Some Integrals

(feat. some theorems)

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November 17, 2020

Definition

Let $f : \mathbb{C} \to \mathbb{C}$ be a function. We say f is of *exponential type* if $f(s) = ab^s$, where a lies in $\mathbb{C}^\times$ and b lies in $\mathbb{R}$.

Proposition

Suppose v is archimedean and f_v is analytic. Then $Z_v(s, \chi_v, f_v)/L_v(s, \chi_v)$ is entire.

Proof.

As f_v is Schwartz, we see the integral over $\{x_v \in F_v^\times \mid \|x_v\|_v > 1\}$ converges. Since $L_v(s, \chi_v)$ vanishes nowhere, we only need to analyze poles resulting from the integral over $\{x_v \in F_v^\times \mid \|x_v\|_v \leq 1\}$.

If $F_v = \mathbb{R}$, then $\chi_v(x) = (x/|x|)^\varepsilon \|x\|_v^\nu$ for ε in $\{0, 1\}$ and ν in $i\mathbb{R}$. Write $f_v(x) = \sum_{n=0}^\infty a(n)x^n$ for small enough x, where the $a(n)$ lie in $\mathbb{C}$. Because $Z_v(s, \chi_v, f_v)$ is linear in f_v, it suffices to consider f_v even or odd. We see the integral over $\{x \in \mathbb{R} \mid \|x\|_v \leq 1\}$ vanishes when the parity of f_v and ε aren't equal, by substituting $x' = -x$.

Proposition

Suppose v is archimedean and f_v is analytic. Then $Z_v(s, \chi_v, f_v)/L_v(s, \chi_v)$ is entire.

Proof (continued).

If f_v is even, then $a(n) = 0$ for odd n. Our lemma implies that the integral over $\{x \in \mathbb{R} \mid \|x\|_v \leq 1\}$ has at worst simple poles at $s = -\nu, -2-\nu, \ldots$. When $\varepsilon = 0$, these are the poles of $L_v(s, \chi_v) = \pi^{-s-\nu}\Gamma((s+\nu)/2)$. We obtain analogous cancellation when f_v and ε are odd.

If $F_v = \mathbb{C}$, then $\chi_v(z) = (z/|z|)^k \|z\|_v^\nu$ for k in $\mathbb{Z}$ and ν in $i\mathbb{R}$. Write $f_v(z) = \sum_{n,m=0}^\infty a(n,m) z^n \overline{z}^m$ for small enough z, where the $a(n,m)$ lie in $\mathbb{C}$. Using polar coordinates $z = re^{2\pi i\theta}$ turns this integral into

$$4\pi \int_0^1 \mathrm{d}^\times r \, r^{2\nu+2s} \int_0^1 \mathrm{d}\theta \, f_v(re^{2\pi i\theta}) e^{2\pi k i\theta}.$$

Now $\int_0^1 \mathrm{d}\theta \, f_v(re^{2\pi i\theta}) e^{2\pi kix} = \sum_{n,m=0}^\infty a(n,m) r^{n+m} \int_0^1 \mathrm{d}\theta \, e^{2\pi(n-m+k)i\theta}$.

Proposition

Suppose v is archimedean and f_v is analytic. Then $Z_v(s, \chi_v, f_v)/L_v(s, \chi_v)$ is entire.

Proof (continued).

This equals $\sum_{n=-k}^{\infty} a(n, n+k) r^{2n+k}$. Our lemma implies that the integral $\int_0^1 \mathrm{d}r$ has at worst simple poles at $s = -\nu - |k|/2, -1 - \nu - |k|/2, \ldots$. These are the poles of $L_v(s, \chi_v) = (2\pi)^{-s-\nu-|k|/2+1}\Gamma(s + \nu + |k|/2)$. □

Proposition

Suppose v is nonarchimedean. Then $Z_v(s, \chi_v, f_v)$ is a $\mathbb{C}$-rational function in q_v^{-s}, and $Z_v(s, \chi_v, f_v)/L_v(s, \chi_v)$ is entire.

Proof.

Homework problem. □

Key Proposition

We can choose f_v such that $Z_v(s, \chi_v, f_v)/L_v(s, \chi_v)$ is of exponential type.

Proof.

