\mathbb{C}), \mathscr{F}') = H^\bullet(\mathscr{F}) \otimes H^\bullet(\mathscr{F}'),\end{aligned}$$

as we are working with derived pushforwards.

Next, we would like to show that $*$ is symmetric (i.e. prove part 3) in Theorem 13.9). In the classical case, one option was to use Gelfand's trick as in the proof of Theorem 2.9. However, it won't work here—to see this, let's try it out anyways and observe what fails.

Let $\sigma : G \longrightarrow G$ be a Cartan anti-involution corresponding to the choice of our maximal compact subgroup K. Then $\sigma(K) = K$. Suppose additionally that $\sigma(z^\lambda) = z^\lambda$ for all λ in $X_\bullet(T)$. Now the anti-involution σ acts on $G(\!(z)\!)$ while preserving $G[\![z]\!]$, so it descends to an action on Gr_G.

Our assumption that σ fixes z^λ implies that σ preserves Gr_λ, so the theory of perverse sheaves tells us that $\sigma^* \mathrm{IC}_\lambda$ must be isomorphic (though not canonically) to IC_λ for any λ in $X_\bullet(T)^{\mathrm{dom}}$. If we assume that $\mathscr{P}$ is semisimple (i.e. part 1) in Theorem 13.9), then we can write any object $\mathscr{F}$ in $\mathscr{P}$ in the form

$$\mathscr{F} \approx \bigoplus_{\lambda} \mathrm{IC}_\lambda,$$

where λ ranges over a finite multiset valued in $X_\bullet(T)^{\mathrm{dom}}$. Summing the isomorphisms on intersection complexes yields an abstract isomorphism $c_{\mathscr{F}} : \sigma^*\mathscr{F} \xrightarrow{\sim} \mathscr{F}$.

Since the anti-involution σ preserves K, we get yields a natural identification $(\sigma^*\mathscr{F}) * (\sigma^*\mathscr{F}') = \sigma^*(\mathscr{F}' * \mathscr{F})$ for all $\mathscr{F}$ and $\mathscr{F}'$ in $\mathscr{P}$. Following Gelfand's trick, we now obtain an isomorphism

$$\mathscr{F} * \mathscr{F}' \xrightarrow{c_{\mathscr{F}}^{-1} * c_{\mathscr{F}'}^{-1}} (\sigma^*\mathscr{F}) * (\sigma^*\mathscr{F}') = \sigma^*(\mathscr{F}' * \mathscr{F}) \xrightarrow{c_{\mathscr{F}' * \mathscr{F}}} \mathscr{F}' * \mathscr{F}.$$

The main problem is to prove this isomorphism is natural in $\mathscr{F}$ and $\mathscr{F}'$. In order to probe the difficulties of this problem, let $i_\lambda : \{z^\lambda G(\mathcal{O})\} \hookrightarrow \mathrm{Gr}_G$ denote the inclusion map. The fact that σ fixes z^λ yields a canonical isomorphism $i_\lambda^*\sigma^*\mathscr{F} \xrightarrow{\sim} i_\lambda^*\mathscr{F}$, and composing with $i_\lambda^* c_{\mathscr{F}}$ gives us an isomorphism

$$i_\lambda^*\mathscr{F} \xrightarrow{i_\lambda^* c_{\mathscr{F}}^{-1}} i_\lambda^*\sigma^*\mathscr{F} = i_\lambda^*\mathscr{F}.$$

Since $\sigma^2 = \mathrm{id}$, if we want $c_{\mathscr{F}}$ to be natural, we should expect the above composition to be multiplication by ± 1 on any given degree.

When $\mathscr{F} = \mathrm{IC}_\lambda$, the pullback $i_\lambda^* \mathrm{IC}_\lambda$ is 1-dimensional over $\mathbb{C}$. In this case, it turns out that a single choice of ± 1 uniquely determines all of c_{IC_λ}. Complications arise from the fact the the choice of sign for c_{IC_λ} also affects the sign of

$$i_\mu^* \mathrm{IC}_\lambda \xrightarrow{i_\mu^* c_{\mathrm{IC}_\lambda}^{-1}} i_\mu^*\sigma^* \mathrm{IC}_\lambda = i_\mu^* \mathrm{IC}_\lambda$$

for all μ such that z^μ lies in $\overline{\mathrm{Gr}}_\lambda(k)$, and we somehow have to choose all the signs of the c_{IC_λ} in a way that is compatible with this dependency. What's worse is that our perverse sheaves $\mathscr{F}$ are complexes, so the signs can (and will) have different signs in different degrees.

Thus Gelfand's trick fails, and we must resort to a strategy of Drinfeld instead.

14.2 Remark. Along the lines considered above, one can boil this problem down to certain combinatorial identities. In general, these identities have only been proven by solving the problem itself, which appeals to Drinfeld's method.

Drinfeld's method itself becomes unavailable in the situation of mixed-characteristic affine Grassmannians from Lecture 4, but one can cleverly rectify this problem as follows: one can use the situation over $\mathbb{C}$ to prove our combinatorial identities of interest and then implement these identities directly in the mixed-characteristic case. One must do more work to show that the result behaves well, and this is exactly what Zhu does in his work on geometric Satake in the mixed-characteristic setting.

In order to work towards proving part 3) in Theorem 13.9, let's switch it up and work in an abstract setup. Let $\mathscr{A}$ be a k-linear abelian category such that

1) Every object in $\mathscr{A}$ has finite length,

2) The isomorphism classes simple objects of $\mathscr{A}$ are indexed by a finite set S (we denote the object corresponding to s in S using L_s),

3) The finite set S is equipped with a partial order $\leq$ (call a subset T of S *closed* if it is downwards closed, and for any subset T of S, write $\mathscr{A}_T$ for the smallest full subcategory of $\mathscr{A}$ that is stable by extensions and contains L_t for all t in T),

4) Given any s in S, there exist objects Δ_s and ∇_s in $\mathscr{A}$ along with morphisms $\Delta_s \twoheadrightarrow L_s$ and $L_s \hookrightarrow \nabla_s$ (where Δ_s and ∇_s are meant to mimic the Verma module and its dual, respectively),

5) For any closed subset T of S and maximal element s of S, Δ_s (respectively ∇_s) is an indecomposable projective (respectively injective) object of $\mathscr{A}_T$.

6) For any s in S, the objects $\ker(\Delta_s \twoheadrightarrow L_s)$ and $\operatorname{coker}(L_s \hookrightarrow \nabla_s)$ lie in $\mathscr{A}_{<s}$,

7) For all s and t in S, we have $\operatorname{Ext}^2_{\mathscr{A}}(\Delta_s, \nabla_t) = 0$.

It turns out that condition 7) can be deduced from the previous six conditions, but I will not prove this. A k-linear abelian category $\mathscr{A}$ satisfying conditions 1)–5) is often called a *highest weight category*.

14.3 Example. If you know anything about Bernstein–Gelfand–Gelfand's category $\mathbb{O}$, this is the prototypical example of a category satisfying conditions 1)–7).

