MATH 99R PROBLEM SET 5

Due at 9am on Thursday, October 15.

Problem (1).(c) was taken from Dinakar Ramakrishnan and Robert Valenza's *Fourier Analysis on Number Fields*.

(1) Let G be a topological group, and let m be a left Haar measure on G.
 (a) Let g be in G. Show that $E \mapsto m(Eg)$ yields a left Haar measure on G.
 (b) Show that $E \mapsto m(E^{-1})$ yields a right Haar measure on G.
 (c) Assume G is abelian and locally compact. Prove that $m(E) = m(E^{-1})$ for all Borel subsets $E \subseteq G$. (Hint: use Haar's theorem and part (b).)

(2) Prove the only continuous group homomorphisms $\mathbb{R} \to \mathbb{R}$ are given by multiplication by a for some a in $\mathbb{R}$.

(3) Let G_1 and G_2 be finite groups. Prove that the map $\mathbb{C}[G_1] \otimes_{\mathbb{C}} \mathbb{C}[G_2] \to \mathbb{C}[G_1 \times G_2]$ given by $f_1(g_1) \otimes f_2(g_2) \mapsto f_1(g_1)f_2(g_2)$ is an isomorphism of (not necessarily commutative) $\mathbb{C}$-algebras.

(4) Let G be a finite abelian group with the discrete topology, and let m be $(\#G)^{-1/2}$ times the counting measure on G.
 (a) Show that $\widehat{G}$ is a finite abelian group with the discrete topology.
 (b) Prove the Plancherel theorem for G, and show that $\widehat{m}$ is $(\#\widehat{G})^{-1/2}$ times the counting measure on $\widehat{G}$.
 (c) Show Pontryagin duality for G.
 (d) Prove the Fourier inversion formula for G.
 (Hint: use the classification of finite abelian groups to reduce each part to the case $G = \mathbb{Z}/n\mathbb{Z}$.)

(5) Let p be a prime number, and let $\psi : \mathbb{Q}_p/\mathbb{Z}_p \to S^1$ be a nontrivial continuous homomorphism. Prove that the map $\psi_\cdot : \mathbb{Z}_p \to \widehat{\mathbb{Q}_p/\mathbb{Z}_p}$ given by $a \mapsto (x \mapsto \psi(ax))$ is a well-defined isomorphism of topological groups. (Hint: emulate our computation of the Pontryagin dual for local fields.)