Some Almost Commutative Algebra

Siyan Daniel Li

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Our setup

Let R be a ring, and let I be a flat ideal of R such that $I^2 = I$.

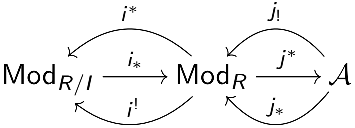
Example

- Let R be any ring, and let $I = R$.
- Choose a nonarchimedean field K with non-discrete value group (e.g. a perfectoid field), let $R = K^\circ$, and let $I = K^{\circ\circ}$.
- Let R be a perfect $\mathbb{F}_p$-algebra, choose $f \in R$, and let $I = (f^{1/p^\infty})$.

Write $\mathcal{A}$ for the full subcategory of Mod_R spanned by M such that $I \otimes_R M \to M$ is an isomorphism. Recall that $\mathcal{A}$ is an abelian category, with kernels and cokernels computed in Mod_R.

Pullbacks and pushforwards

Recall that we have a diagram of categories



where the functors are given by

$$i^* M = R/I \otimes_R M, \qquad j_! N = N,$$
$$i_* L = L, \qquad j^* M = I \otimes_R M,$$
$$i^! M = \mathrm{Hom}_R(R/I, M), \qquad j_* N = \mathrm{Hom}_R(I, N).$$

The composition of any row is zero, any functor is right adjoint to the one above it, $j_!$ is exact, and i_*, $j_!$, and j_* are fully faithful.

Geometric intuition

Write $X = \operatorname{Spec} R$, and write $Z = \operatorname{Spec} R/I$. Note that $\mathrm{Mod}_R = \mathrm{QCoh}(X)$ and $\mathrm{Mod}_{R/I} = \mathrm{QCoh}(Z)$.

Write $U = X \smallsetminus Z$. Now $\mathcal{A}$ is *not* the category $\mathrm{QCoh}(U)$, but think of $\mathcal{A}$ as $\mathrm{QCoh}(\overline{U})$ for some (probably imaginary) open subset $\overline{U} \supseteq U$ that remains disjoint from Z. So think of our diagram of categories as

$$\begin{tikzcd} \mathrm{QCoh}(Z) \arrow[r, "i_*"] & \mathrm{QCoh}(X) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathrm{QCoh}(\overline{U}) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

Example

Choose a perfectoid field K, let $R = K^\circ$ and $I = K^{\circ\circ}$, and choose a nonzero $\varpi \in I$. Then $j^*(R/\varpi) = I/\varpi I \neq 0$, but R/ϖ is 0 when restricted to U. So $j^*(R/\varpi)$ is "supported on $\overline{U} \smallsetminus U$."

Tensor products and internal Homs

Write $\mathcal{I}$ for the full subcategory of Mod_R spanned by M that are I-torsion. Recall that $j^* : \operatorname{Mod}_R \to \mathcal{A}$ realizes $\mathcal{A}$ as the quotient category $\operatorname{Mod}_R / \mathcal{I}$. Since $\mathcal{I}$ is an ideal for $\otimes_R$, we see $\otimes_R$ descends to a tensor product $\otimes$ on $\mathcal{A}$.
Furthermore, recall we have an internal Hom given by

$$\operatorname{alHom}(j^* M_1, j^* M_2) = j^* \operatorname{Hom}_R(M_1, M_2)$$

for all M_1 and M_2 in Mod_R. It satisfies a tensor-hom adjunction

$$\operatorname{Hom}_{\mathcal{A}}(N_1 \otimes N_2, N_3) = \operatorname{Hom}_{\mathcal{A}}(N_1, \operatorname{alHom}(N_2, N_3))$$

for all N_1, N_2, and N_3 in $\mathcal{A}$.

Notation and examples

We write $(-)_!$ for $j_!$, $(-)^a$ for j^*, and $(-)_*$ for j_*. We call $(-)^a$ *almostification*, and we call $(-)_*$ the functor of *almost elements*. We often abuse notation and view $M \in \mathrm{Mod}_R$ as an object of $\mathcal{A}$ by applying $(-)^a$. We often denote $\mathcal{A}$ using Mod_R^a or Mod_{R^a}.

Remark

By $(j_!, j^*)$ adjointness, we have
$M_* = j_* j^* M = \mathrm{Hom}_R(I, I \otimes_R M) = \mathrm{Hom}_R(I, M)$.

