MATH 99R PROBLEM SET 6

Due at 9am on Thursday, October 22.

Write m for the Lebesgue measure on $\mathbb{R}^n$. The goal of this problem set is to prove the following theorem using Poisson summation:

Theorem (Minkowski convex body)**.** *Let C be a convex compact subset of $\mathbb{R}^n$ such that $C = -C$. If $m(C) > 2^n$, then C contains an element of $\mathbb{Z}^n$ other than the origin.*

(1) First, we need to prove Poisson summation for $\mathbb{R}^n$. Let f be in $\mathcal{S}(\mathbb{R}^n)$.
  (a) Prove that $F(x) = \sum_{k\in\mathbb{Z}^n} f(x+k)$ converges uniformly on compact subsets of $\mathbb{R}^n$ and defines a continuous function $F : (S^1)^n \to \mathbb{C}$.
  (b) Prove that $\sum_{k\in\mathbb{Z}^n} f(k) = \sum_{k\in\mathbb{Z}^n} \widehat{f}(k)$.

(2) Next, we need to introduce the *convolution product*. Let G be a locally compact topological group, and let m be a left Haar measure on G. For any f_1 and f_2 in $C_c(G)$, write $f_1 * f_2$ for the function $G \to \mathbb{C}$ given by

$$g \mapsto \int_G \mathrm{d}x\, f(x) g(x^{-1}g).$$

This is the convolution product.
  (a) Prove that $f_1 * f_2$ is continuous.
  (Hint: use a fact from September 8's lecture.)
  (b) Prove that $\operatorname{supp}(f_1 * f_2)$ lies in $(\operatorname{supp} f_1)(\operatorname{supp} f_2)$. Deduce that $f_1 * f_2$ lies in $C_c(G)$.
  (c) Suppose that G is finite. Upon identifying $C_c(G)$ with the group ring $\mathbb{C}[G]$, show that the convolution product corresponds to the usual multiplication law on $\mathbb{C}[G]$.
  (d) Suppose that G is abelian. Prove that $(f_1 * f_2)^\wedge = \widehat{f_1} \cdot \widehat{f_2}$.
  (e) Suppose that $G = \mathbb{R}^n$, and say that f_2 is smooth. Prove that $f_1 * f_2$ is smooth, and that $\frac{\partial(f_1 * f_2)}{\partial x_i} = f_1 * (\frac{\partial f_2}{\partial x_i})$ for all integers $1 \le i \le n$.

(3) We now gather some preliminary observations.
  (a) Suppose that for all ϵ in $(0,1)$, the set $(1+\epsilon)C$ contains an element of $\mathbb{Z}^n$ other than the origin. Explain how to deduce the Theorem from this.
  (b) Let f be in $C_c(\mathbb{R}^n)$, and suppose that f is smooth. Show that f is in $\mathcal{S}(\mathbb{R}^n)$.
  (c) Write $K = \frac{1}{2}C$. Show that $m(K) > 1$.

Because K is convex and compact, one can show that there exists a smooth function $h : \mathbb{R}^n \to \mathbb{R}$ such that $h(x) = h(-x)$ for all x in $\mathbb{R}^n$, we have $\mathbf{1}_{(1-\epsilon)K} \le h \le \mathbf{1}_{(1+\epsilon)K}$, and $\int_{\mathbb{R}^n} \mathrm{d}x\, h(x) = m(K)$. Let's take this for granted.

(4) Finally, we wrap up the proof.
  (a) Show that $\widehat{h}$ is real-valued.
  (b) Set $f = h * h$. Show that f as well as $\widehat{f}$ are real-valued and non-negative.
  (c) Show that $\operatorname{supp} f$ lies in $(1+\epsilon)C$, and we have $f \le m(K)$. Deduce that f and hence $\widehat{f}$ lie in $\mathcal{S}(\mathbb{R}^n)$.
  (d) Apply Problem (1).(b) to f to show that $\#((1+\epsilon)C \cap \mathbb{Z}^n) > 1$, and conclude the proof of the Theorem.