MATH 99R PROBLEM SET 4

Due at 9am on Thursday, October 8.

In problems (1)–(4), let F be a nonarchimedean local field, write q for the cardinality of its residue field, and write $|\cdot|$ for its normalized absolute value. All integrals on F are taken with respect to the Lebesgue measure m.

(1) Let $m \geq 1$ be an integer.
  (a) Show that the group $\mathfrak{m}^m/\mathfrak{m}^{m+1}$ is isomorphic to $\mathcal{O}/\mathfrak{m}$.
  (b) Show that $1+\mathfrak{m}^m$ is an open subgroup of $\mathcal{O}^\times$.
  (c) Show that $(1+\mathfrak{m}^m)/(1+\mathfrak{m}^{m+1})$ is isomorphic to $\mathcal{O}/\mathfrak{m}$, while $\mathcal{O}^\times/(1+\mathfrak{m})$ is isomorphic to $(\mathcal{O}/\mathfrak{m})^\times$.

(2) Show that $m(\mathcal{O}^\times) = 1 - \frac{1}{q}$.

(3) Choose a uniformizer π of F, and let $\chi : F^\times \to S^1$ be a continuous homomorphism that is *unramified*, i.e. $\chi(\mathcal{O}^\times) = 1$. For any complex number z with $\operatorname{Re} z > -1$, show that

$$\int_{\mathcal{O}} \mathrm{d}x\, \chi(x)|x|^z = \left(1 - \frac{1}{q}\right)\left(\frac{1}{1-\chi(\pi)q^{-z-1}}\right).$$

(4) Let f in $\mathcal{O}[t_1, \ldots, t_n]$ be a polynomial in n variables. Prove that $f = 0$ has a solution in $\mathcal{O}^n$ if and only if $f \equiv 0 \mod \mathfrak{m}^m$ has a solution in $\mathcal{O}/\mathfrak{m}^m$ for all $m \geq 1$.
(Hint: use $\mathcal{O} = \varprojlim_m \mathcal{O}/\mathfrak{m}^m$ for one direction, and use the finitude of the $\mathcal{O}/\mathfrak{m}^m$ in the other direction.)

(5) Let G be an abelian topological group. Prove that, if G is discrete, then $\widehat{G}$ is compact.
(If you follow Ramakrishnan–Valenza's proof, please give more detail than them!)

(6) Let G_1 and G_2 be abelian topological groups. Prove that the Pontryagin dual $(G_1 \times G_2)^\wedge$ is naturally isomorphic to $\widehat{G}_1 \times \widehat{G}_2$ as topological groups.