Suppose v is nonarchimedean. If χ_v is unramified, taking $f_v = \mathbf{1}_{\mathcal{O}_v}$ makes this ratio equal $\#(\mathcal{O}_v/\mathfrak{d}_{F_v/\mathbb{Q}_p})^{-1/2}$. If χ_v is ramified, taking $f_v = \mathbf{1}_{\mathcal{O}_v^\times}\chi_v^{-1}$ makes this ratio equal $\#(\mathcal{O}_v/\mathfrak{d}_{F_v/\mathbb{Q}_p})^{-1/2}$ again.

Suppose $F_v = \mathbb{R}$. If $\varepsilon = 0$, taking $f_v(x) = e^{-\pi x^2}$ yields

$$Z_v(s, \chi_v, f_v) = 2\int_0^\infty \mathrm{d}x\, e^{-\pi x^2} x^{s+\nu-1} = \pi^{-(s+\nu)/2} \int_0^\infty \mathrm{d}x'\, e^{-x'} x'^{(s+\nu)/2-1}$$
$$= \pi^{-(s+\nu)/2}\Gamma((s+\nu)/2) = L_v(s, \chi_v),$$

where $x' = \pi x^2$. If $\varepsilon = 1$, taking $f_v(x) = xe^{-\pi x^2}$ yields

$$Z_v(s, \chi_v, f_v) = 2\int_0^\infty \mathrm{d}x\, e^{-\pi x^2} x^{s+\nu} = \pi^{-(s+\nu+1)/2}\Gamma((s+\nu+1)/2) = L_v(s, \chi_v)$$

Key Proposition

We can choose f_v such that $Z_v(s, \chi_v, f_v)/L_v(s, \chi_v)$ is of exponential type.

Proof (continued).

Suppose $F_v = \mathbb{C}$. Using polar coordinates $z = re^{2\pi i\theta}$ and taking $f_v(z) = r^{|k|}e^{-2\pi k i\theta}e^{-2\pi r^2}$ gives us

$$\begin{aligned}
Z_v(s, \chi_v, f_v) &= 4\pi \int_0^\infty \mathrm{d}r \int_0^1 \mathrm{d}\theta\, e^{-2\pi r^2} r^{2s+2\nu-1+|k|} \\
&= 4\pi \int_0^\infty \mathrm{d}r\, e^{-2\pi r^2} r^{2s+2\nu-1+|k|} \\
&= (2\pi)^{-s-\nu-|k|/2+1} \int_0^\infty \mathrm{d}r'\, e^{-r'} r'^{s+\nu+|k|/2-1} \\
&= (2\pi)^{-s-\nu-|k|/2+1}\Gamma(s+\nu+|k|/2) = L_v(s, \chi_v),
\end{aligned}$$

where $r' = 2\pi r^2$. □

With this, we can now define ϵ-*factors*.

Proposition

Set $\epsilon_v(s,\chi_v) = \gamma_v(s,\chi_v)L_v(s,\chi_v)/L_v(1-s,\chi_v^{-1})$. Then $\epsilon_v(s,\chi_v)$ is of exponential type. Furthermore, for v not in S, we have $\epsilon_v(s,\chi_v) = 1$.

Proof.

Since $\gamma_v(s,\chi_v) = Z_v(1-s,\chi_v^{-1},\widehat{f_v})/Z_v(s,\chi_v,f_v)$, we see that

$$\epsilon_v(s,\chi_v) = \frac{Z_v(1-s,\chi_v^{-1},\widehat{f_v})}{L_v(1-s,\chi_v^{-1})} \cdot \frac{L_v(s,\chi_v)}{Z_v(s,\chi_v,f_v)}.$$

The left factor is entire. By choosing f_v such that $Z_v(s,\chi_v,f_v)/L_v(s,\chi_v)$ is of exponential type, we see the right factor is also entire.

For nonarchimedean v, this is an entire $\mathbb{C}$-rational function of q_v^{-s} whose inverse is also entire. Hence it must be of exponential type. If v is unramified and $f_v = \mathbf{1}_{\mathcal{O}_v}$, previous calculations show that $\widehat{f_v} = \mathbf{1}_{\mathcal{O}_v}$. If χ_v is unramified, then so is χ_v^{-1}. We already know $L_v(s,\chi_v) = Z(s,\chi_v,f_v)$ in this situation, and similarly for χ_v^{-1} and $\widehat{f_v}$. Thus $\epsilon_v(s,\chi_v) = 1$ here.

