MATH 99R PROBLEM SET 1

Due at 9am on Thursday, September 17.

Problems (2) and (3) were taken from Sophie Morel's fall 2018 representation theory class.

(1) Let G be a topological group, and let H be a subgroup. Prove that $\overline{H}$ is also a subgroup. (If you follow Ramakrishnan–Valenza's proof, please give much more detail than them!)

(2) Let $(G_i)_{i\in I}$ be a collection of topological groups.
  (a) Show that $\prod_{i\in I} G_i$ equipped with the product topology is a topological group.
  (b) Prove or give a counterexample: if all the G_i are locally compact, is the same true for $\prod_{i\in I} G_i$?

(3) Let I be a partially ordered set, let $(X_i)_{i\in I}$ be a collection of sets, and let $(u_{ij} : X_i \to X_j)_{i\geq j\in I}$ be a collection of maps such that
  - for every i in I, we have $u_{ii} = \mathrm{id}_{X_i}$,
  - for all $i \geq j \geq k$ in I, we have $u_{ik} = u_{ij} \circ u_{jk}$.

  This is called a *projective system* indexed by I. Its *(projective) limit* is the set
$$\varprojlim_{i\in I} X_i = \{(x_i)_{i\in I} \in \prod_{i\in I} X_i \mid \text{for all } i \geq j \in I, \text{ we have } u(x_i) = x_j\}.$$
  (a) If all the X_i are Hausdorff topological spaces and all the u_{ij} are continuous, show that $\varprojlim_{i\in I} X_i$ is a closed subset of $\prod_{i\in I} X_i$ with the product topology. We henceforth always give $\varprojlim_{i\in I} X_i$ the subspace topology.
  (b) If all the X_i are compact Hausdorff topological spaces and all the u_{ij} are continuous, show that $\varprojlim_{i\in I} X_i$ is also compact Hausdorff. (Hint: use Tychonoff's theorem.)
  (c) If all the X_i are groups (respectively rings) and all the u_{ij} are group homomorphisms (respectively ring homomorphisms), show that $\varprojlim_{i\in I} X_i$ is a subgroup (respectively a subring) of $\prod_{i\in I} X_i$.
  (d) If all the X_i are topological groups and all the u_{ij} are continuous group homomorphisms, show that $\varprojlim_{i\in I} X_i$ is a topological group.

(4) Let p be a prime number. Prove that, for all x and y in $\mathbb{Q}$, we have $|x + y|_p \leq \max\{|x|_p, |y|_p\}$.