Example

Choose a perfectoid field K, let $R = K^\circ$, and let $I = K^{\circ\circ}$.

- For I-torsion M, we have $M_* = 0$.
- For torsion-free M, we have $M_* = \{m \in M \otimes_R K \mid \epsilon m \in M \quad \forall \epsilon \in I\}$.
- For $0 < c \leq 1$ and $J = \{r \in R \mid |r| < c\}$, we have $J_* = \{r \in R \mid |r| \leq c\}$. So $I_* = R$.

More examples

Example

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- For $0 < c \leq 1$ and $J = \{r \in R \mid |r| < c\}$, we have $J_* = \{r \in R \mid |r| \leq c\}$. So $I_* = R$.
- For $0 < c \leq 1$ and $J = \{r \in R \mid |r| \leq c\}$, we have $J_* = J$. So $R_* = R$.
- Choose a nonzero $\varpi \in I$. By applying the left-exact $(-)_*$ to the short exact sequence

$$0 \longrightarrow R \xrightarrow{\varpi} R \longrightarrow R/\varpi \longrightarrow 0\ ,$$

we see that $R/\varpi \subseteq (R/\varpi)_*$. It turns out this inclusion is strict if K is not spherically complete.

Almost homological algebra

Proposition

The category Mod_{R^a} has enough injectives.

Proof.

Since $(-)^a$ has a left-exact left adjoint $(-)_!$, it preserves injective envelopes. So lift to Mod_R, find an injective envelope, and apply $(-)^a$. □

Definition

Let $N \in \mathrm{Mod}_{R^a}$. We say N is

- *almost flat* if $N \otimes -$ is exact,
- *almost projective* if $\mathrm{alHom}(N, -)$ is exact.

Remark

As $M^a \otimes - = (M \otimes_R (-)_!)^a$, the exactness of $(-)_!$ and $(-)^a$ shows that M^a being almost flat is equivalent to $\mathrm{Tor}^R_{>0}(M, -)$ being almost zero.

More almost notions

Remark

As $M^a \otimes - = (M \otimes_R (-)_!)^a$, the exactness of $(-)_!$ and $(-)^a$ shows that M^a being almost flat is equivalent to $\mathrm{Tor}^R_{>0}(M, -)$ being almost zero. Similar reasoning shows that M^a being almost projective is equivalent to $\mathrm{Ext}_R^{>0}(M, -)$ being almost zero.

Definition

Let $M \in \mathrm{Mod}_R$. We say M^a is *almost finitely generated* if for all $\epsilon \in I$, there exists a finitely generated R-module M_ϵ and a map $M_\epsilon \to M$ whose kernel and cokernel are annihilated by ϵ. If the number of generators of M_ϵ can be bounded independently of ϵ, we say M^a is *uniformly almost finitely generated*.

We can make the same definition with finite presentation instead.

Typical example

Example

Assume $p \neq 2$, let $K = \mathbb{Q}_p[p^{1/p^\infty}]^\wedge$, and let $L = K[p^{1/2}]$. Because L° has elements of valuation $1/2p^n$ for all n, we see L° is *not* a finitely generated K°-module. But I claim L° is an *almost* finitely generated K°-module.

Set $M_n = K^\circ \oplus K^\circ \cdot p^{1/2p^n}$. It suffices to show that $M_n \hookrightarrow L^\circ$ has cokernel annihilated by $p^{1/p^{n-1}}$. First, note that $M_n \hookrightarrow M_{n+1}$ has cokernel killed by p^{1/p^n}, as $p^{1/p^n} p^{1/2p^{n+1}} = p^{(2p+1)/2p^{n+1}} = p^{((p+1)/2)/p^{n+1}} p^{1/2p^n}$. Thus $M_n \hookrightarrow \bigcup_m M_m$ has cokernel annihilated by any element of valuation at least $\sum_{m=n}^\infty 1/p^m = 1/(p-1)p^{n-1}$. In particular, this cokernel is annihilated by $p^{1/p^{n-1}}$ and is p-adically complete. Since M_n is p-adically complete, we see $\bigcup_m M_m$ is p-adically complete.

But $(\bigcup_m M_m)^\wedge = L^\circ$, which concludes our argument. We see that L° is a uniformly almost finitely presented K°-module.

Étale and unramified morphisms

Recall that a ring morphism $A \to B$ is *étale* if it is

- *flat*, i.e. $B \otimes_A -$ is exact, and
- *unramified*, i.e. finite type and $\Omega^1_{B/A} = 0$.