14.4 Theorem. *For any k-linear abelian category $\mathscr{A}$ that satisfies conditions 1)–7),*

1) $\mathscr{A}$ has enough projectives as well as finite homological dimension,

2) Any projective object of $\mathscr{A}$ has a Verma flag*, i.e. it has a filtration whose successive subquotients are of the form Δ_s.*

14.5 Remark. Theorem 14.4 has the following corollaries:

1) Every injective object of $\mathscr{A}$ has a dual Verma flag,

2) *Bernstein–Gelfand–Gelfand reciprocity*: Let $P_s \twoheadrightarrow L_s$ be any projective cover of L_s. Then for any t in S, the multiplicity of ∇_t in P_s (where we define multiplicities by using a Verma flag of P_s from Theorem 14.4.2) equals the multiplicity of L_s in ∇_t. This is named after the analogous result for the category $\mathbb{O}$.

Let us connect the category theory of $\mathscr{A}$ to geometry. Let X be an algebraic variety over k, and let $X = \coprod_{s\in S} X_s$ be a stratification of X. Write $j_s : X_s \hookrightarrow X$ for the associated locally closed embeddings, and write $\overline{X}_s$ for the Zariski closure of X_s in X.

14.6 Lemma. *If X_s is isomorphic to $\mathbb{A}^N$, then $\overline{X}_s \smallsetminus X_s$ is a divisor in $\overline{X}_s$. In particular, j_s is an affine morphism.*

Next, we introduce a lemma that comprises a part of our black-boxing of perverse sheaves.

14.7 Lemma. *Let Y be a smooth variety over k. For any affine locally closed embedding $j : Y \hookrightarrow X$, the objects $j_*\mathbb{C}_Y[\dim Y]$ and $j_!\mathbb{C}_Y[\dim Y]$ are perverse sheaves on X.*

We shall apply our category theory to perverse sheaves as follows.

14.8 Theorem. *Let $X = \coprod_{s\in S} X_s$ be a stratification of X such that every X_s is isomorphic to $\mathbb{A}^{N_s}$ for some non-negative integer N_s. Write $\nabla_s \coloneqq (j_s)_*\mathbb{C}_{X_s}[N_s]$ and $\Delta_s \coloneqq (j_s)_!\mathbb{C}_{X_s}[N_s]$ for all s in S, and let $\mathscr{P}_S$ be the category of perverse sheaves on X that are $\mathbb{C}$-local systems along the stratification $X = \coprod_{s\in S} X_s$. Then*

1) $\mathscr{P}_S$ satisfies conditions 1)–7),

2) For all M and N in $\mathscr{P}_S$, we have an isomorphism $\mathrm{Ext}^i_{\mathscr{P}_S}(M, N) \xrightarrow{\sim} \mathrm{Ext}^i_{D^b_c(X)}(M, N)$ for all i.

15 May 15, 2018

Recall Theorem 14.8. Before proving it, let's make a few comments:

- A better statement would be that one has an equivalence of categories $D^b(\mathscr{P}_S) \xrightarrow{\sim} D^b_S(X)$, the latter of which refers to the full subcategory of $D^b_c(X)$ formed by complexes whose cohomology consists of local systems along the stratification S. However, this is *false*! The perverse sheaves in $\mathscr{P}_S$ are already in $D^b_S(X)$, but this inclusion does not extend naturally to a functor on $D^b(\mathscr{P}_S)$.

  One could instead refine this statement via *dg-enhancement* or via *filtered triangulated categories*. Indeed, the latter are thoroughly discussed near the beginning of Beilinson-Bernstein-Deligne. This is one of the major reasons why one cares about dg-categories.

- When we do not have enough projectives, we cannot define Ext groups in terms of projective resolutions. However, we can use the Yoneda definition of Ext groups instead, which is given as follows: $\mathrm{Ext}^i(M, N)$ is the set of isomorphism classes of short exact sequences

  $$0 \longrightarrow N \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_i \longrightarrow M \longrightarrow 0,$$

  where two such sequences are isomorphic if there exists a chain of commutative diagrams of the form

  $$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] \arrow[d, equal] & X_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & X_i \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & N \arrow[r] & Y_1 \arrow[r] & \cdots \arrow[r] & Y_i \arrow[r] & M \arrow[r] & 0, \end{tikzcd}$$

  linking them.

Proof of Theorem 14.8. Let us begin by verifying conditions 1)–7):

1) This is a general property of perverse sheaves.

2) This is another general property of perverse sheaves—the simple objects of $\mathscr{P}_S$ are given by $\mathrm{IC}(\overline{X}_s)$, as s ranges over S.

3) For any t and s in S, we say $t \leq s$ if X_t lies in $\overline{X}_s$.

4) This is also general property of perverse sheaves (where we take $\nabla_s := (j_s)_*\mathbb{C}_{X_s}[N_s]$ and $\Delta_s := (j_s)_!\mathbb{C}_{X_s}[N_s]$ as in the statement of the Theorem 14.8).

5) Let T be a closed subset of S. Then $Y := \bigcup_{s\in T} X_s$ is closed in X, and for any maximal element s of T, the stratum X_s is open in Y. Furthermore, the category $\mathscr{A}_T$ equals the category $\mathscr{P}_T$ of perverse sheaves on Y that are $\mathbb{C}$-local systems along the stratification given by T.

   It's a fact from the theory of perverse sheaves that ∇_s and Δ_s are indecomposable objects of $\mathscr{P}_T$. We shall only prove that Δ_s is projective in $\mathscr{P}_T$, as the proof that ∇_s is injective goes similarly. This amounts to showing that the functor

$$\mathrm{Hom}_{\mathscr{P}_T}(\Delta_s, -) = \mathrm{Hom}_{D^b_c(Y)}(\Delta_s, -) = \mathrm{Hom}_{D^b_c(Y)}((j_s)_!\mathbb{C}_{X_s}[N_s], -) = \mathrm{Hom}_{D^b_c(X_s)}(\mathbb{C}_{X_s}, j_s^*(-)[-N_s])$$

   is exact. But this follows from the projectivity of $\mathbb{C}_{X_s}$ and the exactness of $[-N_s]$ and j_s^*.

6) This is yet another general property coming from the theory of perverse sheaves.

7) For arbitrary t and s in S, the adjunction gives us

$$\mathrm{Ext}^i_{D^b_c(X)}((j_s)_!\mathcal{F}, (j_t)_*\mathcal{G}) = \mathrm{Ext}^i_{D^b_c(X_t)}((j_t^*(j_s)_!\mathcal{F}, \mathcal{G})$$

   for all $\mathcal{F}$ in $D^b_c(X_s)$, $\mathcal{G}$ in $D^b_c(X_t)$, and non-negative integers i. This vanishes unless $s = t$, because the X_s and X_t are disjoint for $s \neq t$. Furthermore, when $s = t$ and $\mathcal{F} = \mathcal{G} = \mathbb{C}_{X_s}$, this vanishes for $i \geq 1$ since X_s is isomorphic to $\mathbb{A}^{N_s}$.

   However, we want to prove this Ext vanishing in the category $\mathscr{P}_S$ rather than the category $D^b_c(X)$. To achieve this, we use the following lemma.

   15.1 Lemma. *Let $\mathscr{A}$ be a full abelian subcategory of a triangulated category $\mathscr{D}$ that is closed under extensions. Then the natural map*

$$\mathrm{Ext}^2_{\mathscr{A}}(M, N) \longrightarrow \mathrm{Ext}^2_{\mathscr{D}}(M, N)$$

   is injective for all M and N in $\mathscr{A}$.

