MATH 99R PROBLEM SET 9

Due at 9am on Thursday, November 12.

Throughout, let F be a number field.

(1) Let G be an abelian locally compact topological group, let m be a Haar measure on G, let H be a countable closed subgroup of G, and let D be a Borel subset of G. If D has compact closure, nonempty interior, and maps bijectively to G/H, then the pushforward of m via $D \to G/H$ yields a Haar measure on G/H.

(2) Show that $D = \{(x_v)_v \in \mathbb{A}_{\mathbb{Q}} \mid \|x_v\|_v \leq 1 \text{ for } v \neq \infty \text{ and } 0 \leq x_\infty < 1\}$ is a fundamental domain for $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}$. Conclude directly that $m(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}) = 1$.

(3) For any positive integer m, consider the subgroup $C_m = \{(x_v)_v \in \mathbb{A}_{\mathbb{Q}} \mid x_v \in m\mathbb{Z}_v \text{ for } v \neq \infty \text{ and } v_\infty = 0\}$.
  (a) Prove that the natural map $\mathbb{A}_{\mathbb{Q}} \to \varprojlim_m \mathbb{A}_{\mathbb{Q}}/C_m$ is an isomorphism of topological groups. Conclude that this yields an isomorphism $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q} \to \varprojlim_m \mathbb{A}_{\mathbb{Q}}/(\mathbb{Q} + C_m)$ of topological groups.
  (b) Show that the inclusion $\mathbb{R} \to \mathbb{A}_{\mathbb{Q}}$ induces a well-defined map $\mathbb{R}/m\mathbb{Z} \to \mathbb{A}_{\mathbb{Q}}/(\mathbb{Q} + C_m)$.
  (c) Prove that $\mathbb{R}/m\mathbb{Z} \to \mathbb{A}_{\mathbb{Q}}/(\mathbb{Q} + C_m)$ is an isomorphism of topological groups. (Hint: for surjectivity, use strong approximation.)
  (d) Conclude that $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}$ is isomorphic to $\varprojlim_m \mathbb{R}/m\mathbb{Z}$ as topological groups.

(4) Prove that the Dedekind zeta function $\zeta_F(s) = \prod_{\mathfrak{p}}(1 - \mathrm{Nm}(\mathfrak{p})^{-s})^{-1}$ converges for $\mathrm{Re}\, s > 1$. (Hint: show that $(1 - p^{-r\sigma})^{-1} \leq (1 - p^{-\sigma})^{-r}$ for positive integers r and real $\sigma > 0$. Then bootstrap from the case of the Riemann zeta function.)