Proposition

Say $A \to B$ is unramified, and write $\mu : B \otimes_A B \to B$ for multiplication. Then there exists $e \in B \otimes_A B$ such that $e^2 = e$, $\mu(e) = 1$, and $(\ker \mu)e = 0$.

We call such an e a *diagonal idempotent*.

Proof.

As B is finite type over A, we see $\ker \mu$ is finite over $B \otimes_A B$. Now $(\ker \mu)/(\ker \mu)^2 = \Omega^1_{B/A} = 0$, so Nakayama's lemma yields $e \in B \otimes_A B$ such that $e = 1 + i$ for $i \in \ker \mu$ and $(\ker \mu)e = 0$. Hence $\mu(e) = 1$, and $0 = ei = i + i^2$. Thus $e^2 = 1 + i + i + i^2 = 1 + i = e$. $\square$

Almost étale algebras

We use $\otimes$ to define *algebra objects* in Mod_{R^a}, i.e. objects $A \in \mathrm{Mod}_{R^a}$ along with morphisms $\mu : A \otimes A \to A$ and $\iota : R^a \to A$ satisfying identity, commutativity, and associativity. Write CAlg_{R^a} for the category of such objects.

Because $(-)^a$ sends $\otimes_R$ to $\otimes$, it yields a functor $(-)^a : \mathrm{CAlg}_R \to \mathrm{CAlg}_{R^a}$. Furthermore, since $(N_{1,*} \otimes_R N_{2,*})^a = N_1 \otimes N_2$, adjunction yields a natural map $N_{1,*} \otimes_R N_{2,*} \to (N_1 \otimes N_2)_*$. Noting that $(R^a)_* = R$, we see that applying $(-)_*$ and then composing μ_* with this natural map yields a functor $(-)_* : \mathrm{CAlg}_{R^a} \to \mathrm{CAlg}_R$. As before, $(-)^a$ is left adjoint to $(-)_*$.

Definition

Let $A \in \mathrm{CAlg}_{R^a}$. We say A is

- *almost unramified* if $(A \otimes A)_*$ has a diagonal idempotent,
- *almost étale* if A is almost flat and almost unramified,
- *almost finite étale* if A is almost étale and almost finitely presented.

Typical example, revisited

Example

Assume $p \neq 2$, let $K = \mathbb{Q}_p[p^{1/p^\infty}]^\wedge$, and let $L = K[p^{1/2}]$. Then L/K is a ramified extension of valued fields, so we expect L°/K° to be ramified. But I claim L° is an *almost* unramified K°-algebra.

Write $\sigma : L \to L$ for the nontrivial automorphism over K. We have an isomorphism $\psi : L \otimes_K L \to L \times L$ given by $a \otimes b \mapsto (ab, a\sigma(b))$. Under this isomorphism, μ corresponds to projection to the first factor, and we see $e = \psi^{-1}(1, 0)$ equals $(1 \otimes p^{1/2} + p^{1/2} \otimes 1)/2p^{1/2}$. Note that $e^2 = e$, $\mu(e) = 1$, and $(\ker \mu)e = 0$. Now $pe \in L^\circ \otimes_{K^\circ} L^\circ$, and we can also write $e = (1 \otimes p^{1/2p^n} + p^{1/2p^n} \otimes 1)/2p^{1/2p^n}$. We then see $p^{1/p^n} e \in L^\circ \otimes_{K^\circ} L^\circ$, so e yields a diagonal idempotent in $(L^{\circ,a} \otimes L^{\circ,a})_*$.

We already saw L° is almost finitely presented, and as it's torsionfree and hence flat, it's also almost flat. So L° is an almost finite étale algebra over K°.

Do you feel the magic?

Remark

One can show that being almost flat and almost finitely presented is equivalent to being almost projective and almost finitely generated.

In our example, we had an algebra over K° that was finite étale over $K^\circ[1/p]$ but not over K°. However, it was *almost* finite étale over K°. Somehow, the finite étale cover spread from U to $\overline{U}$! This phenomenon is called *almost purity*.

Theorem (Almost purity over $\mathbb{F}_p$, first edition)

Let R be a perfect $\mathbb{F}_p$-algebra, choose $f \in R$, and let $I = (f^{1/p^\infty})$. Say $R \to S$ is an integral map, where S is perfect. If $S[1/f]$ is étale over $R[1/f]$, then S is almost finite étale over R.

Thank you!