   Proof of Lemma 15.1. How do we get these natural maps in the first place? The families $\{\mathrm{Ext}^i_{\mathscr{A}}(M, -)\}_i$ and $\{\mathrm{Ext}^i_{\mathscr{D}}(M, -)\}_i$ are δ-functors on $\mathscr{A}$, and $\{\mathrm{Ext}^i_{\mathscr{A}}(M, -)\}_i$ is universal. Therefore the identity map

$$\mathrm{Ext}^0_{\mathscr{A}}(M, N) = \mathrm{Hom}_{\mathscr{A}}(M, N) = \mathrm{Hom}_{\mathscr{D}}(M, N) = \mathrm{Ext}^0_{\mathscr{D}}(M, N)$$

   induces a map on all higher Ext groups. Suppose that we have a nonzero element e of $\mathrm{Ext}^2_{\mathscr{A}}(M, N)$. By using the Yoneda interpretation of $\mathrm{Ext}^2_{\mathscr{A}}(M, N)$, there exists a projective covering $P \twoheadrightarrow M$ in $\mathscr{A}$ such that the image of e in $\mathrm{Ext}^2_{\mathscr{A}}(P, N)$ is zero.[22]

[22]This step works for all $\mathrm{Ext}^i_{\mathscr{A}}$, not just the $i = 2$ case.

Form a short exact sequence $0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$, and consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}^1_{\mathscr{A}}(P,N) \arrow[r] \arrow[d, "\wr"] & \mathrm{Ext}^1_{\mathscr{A}}(K,N) \arrow[r] \arrow[d, "\wr"] & \mathrm{Ext}^2_{\mathscr{A}}(M,N) \arrow[r] \arrow[d] & \mathrm{Ext}^2_{\mathscr{A}}(P,N) \arrow[d] \\
\mathrm{Ext}^1_{\mathscr{D}}(P,N) \arrow[r] & \mathrm{Ext}^1_{\mathscr{D}}(K,N) \arrow[r] & \mathrm{Ext}^2_{\mathscr{D}}(M,N) \arrow[r] & \mathrm{Ext}^2_{\mathscr{D}}(P,N)
\end{tikzcd}$$

whose rows are induced long exact sequences.

Then e comes from an element of $\mathrm{Ext}^1_{\mathscr{A}}(K,N)$. □

Applying Lemma 15.1 immediately concludes the proof of 7).

Next, we want to show that $\mathrm{Ext}^i_{\mathscr{P}_S}(M,N) \xrightarrow{\sim} \mathrm{Ext}^i_{D^b_c(X)}(M,N)$:

- The proof of 7) indicates this holds for $M = \Delta_s$ and $N = \nabla_t$ for any s and t in S.
- Theorem ASDF indicates that every projective M has a Verma flag and every injective N has a dual Verma flag, so this holds for such M and N. ASDF
- To prove this for arbitrary M and injective N, we induct on the length ℓ of a projective resolution of M. The previous step takes care of the $\ell = 1$ case, and for the induction step let $P \twoheadrightarrow M$ be a projective covering of M. Then the short exact sequence

$$0 \longrightarrow M' \longrightarrow P \longrightarrow M \longrightarrow 0$$

induces a long exact sequence

$$\mathrm{Ext}^{\ell-1}_{\mathscr{A}}(M', I) ASDF$$

□

Recall that our goal is to prove that the Satake category $\mathscr{P}$ is semisimple. For this, we shall need to introduce *parity vanishing* for the Satake category. Fix a Borel subgroup B of G, and write $p : G[\![z]\!] \longrightarrow G$ for the reduction modulo z map (i.e. evaluation at $z = 0$).

15.2 Definition. The *Iwahori* subgroup $\mathcal{I}$ of $G[\![z]\!]$ is the inverse image of B under p, and the *(maximal) affine flag variety* is the fpqc quotient $\mathcal{B} := G(\!(z)\!)/\mathcal{I}$.

- The group $\mathcal{I}$ naturally acts on $\mathcal{B}$. It turns out that the $\mathcal{I}$-orbits of $\mathcal{B}$ are parameterized by the *affine Weyl group*, which is generally an infinite group. We denote the affine Weyl group using W_{aff}. There is a notion of length for elements w of W_{aff} as well, which we denote using $\ell(w)$.
- There exists an affine version of the Bruhat decomposition:

$$\mathcal{B} = \coprod_{w \in W_{\text{aff}}} \mathcal{B}_w,$$

where the $\mathcal{B}_w$ are locally closed subvarieties such that $\mathcal{B}_w$ is isomorphic to $\mathbb{A}^{\ell(w)}$. Write IC_w for the intersection complex $\mathrm{IC}(\overline{\mathcal{B}}_w)$.

15.3 Theorem. *For all y in $\overline{\mathcal{B}}_w$, we have $H^i(\mathrm{IC}_w)\big|_y = 0$ unless $i \equiv \ell(w) \pmod 2$.*

Proof. We shall use the *Bott–Samelson resolution* (which is also important elsewhere): begin by writing $w = s_{i_1} \cdots s_{i_n}$ be a reduced expression for w, where the s_{i_k} are simple reflections. In the case when $w = s$ itself is a simple reflection, one knows that

$$\mathbb{P}^1 = \overline{\mathcal{B}}_s = \underbrace{I/I}_{\text{a point}} \cup \underbrace{IsI/I}_{\mathbb{A}^1}.$$

ASDF MINIMAL "PARABOLIC" □

16 May 17, 2018

Today, we shall prove part 3) of Theorem 13.9 using our work from the last few days.

Proof of 3) in Theorem 13.9. This proceeds in seven steps:

Step 1. Consider the $\mathcal{I}(k)$-orbits on $\mathrm{Gr}_G(k)$. It is a fact that they form a stratification

$$\mathrm{Gr}_G(k) = \coprod_{s \in S} X_s(k),$$

where the X_s are varieties that are isomorphic to $\mathbb{A}^{N(s)}$ for some non-negative integers $N(s)$. Since $\mathcal{I}(k)$ is a subgroup of $G(\mathcal{O})$, we see that this stratification refines the Cartan decomposition. Therefore the category $\mathscr{P}_{\mathcal{I}(k)\text{-constr}}$ of perverse sheaves on $\mathrm{Gr}_G(\mathbb{C})$ that are constant along the $\mathcal{I}(\mathbb{C})$-stratification and supported on finitely many strata contains $\mathscr{P}$ as a full subcategory. ASDF SOMETHING ABOUT Ext^1 MATCHING

Step 2. The Cartan decomposition tells us that $G(\mathcal{O})$-orbits on $\mathrm{Gr}_G(k)$ are labeled by dominant coweights. Write $\mathrm{Gr}_\lambda(k)$ for the $G(\mathcal{O})$-orbit corresponding to λ in $X_\bullet(T)^{\mathrm{dom}}$. Then $\mathrm{Gr}_\lambda(k)$ is a finite union of $\mathcal{I}(k)$-orbits, so it contains a unique open dense $\mathcal{I}(k)$-orbit $\mathfrak{O}(k)$. Therefore we can write the simple objects of $\mathscr{P}$ as

$$\mathrm{IC}(\overline{\mathrm{Gr}_\lambda}(\mathbb{C})) = \mathrm{IC}(\overline{\mathfrak{O}}(\mathbb{C}))$$

for some $\mathcal{I}(k)$-orbits $\overline{\mathfrak{O}}(k)$.

Step 3. The connected components of $\mathrm{Gr}_G(k)$ are in bijection with $X_\bullet(T)/Q^\vee$, where $Q^\vee$ denotes the coroot lattice associated to our data (G, T). Furthermore, we have

$$\dim \mathrm{Gr}_\lambda(k) = \dim \mathrm{Gr}_\mu(k) \pmod 2$$

for any λ and μ in $X_\bullet(T)^{\mathrm{dom}}$ if and only if $\lambda - \mu$ lies in $Q^\vee$.