Proposition

Set $\epsilon_v(s, \chi_v) = \gamma_v(s, \chi_v)L_v(s, \chi_v)/L_v(1-s, \chi_v^{-1})$. Then $\epsilon_v(s, \chi_v)$ is of exponential type. Furthermore, for v not in S, we have $\epsilon_v(s, \chi_v) = 1$.

Proof (continued).

For archimedean v, it'll be a homework problem to explicitly compute $\epsilon_v(s, \chi_v)$, which will verify it is of exponential type. □

Write $\epsilon(s, \chi) = \prod_{v \in S} \epsilon_v(s, \chi_v)$. Since the $\epsilon_v(s, \chi_v)$ are of exponential type, we see $\epsilon(s, \chi)$ is as well.

Theorem (Hecke, Tate)

Our $L(s, \chi)$ has meromorphic continuation to all s in $\mathbb{C}$. It is entire unless $\chi = \|\cdot\|^\nu$ for some ν in $i\mathbb{R}$, in which case its only poles are at $s = -\nu$ and $s = 1 - \nu$. Furthermore, we have $L(s, \chi) = \epsilon(s, \chi)L(1-s, \chi^{-1})$.

Theorem (Hecke, Tate)

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Proof (Tate).

We have $L(s,\chi)\prod_{v\in S} Z_v(s,\chi_v,f_v)/L_v(s,\chi_v) = Z(s,\chi,f)$. For v in S, we can choose f_v such that $Z_v(s,\chi_v,f_v)/L_v(s,\chi_v)$ is of exponential type. Hence $L(s,\chi)$ has meromorphic continuation to all s in $\mathbb{C}$, and it also has the same poles as $Z(s,\chi,f)$. Finally, we have

$$\begin{aligned} L(s,\chi) &= Z(s,\chi,f)\prod_{v\in S} L_v(s,\chi_v)/Z_v(s,\chi_v,f_v) \\ &= Z(1-s,\chi_v^{-1},\widehat{f})\prod_{v\in S} \epsilon_v(s,\chi_v)L_v(1-s,\chi_v^{-1})/Z_v(1-s,\chi_v^{-1},\widehat{f_v}) \\ &= \epsilon(s,\chi)L(1-s,\chi^{-1}). \end{aligned}$$

□

Example (Class number formula)

Let $\chi = 1$. For nonarchimedean v, taking $f_v = \mathbf{1}_{\mathcal{O}_v}$ makes the local zeta integral $Z_v(s, \chi_v, f_v) = \#(\mathcal{O}_v/\mathfrak{d}_{F_v/\mathbb{Q}_p})^{-1/2} L_v(s, \chi_v)$. For archimedean v, taking f_v as in the Key Proposition gives $Z_v(s, \chi_v, f_v) = L_v(s, \chi_v)$. Therefore $Z(s, \chi, f) = |\mathcal{D}_{F/\mathbb{Q}}|^{-1/2} \zeta_F(s) \prod_{v \in M_{F,\infty}} L_v(s, \chi_v)$.

Recall $Z(s, \chi, f)$ has a simple pole at $s = 1$ with residue $\widehat{f}(0) m(\mathbb{A}_F^{\times,1}/F^\times)$. For nonarchimedean v, recall that $\widehat{f_v} = \#(\mathcal{O}_v/\mathfrak{d}_{F_v/\mathbb{Q}_p})^{-1/2} \mathbf{1}_{\mathfrak{d}^{-1}_{F_v/\mathbb{Q}_p}}$. For archimedean v, recall that f_v is self-dual. Therefore

$$\widehat{f}(0) = \prod_{v \in M_F} \widehat{f_v}(0) = |\mathcal{D}_{F/\mathbb{Q}}|^{-1/2}.$$

In our situation, $L_v(1, \chi_v) = 1$ for archimedean v. Altogether, we see that $\zeta_F(s)$ has a simple pole at $s = 1$ with residue

$$m(\mathbb{A}_F^{\times,1}/F^\times) = \frac{2^{r_1}(2\pi)^{r_2} h_F \mathcal{R}_F}{|\mathcal{D}_{F/\mathbb{Q}}|^{1/2} w_F}.$$