MATH 99R PROBLEM SET 2

Due at 9am on Thursday, September 24.

Problems (4)–(6) were taken from Jürgen Neukirch's *Algebraic Number Theory*.

Throughout, let p be a prime number.

(1) Let F be a field, and let $|\cdot|$ be a nonarchimedean norm on F.
   (a) Let $r > 0$. Show that the closed ball $B_c(0,r) = \{x \in F \mid |x| \leq r\}$ as well as the open ball $B_o(0,r) = \{x \in F \mid |x| < r\}$ are subgroups of F.
   (b) Show that the closed unit ball $\mathcal{O} = B_c(0,1)$ is a subring of F, and prove that its only maximal ideal is the open unit ball $\mathfrak{m} = B_o(0,1)$.

(2) Let F be a field, and let $|\cdot|$ be a nonarchimedean norm on F. Prove that, if x and y in F satisfy $|x| \neq |y|$, then $|x+y| = \max\{|x|,|y|\}$.

(3) Let F be a field, let $|\cdot|$ be a discretely valued norm on F, and write v for the corresponding normalized valuation. Let π in F be a uniformizer. Show that $\pi^m\mathcal{O} = \{x \in F \mid v(x) \geq m\}$ for all non-negative integers m.

(4) Prove that the only field automorphism of $\mathbb{Q}_p$ is the identity map.

(5) Compute the 5-adic expansions of $\frac{2}{3}$ and $-\frac{2}{3}$.

(6) Let x be in $\mathbb{Q}_p$, and write $a_N p^N + a_{N+1}p^{N+1} + \cdots$ for its p-adic expansion. Prove that x lies in $\mathbb{Q}$ if and only if its sequence of digits $a_N, a_{N+1}, \cdots$ is eventually periodic.