Step 4. The category $\mathscr{P}_{\mathcal{I}(k)\text{-constr}}$ satisfies the conditions of ASDF, so in particular we have

$$\mathrm{Ext}^i_{\mathscr{P}_{\mathcal{I}(k)\text{-constr}}}(M, N) \xrightarrow{\sim} \mathrm{Ext}^i_{D^b_c(\mathrm{Gr}_G(\mathbb{C}))}(M, N)$$

for all M and N in $\mathscr{P}_{\mathcal{I}(k)\text{-constr}}$ and non-negative integers i.

Step 5. ASDF

Step 6. Write $p : \mathcal{B} = G(\!(z)\!)/\mathcal{I} \longrightarrow G(\!(z)\!)/G[\![z]\!] = \mathrm{Gr}_G$ for the quotient map. As an aside, it is instructive to reinterpret p on $\mathbb{C}$-points in terms of our loop group $\Omega(K)$. The Iwasawa decomposition for complex Lie groups tells us that $T_c := B(\mathbb{C}) \cap K$ is a maximal (compact) torus of K, and we have $(G/B)(\mathbb{C}) = K/T_c$.

16.1 Definition. The space of *free loops in* K, denoted using $L(K)$, is the set of polynomial maps $g : S^1 \subset \mathbb{C}^\times \longrightarrow K$.

We have an isomorphism of ind-smooth (real) manifolds given by

$$\begin{aligned} L(K) &\xrightarrow{\sim} K \times \Omega(K) \\ g &\longmapsto (g(1), g(1)^{-1}g(\theta)). \end{aligned}$$

ASDF $\mathcal{B} = L(K)/T_c$ THING, SEE ASDF

Step 7. ASDF THE PAVING CAN BE PROVED INDUCTIVELY, STEMMING FROM THE FACT THAT $\mathbb{P}^1$ IS PAVED (SIMPLE REFLECTIONS, COXETER GROUPS) □

In order to proceed further with the proof of Theorem 13.9, we want to formulate our convolution product algebraically. Let K now denote any linear algebraic group over k, and let X be a variety over k with a K-action. Write $a : K \times X \longrightarrow X$ for the action map, and write $p : K \times X \longrightarrow X$ for the projection map. We begin by defining *equivariant sheaves* in an algebraic manner.

16.2 Definition. Let $\mathscr{F}$ be a constructible sheaf on X. A *K-equivariant structure on $\mathscr{F}$* is an isomorphism $\theta : a^*\mathscr{F} \xrightarrow{\sim} p^*\mathscr{F}$ such that

- $\theta|_{1\times X}$ is the identity on $\mathscr{F}$, where we naturally identify $a^*\mathscr{F}|_{1\times X}$ and $p^*\mathscr{F}|_{1\times X}$ with $\mathscr{F}$,
- Write $m : K \times K \longrightarrow K$ for the multiplication map. Then $(m \times \mathrm{id}_X)^* = \mathrm{pr}_{23}^*(\theta) \circ (\mathrm{id}_K \times a)^*(\theta)$, where $\mathrm{pr}_{23} : K \times K \times X \longrightarrow K \times X$ denotes projection onto the second and third components.

A constructible sheaf equipped with a K-equivariant structure is a *K-equivariant sheaf*.

While equivariance for functions was merely a property, for sheaves it requires additional data.

16.3 Remark.

1) The constant sheaf $\mathbb{C}_X$ on X has a canonical K-equivariant structure.

2) Assume that K is connected. Then if such a θ exists, it must be unique. ASDF USE CONSTRUCTIBILITY

3) Suppose that K is a closed subgroup of another linear algebraic group L over k. Then one can show that taking $L \times_K (-)$ yields an equivalence of categories

$$\{K\text{-equivariant sheaves on } X\} \xrightarrow{\sim} \{L\text{-equivariant sheaves on } L \times_K X\}$$

This process is analogous to (and generalizes) induction for representations.

4) In the other direction, we have a statement that is analogous to descent: let $p : P \longrightarrow X$ be a principal K-bundle. Then p^* induces an equivalence of categories

$$\{\text{constructible sheaves on } X\} \xrightarrow{\sim} \{K\text{-equivariant sheaves on } P\}.$$

Let us conclude with a remark on equivariant derived categories. The right way to think about equivariant derived categories is *not* to just take the derived category of equivariant sheaves, for the following reason: Remark 16.3.1) indicates that $\mathbb{C}_X$ is always equivariant. However, taking Ext groups of $\mathbb{C}_X$ would then result in normal cohomology, but we want the answer to be equivariant cohomology.

One possible fix for this is to impose Definition 16.2 on the level of objects in derived categories. However, one needs to extend this definition to include coherence of pullbacks to $K^n \times X$ for any n, because this coherence is no longer automatically implied by Definition 16.2 (and we certainly want this coherence to be true!). This infinite number of coherence relations is reminiscent of an ∞-categorical setup, which inspires us to turn to $B(K(\mathbb{C}))$.

Thus we are led to the following definition of Bernstein–Lusztig. Write $E(K(\mathbb{C})) \longrightarrow B(K(\mathbb{C}))$ for the universal bundle of $K(\mathbb{C})$. For any K-variety X, we can form the usual Borel construction $X_K := E(K(\mathbb{C})) \times_{K(\mathbb{C})} X(\mathbb{C})$ using $X(\mathbb{C})$ and $K(\mathbb{C})$. This yields arrows

$$\begin{tikzcd} & E(K(\mathbb{C})) \times X(\mathbb{C}) \arrow[dl, "q"'] \arrow[dr, "\mathrm{pr}_2"] & \\ X_K & & X(\mathbb{C}), \end{tikzcd}$$

where q is the quotient map, and pr_2 is projection to the second factor.

16.4 Definition. The category of *K-equivariant derived sheaves on X*, denoted using $D^b_K(X)$, has objects of the form $(\mathscr{F}, \mathscr{F}_K, \theta)$, where $\mathscr{F}$ lies in $D^b_c(X)$, $\mathscr{F}_K$ lies in $D^b_c(X_K)$, and θ is an isomorphism $q^*\mathscr{F}_K \xrightarrow{\sim} \mathrm{pr}_2^* \mathscr{F}$.

Note that Definition 16.4 does not work in the étale setting, since we have no easy access to $E(K(\mathbb{C}))$ or $B(K(\mathbb{C}))$. Furthermore, we remark that parts 1), 3), and 4) of Remark 16.3 are also valid in the derived setting.

Next time, I will discuss equivariant perverse sheaves.

17 May 22, 2018

Unfortunately, something has come up next week, so I won't be here then. Therefore this shall be the last week of classes, so I'll change my plans for what to cover. We'll start with alternative, choice-independent definition of convolution!

As before, let $k = \mathbb{C}$. Let L be a linear algebraic group over k, and let K be an algebraic subgroup of L over k. Write X for the quotient variety L/K. Recall that L has a left K-action via inverse right multiplication. Consider the important diagram

$$X \times X \xleftarrow{p \times \mathrm{id}_X} L \times X \xrightarrow{\quad q \quad} L \times_K X \xrightarrow{\quad a \quad} X,$$

where p and q are the canonical quotient morphisms, and a is the action morphism. In what follows, we will only work in the derived category, so assume that everything is derived.

