MATH 99R: FOURIER ANALYSIS ON NUMBER FIELDS

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Course Website: `http://people.math.harvard.edu/~sli/math99r_f20/`
Office Hours: 9am–10:15am on Wednesdays and by appointment.

Prerequisites: Real analysis (e.g. Math 114), complex analysis (e.g. Math 113), and a one-year course in algebra (e.g. Math 122–123). Familiarity with algebraic number theory is highly recommended but not required; we will cover background as needed.

Course Description: The Riemann zeta function is ubiquitous in number theory, and certain generalizations thereof called *Hecke L-functions* play a similarly important role. Their analytic properties encode subtle information about arithmetic, but early proofs (and even statements!) of these facts were complicated, bouncing back and forth between algebra and analysis. In this tutorial, our ultimate goal is to explain the approach taken in Tate's PhD thesis to studying these L-functions.

Tate's thesis uses *Pontryagin duality* (a generalization of Fourier analysis) to present a streamlined proof of the functional equation and meromorphic continuation for Hecke L-functions. This approach is remarkable for thorough use of the *adele ring*, which smoothly unifies algebraic aspects of number theory with arguments from harmonic analysis. Not only is Tate's approach more elegant than previous proofs, but it also naturally leads into many important results in the theory of *automorphic forms* (a vast simultaneous generalization of modular forms and Hecke characters).

Textbooks: We will not adhere to any particular source, but here are some recommended references:

- Ramakrishnan–Valenza's *Fourier Analysis on Number Fields* is a very complete reference, including lots of background on topological groups, functional analysis, number fields, and even a summary of class field theory. It also covers many applications of the ideas we'll discuss. However, its explanations are somewhat roundabout at times.
- Bump's *Automorphic Forms and Representations* has a section on Tate's thesis. This book is a great reference if you're interested in continuing to study automorphic forms. However, it's fairly advanced, so its treatment offers little to no background and is rather streamlined. If you want a succinct account of the topic, this can also be a good thing.
- Cassels–Frölich's *Algebraic Number Theory*, which is actually a collection of articles by various authors, contains a copy of Tate's original PhD thesis. Tate's notation is old, and modern treatments have slightly cleaned up his approach, but his original thesis remains a classic.

Homework: On most weeks, there will be homework assigned on Thursday afternoon, and it is due via email by 9am on the following Thursday. *Late homework will not be accepted.* The two lowest homework grades will be dropped.

You are strongly encouraged to collaborate with fellow students on homework. However, you must *write up solutions on your own*, as well as *list your collaborators* on your homework. You are also welcome to consult whatever references you like, as long as you *list what references* contributed to finishing your homework. For example, **I consider copying proofs without reference to be plagiarism.**

Final project: There will be a final project due by 5pm on December 9 (the last day of Reading Period). This consists of an expository paper, roughly 10–15 pages in length, on a topic related to the course content. I welcome you to discuss potential topics with me; this material has fascinating connections to other areas in math, so there are many possibilities!

Grading: The course grade is 65% homework, 20% final project, and 15% course participation.