MATH 99R PROBLEM SET 3

Due at 9am on Thursday, October 1.

(1) Let F be a field, let $|\cdot|$ be a discretely valued norm on F, and let π in F be a uniformizer. Show that the map $\mathbb{Z} \times \mathcal{O}^\times \to F^\times$ given by $(n, x) \mapsto \pi^n x$ is an isomorphism of topological groups.

(2) Let F be a field, and let $|\cdot|$ be a nonarchimedean norm on F. Consider two closed balls $B_c(a, r) = \{x \in F \mid |x - a| \leq r\}$ and $B_c(b, s) = \{x \in F \mid |x - b| \leq s\}$, where a and b lie in F, and $r \geq s$ lie in $\mathbb{R}_{\geq 0}$. If $B_c(a, r)$ and $B_c(b, s)$ intersect, prove that $B_c(a, r)$ contains $B_c(b, s)$.

(3) Let F be a field, and let $|\cdot|$ be a nonarchimedean norm on F. Recall that also we denote the the Gauss norm on the polynomial ring $F[t]$ by $|\cdot|$.
   (a) Show that, for all f in $F[t]$, we have $|f| = 0$ if and only if $f = 0$.
   (b) Show that, for all f and g in $F[t]$, we have $|f + g| \leq \max\{|f|, |g|\}$.
   (c) Prove that, for all f and g in $F[t]$, we have $|fg| = |f||g|$.
   (d) Show that $|\cdot|$ extends uniquely to a nonarchimedean norm on the rational function field $F(t)$.

(4) Let κ be a field, and write $v : \kappa(\!(t)\!) \to \mathbb{Z} \cup \{\infty\}$ for the map sending $f \mapsto \operatorname{ord}_{t=0} f$.
   (a) Show that v is a valuation on $\kappa(\!(t)\!)$.
   (b) Prove that $\kappa(\!(t)\!)$ is complete with respect to the norm induced by v.
   (c) Prove that $\kappa(\!(t)\!)$ is locally compact if and only if κ is finite.

(5) Let F be a local field of characteristic $p > 0$, and write q for the cardinality of its residue field. Prove that F is isomorphic to $\mathbb{F}_q(\!(t)\!)$ as a topological field.
(Hint: use π-adic expansions, along with a special choice of representatives from Hensel's lemma.)

(6) Let F be field, let $|\cdot|$ be a discretely valued norm on F, and suppose F is complete with respect to $|\cdot|$. Prove that $|\cdot|$ is the only discretely valued norm on F, up to isomorphism.
(Hint: use weak approximation and Hensel's lemma to obtain a contradiction.)