Let $\mathscr{E}$ be a (derived) K-equivariant sheaf on X, and let $\mathscr{F}$ be a (derived constructible) sheaf on X. Since $p : L \longrightarrow X$ is a principal K-bundle, Remark 16.3.4) indicates that $p^*\mathscr{F}$ is a K-equivariant sheaf on X. Thus $p^*\mathscr{F} \boxtimes \mathscr{E}$ is a K-equivariant sheaf on the principal K-bundle $L \times X \longrightarrow L \times_K X$, so applying Remark 16.3.4) again yields a unique constructible sheaf $\mathscr{F} \widetilde{\boxtimes} \mathscr{G}$ on $L \times_K X$ such that $p^*\mathscr{F} \boxtimes \mathscr{E} = q^*(\mathscr{F} \widetilde{\boxtimes} \mathscr{E})$.

17.1 Definition. The *convolution* of $\mathscr{F}$ and $\mathscr{E}$ is the sheaf

$$\mathscr{F} * \mathscr{E} := a_*(\mathscr{F} \widetilde{\boxtimes} \mathscr{E}).$$

This convolution product yields a functor $* : D^b_c(X) \times D^b_K(X) \longrightarrow D^b_c(X)$.

17.2 Remark. Suppose that H is another algebraic subgroup of L. Then H acts on $X = L/K$ via left multiplication. Furthermore, if $\mathscr{F}$ is H-equivariant, then so is $\mathscr{F} * \mathscr{E}$, so the convolution product yields a functor $* : D^b_H(X) \times D^b_K(X) \longrightarrow D^b_H(X)$. (And we recover Definition 17.1 when $H = 1$).

Suppose we could endow $L(\mathbb{C})$, $K(\mathbb{C})$, and $H(\mathbb{C})$ with compatible Haar measures.[23] Then the convolution product would be a sheaf-theoretic analog of the convolution operation

$$C_c^\infty(H(\mathbb{C})\backslash L(\mathbb{C})/K(\mathbb{C})) \times C_c^\infty(H(\mathbb{C})\backslash L(\mathbb{C})/K(\mathbb{C})) \longrightarrow C_c^\infty(H(\mathbb{C})\backslash L(\mathbb{C})/K(\mathbb{C}))$$
$$(e,f) \mapsto \left[x \mapsto \int_{L(\mathbb{C})/K(\mathbb{C})} \mathrm{d}\ell\, e(x\ell) f(\ell^{-1})\right]$$

Anyways, when $H = K$, the convolution product for sheaves provides a monoidal structure on $D^b_K(X)$.

We will be able to make Remark 17.2 work in the setting of loop groups too, even though they are not varieties. Assuming that we can, taking $L = G(\!(z)\!)$ and $K = G[\![z]\!]$ would then give a monoidal structure on $D^b_{G(\mathcal{O})}(\mathrm{Gr}_G)$.

Let's now justify our ability to make Remark 17.2 work. Recall that the Cartan decomposition

$$\mathrm{Gr}_G(k) = \coprod_{\lambda \in X_\bullet(T)^{\mathrm{dom}}} \mathrm{Gr}_\lambda(k)$$

stratifies $\mathrm{Gr}_G(k)$ into quasi-projective varieties $\mathrm{Gr}_\lambda(k)$ over k, and every object in $D^b_{G(\mathcal{O})}(\mathrm{Gr}_G)$ is supported on a finite union of the Gr_λ by definition. So let X be one such finite union of strata.

Next, we need to form some level structure groups. For any positive integer n, write G_n for

$$G_n \coloneqq \ker\left(G[\![z]\!] \longrightarrow \mathrm{Res}_{k[\![z]\!]/(z^n)/k}\, G\right),$$

which is a fpqc subsheaf of $G[\![z]\!]$. These G_n thus form a descending filtration of normal subgroups of $G[\![z]\!]$, and we have the following fact.

17.3 Proposition. *There exists a positive integer n such that G_n acts trivially on X.*

Hence the $G[\![z]\!]$-action on X descends to an $G[\![z]\!]/G_n$-action on X, and this group is the $(n-1)$-th *jet group* of G. In particular, it is a linear algebraic group over k.

Let $\mathscr{E}$ be an object of $\mathrm{Perv}_{G(\mathcal{O})\text{-constr}}$, which is a full subcategory of $D^b_{G(\mathcal{O})}(\mathrm{Gr}_G)$, and set $X = \operatorname{supp} \mathscr{E}$. Consider the following analog of our important convolution diagram:

$$\mathrm{Gr}_G \times X \xleftarrow{p \times \mathrm{id}_X} \left(G(\!(z)\!)/G_n\right) \times X \xrightarrow{\quad q \quad} \left(G(\!(z)\!)/G_n\right) \times_{G[\![z]\!]/G_n} X \xrightarrow{\quad a \quad} \mathrm{Gr}_G,$$

where the p and q are the canonical quotient morphisms, and a is the action morphism. By replacing L with Gr_G and K with $G[\![z]\!]/G_n = \mathrm{Res}_{k[\![z]\!]/(z^n)/k}\, G$, it turns out that we can carry out Definition 17.1 in this setting.

That's all I want to say about the convolution product. At this point, I'd like to review the upshot of the geometric Satake equivalence and what you can do beyond it. Recall that geometric Satake itself gives a tensor equivalence

$$\mathbb{S} : (\mathrm{Rep}_k\, G^\vee, \otimes) \xrightarrow{\sim} (\mathrm{Perv}_{G(\mathcal{O})\text{-constr}}, *).$$

The right-hand side is super complicated, while the left-hand side is relatively simple. Thus we usually expand the right-hand side using the left-hand side.

[23] By this, I mean measures that induce a quotient measure on $H(\mathbb{C})\backslash L(\mathbb{C})/K(\mathbb{C})$ and $K(\mathbb{C})\backslash L(\mathbb{C})/K(\mathbb{C})$.

One possible extension of the geometric Satake equivalence is *equivariant derived Satake*. Write $\mathfrak{g}^\vee$ for the Lie algebra of $G^\vee$, which $G^\vee$ acts on via the adjoint representation. We may reproduce the strategy of Definition 16.4 to define the category $D^b_{G^\vee}\mathrm{Coh}(\mathfrak{g}^\vee)$ of $G^\vee$-equivariant derived coherent sheaves on $\mathfrak{g}^\vee$, and for any V in $\mathrm{Rep}_k\, G^\vee$, the tensor product $V \otimes_k \mathscr{O}_{\mathfrak{g}^\vee}$ is an object of said category.

17.4 Theorem (Equivariant derived Satake)**.** *There exists a natural equivalence of categories*

$$D\mathbb{S} : D^b_{G^\vee}\mathrm{Coh}(\mathfrak{g}^\vee) \xrightarrow{\sim} D^b_{G(\mathcal{O})}(\mathrm{Gr}_G)$$

that sends $V \otimes_k \mathscr{O}_{\mathfrak{g}^\vee}$ to $\mathbb{S}(V)$ for all V in $\mathrm{Rep}_k\, G^\vee$.

Before we proceed further, let us make the following digression on monoidal categories. Let $(\mathscr{D}, *)$ be a monoidal category (which we always take to be abelian, essentially small, and k-linear). For any objects M and N in $\mathscr{D}$, there exists a natural pairing

$$\mathrm{Ext}^i_{\mathscr{D}}(1, M) \times \mathrm{Ext}^j_{\mathscr{D}}(1, N) \longrightarrow \mathrm{Ext}^{i+j}_{\mathscr{D}}(1, M * N)$$

given as follows. For any α in $\mathrm{Ext}^i_{\mathscr{D}}(1, N)$, ASDF applying the functor $M * (-)$ yields an element of $\mathrm{Ext}^i_{\mathscr{D}}(M * 1, M * N)$. Then just take the Yoneda product.

In particular, when R is a ring object in $\mathscr{D}$, we obtain a graded algebra $\mathcal{R} := \bigoplus_{i=0}^\infty \mathrm{Ext}^i(1, R)$. Furthermore, for any M in $\mathscr{D}$, the module $\bigoplus_{i=0}^\infty \mathrm{Ext}^i_{\mathscr{D}}(1, R * M)$ has the natural structure of a graded $\mathcal{R}$-module.

ASDF How do we get ring objects in our rep setting?

ASDF For $\lambda = 0$, $\mathrm{Gr}_\lambda(k)$ is a point, so therefore IC_λ is the skyscraper at this point. Now the irrep corresponding to λ is the trivial rep, so $\mathbb{S}(triv) = \mathbb{C}_{\{1\}}$. ASDF THIS IS THE UNIT OF CONVOLUTION, SINCE $\mathbb{S}$ PRESERVES THE PRODUCTS

17.5 Theorem.

1) There is a natural isomorphism of graded k-algebras

$$\mathbb{C}[\mathfrak{g}^\vee] \xrightarrow{\sim} \mathcal{R},$$

where degree n polynomials on $\mathfrak{g}^\vee$ are sent to $\mathrm{Ext}^{2n}_{G(\mathcal{O})}(1, R)$ ASDF

2) For all V in $\mathrm{Rep}_k\, G^\vee$, there exists a natural graded R-module isomorphism

$$V \otimes \mathbb{C}[\mathfrak{g}^\vee] \xrightarrow{\sim} \mathrm{Ext}^\bullet_{G(\mathcal{O})}(1, \mathbb{S}(V) * R).$$

I will not prove the theorem, but I shall construct the maps. MASSIVELY ASDF

Let's deduce some facts about the cohomology of $\mathrm{Gr}_G(\mathbb{C})$. After fixing a maximal compact subgroup K of $G(\mathbb{C})$, we can replace $\mathrm{Gr}_G(\mathbb{C})$ with $\Omega(K)$. Since $\Omega(K)$ is a topological group, its cohomology ring $H^\bullet(\Omega(K), \mathbb{C})$ is a Hopf algebra. In fact, it is known that

$$H^\bullet(\Omega(K), \mathbb{C}) = \mathrm{Sym}(\mathbb{C} \otimes_{\mathbb{Z}} \pi_\bullet(\Omega(K))^*) = \mathrm{Sym}(\mathbb{C} \otimes_{\mathbb{Z}} \pi_{\bullet-1}(\Omega(K))^*)$$

ASDF primitive elements of Hopf algebras (recall it means $\nabla(a) = a \otimes 1 + 1 \otimes a$), the above sym is generated by primitive elements

c in $H^2_{G(\mathcal{O})}(\mathrm{Gr})$ be $c_1(\det bundle)$

COMBINING TANNAKIAN FORMALISM WITH THIS BEAUTIFUL CONSTRUCTION

18 May 24, 2018

Today, I will talk about Drinfeld's method of proving that the convolution product is symmetric. This strategy revolves using *Beilinson–Drinfeld Grassmannian*, which is a space that pieces together affine Grassmannians in families. To explain this, let us first consider the case when $G = \mathrm{GL}_n$. Recall Recall that we may then interpret $\mathrm{Gr}_G(k)$ as the set of $\mathcal{O}$-lattices L in K^n.

The affine Grassmannian is inherently a local object—if we had a curve over k with a parameter z, our local setting roughly corresponds to the neighborhood of point $z = 0$. But what if we worked over the entire curve, say, $X := \operatorname{Spec} k[z]$ instead? We can begin with the following naive analog of $\mathrm{Gr}_G(k)$.

18.1 Definition. The set of k-points of *Ran space* over X is

$$\mathrm{Gr}^{\mathrm{BD}}(k) := \left\{\text{finitely generated } k[z]\text{-submodules } L \text{ of } k(z)^n \text{ such that } k(z) \otimes_{k[t]} L = k(z)^n\right\}.$$

As in the local setting, we have a standard lattice $L_0 := k[t]^n$ in $\mathrm{Gr}^{\mathrm{BD}}(k)$.

We will not use Ran space. Instead, we add the following restriction. Let ℓ be a positive integer.

18.2 Definition. The set of k-points of the *Beilinson–Drinfeld Grassmannian* over X is

$$\mathrm{Gr}^{\mathrm{BD}}_{X^\ell}(k) := \left\{(L, z_1, \dots, z_\ell) \in \mathrm{Gr}^{\mathrm{BD}}(k) \times k^\ell \text{ such that } L\left[\tfrac{1}{(z-z_1)\cdots(z-z_\ell)}\right] = L_0\left[\tfrac{1}{(z-z_1)\cdots(z-z_\ell)}\right]\right\}.$$

Note that $\mathrm{Gr}^{\mathrm{BD}}(k) = \bigcup_{\ell=1}^\infty \mathrm{Gr}^{\mathrm{BD}}_{X^\ell}(k)$, since one only has to invert finitely many irreducible polynomials in order to trivialize L. In practice, we shall usually only deal with $\ell = 2$.

Let us now view the z_i as points of X. Then we have a canonical map

$$\begin{aligned} \pi_\ell : \mathrm{Gr}^{\mathrm{BD}}_{X^\ell}(k) &\longrightarrow X^\ell(k) \\ (L, z_1, \dots, z_\ell) &\longmapsto (z_1, \dots, z_\ell). \end{aligned}$$

Upon inspecting the preimage of π_ℓ on the *principal diagonal* $\Delta_{\text{prin}}(k)$ of $X^\ell(k)$ (i.e. the image of the diagonal map $X(k) \longrightarrow X^\ell(k)$), Proposition 2.6 indicates that

$$\pi_\ell^{-1}(z, \dots, z) = \mathrm{Gr}_G(k).$$

for any z in $X(k)$. From here, one can show that $\pi_\ell^{-1}(\Delta_{\text{prin}}(k)) = \mathrm{Gr}_G(k) \times X(k)$.

Now let's try to study the fibers of π_ℓ on general points of $X^\ell(k)$. Let L be a point in $\pi_\ell^{-1}(z_1, \dots, z_\ell)$, and suppose that L is contained in L_0. Since $k[z]$ is a PID, the fact that L lies in $\pi_\ell^{-1}(z_1, \dots, z_\ell)$ and the theory of elementary divisors yields

$$L_0/L \approx k[z]/(z - z_1)^{k_1} \oplus \cdots \oplus k[z]/(z - z_\ell)^{k_\ell}$$

for some non-negative integers $k_1, \dots, k_\ell$. Our intuition is that if the $z_1, \dots, z_\ell$ are pairwise distinct, then these elementary factors are "independent."

18.3 Proposition. *If the $z_1, \dots, z_\ell$ are pairwise distinct, we have*[24]

$$\pi_\ell^{-1}(z_1, \dots, z_\ell) = \pi_1^{-1}(z_1) \times \cdots \times \pi_1^{-1}(z_\ell) = \underbrace{\mathrm{Gr}_G(k) \times \cdots \times \mathrm{Gr}_G(k)}_{\ell \text{ times}}.$$

[24] This is a corollary of Proposition 18.6.

Observe that we have an action of $\mathbb{G}_a(k)$ on $\mathrm{Gr}^{\mathrm{BD}}_{X^\ell}(k)$ as follows. For any t in $\mathbb{G}_a(k)$, we obtain an automorphism of $k(z)$ via substituting $z \mapsto z + t$, and this induces an action on $k(z)^n$ via applying it to every coordinate.

At this point, set $\ell = 2$ and write $z = z_1 - z_2$ for a coordinate on the antidiagonal

$$X_{\text{anti}} \coloneqq \{(y, -y)\} \subset X^2,$$

which is isomorphic to X. Write $\mathrm{Gr}^{\mathrm{BD}}_{-}(k)$ for the inverse image of $X_{\text{anti}}(k)$ under π_2. By further taking pullbacks, Proposition 18.3 and the comments preceding it give us a diagram

$$\begin{tikzcd}
\mathrm{Gr}_G(k) \arrow[r, hook, "i"] \arrow[d] & \mathrm{Gr}^{\mathrm{BD}}_{-}(k) \arrow[d] & \mathrm{Gr}_G(k) \times \mathrm{Gr}_G(k) \times \mathbb{G}_m(k) \arrow[l, hook', "j"'] \arrow[d] \\
\{0\} \arrow[r, hook] & X(k) & \mathbb{G}_m(k), \arrow[l, hook']
\end{tikzcd}$$

where all the vertical arrows are induced by π_2. We shall use this diagram in the proof of Theorem 18.8.

18.4 Remark. The Beilinson–Drinfeld Grassmannian gives an example of a morphism of algebro-geometric objects for which the general fiber is larger than fibers of special loci. This is something that *never* happens in classical algebraic geometry—it's always the other way around! But this is not a contradiction, since we're dealing with very infinite-dimensional geometric objects.

For any z in $X(k)$, write K_z for the completion of $k(X)$ at z, and write $\mathcal{O}_z$ for its ring of integers. Any point $(L, z_1, \dots, z_\ell)$ of $\mathrm{Gr}^{\mathrm{BD}}_{X^\ell}(k)$ yields the data of a cocharacter of (a fixed maximal torus of) GL_n at each of the z_i, via comparing the relative positions of L and L_0. When the z_i are pairwise distinct, this yields ℓ pieces of data, but as two distinct z_i and $z_{i'}$ converge to one another, their corresponding pieces of data merge together.

The merging of this data and investigation of how they collide is sometimes called *fusion*, and it's at the heart of our convolution product. And all it really takes is a lack of fear of infinite-dimensional varieties and the new behavior they bring.

The $G = \mathrm{GL}_n$ and $X = \mathbb{A}^1$ case above is crucial for our general understanding, and let us now go to the general case: let G be an arbitrary connected reductive group over k, and let X be a smooth curve over k. For any closed point x of X, an analog of Proposition 2.6 in this setting shows that

$$\mathrm{Gr}_G(k) = \left\{(\mathcal{F}, \nu) \mid \mathcal{F} \text{ is a } G\text{-bundle on } X, \text{ and } \nu \text{ is a trivialization of } \mathcal{F} \text{ on } X \smallsetminus x\right\}.$$

This immediately suggests how to expand our definition to the Beilinson–Drinfeld Grassmannian.

18.5 Definition. The *Beilinson–Drinfeld Grassmannian* is the stack defined by

$$\mathrm{Gr}^{\mathrm{BD}}_{X^\ell} \coloneqq \left\{(\mathcal{F}, \nu, \underline{x} = (x_1, \dots, x_\ell)) \,\middle|\, \mathcal{F} \text{ is a } G\text{-bundle on } X \text{ and } \nu \text{ is a trivialization of } \mathcal{F} \text{ on } X \smallsetminus \underline{x}\right\},$$

where we write $X \smallsetminus \underline{x}$ for $X \smallsetminus \{x_1, \dots, x_\ell\}$. Write $\pi_\ell : \mathrm{Gr}^{\mathrm{BD}}_{X^\ell} \longrightarrow X^\ell$ for the morphism that sends

$$(\mathcal{F}, \nu, \underline{x}) \mapsto \underline{x},$$

and write Gr_X for $\mathrm{Gr}^{\mathrm{BD}}_X$.

- When $X = \mathbb{A}^1$, it is a fact that Gr_X is isomorphic to $\mathrm{Gr}_G \times \mathbb{A}^1$.
- Note that $\pi_\ell^{-1}(x, \dots, x) = \mathrm{Gr}_X$.

Let us now discuss the crucial *factorization* property of Beilinson–Drinfeld Grassmannians.

18.6 Proposition. *Suppose that x_1 and x_2 are distinct closed points of X.*[25] *Then we have a natural isomorphism*

$$\pi_2^{-1}(x_1, x_2) \xrightarrow{\sim} \pi_1^{-1}(x_1) \times \pi_2^{-1}(x_2).$$

Proof. Let's first describe a map going from the right-hand side to the left-hand side. Given a G-bundle $\mathcal{F}_1$ on an open subset $U_1 \subseteq X$ and another G-bundle $\mathcal{F}_2$ on an open subset $U_2 \subseteq X$, we can form a G-bundle on $U_1 \cup U_2$ once we're given an isomorphism $\nu : \mathcal{F}_1|_{U_1 \cap U_2} \xrightarrow{\sim} \mathcal{F}_2|_{U_1 \cap U_2}$. In this setting, the data $((\mathcal{F}_1, \nu_1, x_1), (\mathcal{F}_2, \nu_2, x_2))$ of the right-hand side yields the desired ingredients, by taking U_i to be $X \smallsetminus x_i$ and ν to be $\nu_2 \circ \nu_1^{-1}$.

Let's now describe a map going from the left-hand side to the right-hand side. For any point $(\mathcal{F}, \nu, (x_1, x_2))$ of $\pi_2^{-1}(x_1, x_2)$, form $\mathcal{F}_1$ by gluing the trivial bundle on $X \smallsetminus x_1$ with $\mathcal{F}|_{X \smallsetminus x_2}$ via ν, and form $\mathcal{F}_2$ by gluing the trivial bundle on $X \smallsetminus x_2$ with $\mathcal{F}|_{X \smallsetminus x_1}$ via ν^{-1}. The use of ν^{-1} here ensures that these two maps are indeed mutually inverse, completing our proof. □

Finally, we shall describe how to use Beilinson–Drinfeld Grassmannians to define the convolution product. Write $\mathcal{D}_x := \operatorname{Spf} \mathcal{O}_x$ and $\mathring{\mathcal{D}}_x := \operatorname{Spec} K_x$. The manipulations used to prove Proposition 2.6 allow us to identify

$$G(K_x) = \left\{ (\mathcal{F}, \nu, \mu) \;\middle|\; \begin{array}{c} \mathcal{F} \text{ is a } G\text{-bundle on } X,\ \nu \text{ is a trivialization of } \mathcal{F} \text{ on} \\ X \smallsetminus x, \text{ and } \mu \text{ is a trivialization of } \mathcal{F} \text{ on } \mathcal{D}_x \end{array} \right\} \Big/ \sim .$$

This description of $G(K_x)$ inspires us to make the following version of our convolution diagram from Lecture 17:

$$\mathrm{Gr}_X \times \mathrm{Gr}_X \xleftarrow{\ p\ } \widetilde{\mathrm{Gr}_X \times \mathrm{Gr}_X} \xrightarrow{\ q\ } \mathrm{Gr}_X \mathbin{\widetilde{\times}} \mathrm{Gr}_X \xrightarrow{\ m\ } \mathrm{Gr}^{\mathrm{BD}}_{X^2},$$

where our stacks in question are

$$\widetilde{\mathrm{Gr}_X \times \mathrm{Gr}_X} := \left\{ (\mathcal{F}_1, \mathcal{F}_2, x_1, x_2, \nu_1, \nu_2, \mu_1) \;\middle|\; \begin{array}{c} \text{the } \mathcal{F}_i \text{ are } G\text{-bundles on } X, \text{ the } x_i \text{ are points of } X, \text{ the } \nu_i \text{ are} \\ \text{trivializations of } \mathcal{F}_i \text{ on } X \smallsetminus x_i, \text{ and } \mu_1 \text{ is a trivialization of } \mathcal{F}_1 \text{ on } \mathcal{D}_{x_2} \end{array} \right\}$$

$$\mathrm{Gr}_X \mathbin{\widetilde{\times}} \mathrm{Gr}_X := \left\{ (\mathcal{F}_1, \mathcal{F}_2, x_1, x_2, \nu_1, \eta) \;\middle|\; \begin{array}{c} \text{the } \mathcal{F}_i \text{ are } G\text{-bundles on } X, \text{ the } x_i \text{ are points of } X, \nu_1 \text{ is a trivialization} \\ \text{of } \mathcal{F}_1 \text{ on } X \smallsetminus x_1, \text{ and } \eta \text{ is an isomorphism } \mathcal{F}_1|_{X \smallsetminus x_2} \xrightarrow{\sim} \mathcal{F}_2|_{X \smallsetminus x_2} \end{array} \right\},$$

and the morphisms are given as follows. The morphism p sends

$$(\mathcal{F}_1, \mathcal{F}_2, x_1, x_2, \nu_1, \nu_2, \mu_1) \mapsto ((\mathcal{F}_1, x_1, \nu_1), (\mathcal{F}_2, x_2, \nu_2)),$$

the morphism m sends

$$(\mathcal{F}_1, \mathcal{F}_2, x_1, x_2, \nu_1, \eta) \mapsto (\mathcal{F}_1, \nu_1, (x_1, x_2)),$$

and the morphism q sends

$$(\mathcal{F}_1, \mathcal{F}_2, x_1, x_2, \nu_1, \nu_2, \mu_1) \mapsto (\mathcal{F}_1, \mathcal{F}_2', x_1, x_2, \nu_1, \eta),$$

where $\mathcal{F}_2'$ is obtained from gluing $\mathcal{F}_1|_{X \smallsetminus x_2}$ and $\mathcal{F}_2|_{\mathcal{D}_{x_2}}$ with the isomorphism $\nu_2|_{\mathring{\mathcal{D}}_{x_2}} \circ \mu_1|_{\mathring{\mathcal{D}}_{x_2}}^{-1}$ (this is accomplished using the Beauville–Laszlo theorem), and η is the identity morphism. It turns out that one can use this convolution diagram to implement Definition 17.1, giving us a convolution product.

Now that we have all this notation, we shall now introduce some theorems. The first one is due to Gaitsgory, and we've been using it implicitly for many lectures now.

[25] When interpreting this for general S-valued points, we want x_1 and x_2 to have disjoint graphs in $X \times S$.

18.7 Theorem (Gaitsgory)**.** *Let $\mathscr{F}$ be a perverse sheaf on Gr_G, and let $\mathscr{E}$ be an object of the Satake category $\underline{\mathrm{Sat}} = \mathrm{Perv}_{G(\mathcal{O})\text{-constr}}$. Then $\mathscr{F} * \mathscr{G}$ remains a perverse sheaf on Gr_G.*

Combining Theorem 18.7 with Remark 16.3 then shows that $\underline{\mathrm{Sat}}$ is closed under the convolution product. The proof of Theorem 18.7 uses the Beilinson–Drinfeld Grassmannian, so it's a global proof of a local fact.

Beilinson–Drinfeld Grassmannians also enable us to prove the following purely local fact.

18.8 Theorem. *The monoidal structure $(\mathrm{Perv}_{G(\mathcal{O})\text{-constr}}, *)$ is symmetric.*

Note that we have been taking Theorem 18.8 for granted as well for many lectures now.

Sketch of the proof. The proof is insensitive to the choice of curve X, so we might as well take $X = \mathbb{A}^1$. The goal is to construct a canonical isomorphism $\mathscr{E}_1 * \mathscr{E}_2 \xrightarrow{\sim} \mathscr{E}_2 * \mathscr{E}_1$ for all $\mathscr{E}_1$ and $\mathscr{E}_2$ in $\underline{\mathrm{Sat}}$, and we begin by observing that

$$(\mathcal{F}, \nu, (x_1, x_2)) \mapsto (\mathcal{F}, \nu, (x_2, x_1))$$

yields an automorphism σ of $\mathrm{Gr}^{\mathrm{BD}}_{X^2}$. The diagram

$$\begin{tikzcd} & \mathrm{Gr}_G \arrow[dl, hook', "i"'] \arrow[dr, hook, "i"] & \\ \mathrm{Gr}^{\mathrm{BD}}_{X^2} \arrow[rr, "\sigma"] & & \mathrm{Gr}^{\mathrm{BD}}_{X^2} \end{tikzcd}$$

commutes, where i is from (a geometrized version of) our earlier diagram:

$$\begin{tikzcd} \mathrm{Gr}_G \arrow[r, hook, "i"] \arrow[d] & \mathrm{Gr}^{\mathrm{BD}}_{-} \arrow[d] & \mathrm{Gr}_G \times \mathrm{Gr}_G \times \mathbb{G}_m \arrow[l, hook', "j"'] \arrow[d] \\ \{0\} \arrow[r, hook] & X & \mathbb{G}_m. \arrow[l, hook'] \end{tikzcd}$$

We see that σ acts on the top row of this diagram. It turns out that there exists a functor can from perverse sheaves on $\mathrm{Gr}_G \times \mathrm{Gr}_G \times \mathbb{G}_m$ to perverse sheaves on $\mathrm{Gr}^{\mathrm{BD}}_{-}$ that commutes with σ^* and that satisfies

$$\mathscr{F} * \mathscr{E} = i^* \operatorname{can}(\mathscr{F} \boxtimes \mathscr{E} \boxtimes \mathbb{C}_{\mathbb{G}_m}).$$

As $i^* = i^* \sigma^*$, proceeding from here gives us

$$i^* \operatorname{can}(\mathscr{F} \boxtimes \mathscr{E} \boxtimes \mathbb{C}_{\mathbb{G}_m}) = i^* \sigma^* \operatorname{can}(\mathscr{F} \boxtimes \mathscr{E} \boxtimes \mathbb{C}_{\mathbb{G}_m}) = i^* \operatorname{can}(\mathscr{E} \boxtimes \mathscr{F} \boxtimes \mathbb{C}_{\mathbb{G}_m}) = \mathscr{E} * \mathscr{F}$$

as desired! □

Let us say a few words about what can is. Beilinson–Drinfeld defined it to be the perverse *intermediate extension* functor $\mathrm{can} = j_{!*}$. Gaitsgory has an alternative definition in terms of nearby cycles along the open embedding j, using the fact that nearby cycles preserve perversity.