Global Zeta Integrals

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We'll use the following consequence of adelic Poisson summation.

Lemma

Let f be in $\mathcal{S}(\mathbb{A}_F)$, and let x be in $\mathbb{A}_F^\times$. Then we have $\sum_{\gamma\in F} f(x\gamma) = \|x\|^{-1} \sum_{\gamma\in F} \widehat{f}(x^{-1}\gamma)$.

Proof.

Homework problem. □

Proposition

Our $Z(s,\chi,f)$ has meromorphic continuation to all s in $\mathbb{C}$. It is entire unless $\chi = \|\cdot\|^\nu$ for some ν in $i\mathbb{R}$, in which case its only poles are at $s=-\nu$ and $s=1-\nu$. Furthermore, we have $Z(s,\chi,f) = Z(1-s,\chi^{-1},\widehat{f})$.

Proof.

We know $Z(s,\chi,f)$ converges for $\operatorname{Re} s > 1$, so the integral over $\{x \in \mathbb{A}_F^\times \mid \|x\| > 1\}$ also converges. On this domain, lowering $\operatorname{Re} s$ only shrinks the integral, so the integral here converges for all s.

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Proof (continued).

Hence we focus on the integral over $\{x \in \mathbb{A}_F^{\times} \mid \|x\| < 1\}$. The fundamental domain $D \times \mathbb{R}_{>0}$ of $\mathbb{A}_F^{\times}/F^{\times}$ shows that this integral equals

$$\int_{D\times(0,1)} \mathrm{d}^{\times}x \sum_{\gamma \in F^{\times}} f(x\gamma)\chi(x\gamma)\|x\gamma\|^{s} = \int_{D\times(0,1)} \mathrm{d}^{\times}x \sum_{\gamma \in F^{\times}} f(x\gamma)\chi(x)\|x\|^{s},$$

since χ and $\|\cdot\|$ are trivial on $F^{\times}$. Adding and subtracting a $\gamma = 0$ term turns this integral into

$$\int_{D\times(0,1)} \mathrm{d}^{\times}x \sum_{\gamma \in F} f(x\gamma)\chi(x)\|x\|^{s} - f(0) \int_{D\times(0,1)} \mathrm{d}^{\times}x \, \chi(x)\|x\|^{s}.$$

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Proof (continued).

We first investigate the right term. Decomposing into D and $(0, 1)$ yields

$$-f(0) \int_0^1 \mathrm{d}^{\times}t \, t^s \chi(t) \int_{\mathbb{A}^{\times,1}/F^{\times}} \mathrm{d}^{\times}y \, \chi(y).$$

If χ is nontrivial on $\mathbb{A}_F^{\times,1}/F^{\times}$, the usual trick shows this integral vanishes. If χ is trivial on $\mathbb{A}_F^{\times,1}/F^{\times}$, we must have $\chi = \|\cdot\|^{\nu}$ for some ν in $i\mathbb{R}$. Then our expression becomes

$$-f(0) m(\mathbb{A}_F^{\times,1}/F^{\times}) \int_0^1 \mathrm{d}^{\times}t \, t^{s+\nu} = -\frac{f(0) m(\mathbb{A}_F^{\times,1}/F^{\times})}{s + \nu}.$$

Proposition

Our $Z(s, \chi, f)$ has meromorphic continuation to all s in $\mathbb{C}$. It is entire unless $\chi = \|\cdot\|^{\nu}$ for some ν in $i\mathbb{R}$, in which case its only poles are at $s = -\nu$ and $s = 1 - \nu$. Furthermore, we have $Z(s, \chi, f) = Z(1 - s, \chi^{-1}, \widehat{f})$.

Proof (continued).

Next, we investigate the left term. Using our lemma and setting $x' = x^{-1}$ converts it into

$$\int_{D \times (0,1)} \mathrm{d}^{\times} x \sum_{\gamma \in F} \widehat{f}(x^{-1}\gamma)\chi(x)\|x\|^{s-1}$$
$$= \int_{D^{-1} \times (1,\infty)} \mathrm{d}^{\times} x' \sum_{\gamma \in F} \widehat{f}(x'\gamma)\chi(x')^{-1}\|x'\|^{1-s}.$$

Note that D^{-1} is also a fundamental domain for $\mathbb{A}_F^{\times,1}/F^{\times}$. Next, we'll split off the $\gamma = 0$ term to obtain...

Proposition

Our $Z(s, \chi, f)$ has meromorphic continuation to all s in $\mathbb{C}$. It is entire unless $\chi = \|\cdot\|^\nu$ for some ν in $i\mathbb{R}$, in which case its only poles are at $s = -\nu$ and $s = 1 - \nu$. Furthermore, we have $Z(s, \chi, f) = Z(1 - s, \chi^{-1}, \widehat{f})$.

Proof (continued).

$$\int_{D^{-1}\times(1,\infty)} \mathrm{d}^\times x' \sum_{\gamma \in F^\times} \widehat{f}(x'\gamma)\chi(x')^{-1}\|x'\|^{1-s}$$
$$+ \widehat{f}(0) \int_{D^{-1}\times(1,\infty)} \mathrm{d}^\times x' \chi(x')^{-1}\|x'\|^{1-s}.$$

Our previous argument shows that this new right term doesn't vanish if and only if $\chi = \|\cdot\|^\nu$ for some ν in $i\mathbb{R}$, in which case it equals

$$-\frac{\widehat{f}(0) m(\mathbb{A}_F^{\times,1}/F^\times)}{1 - s - \nu}.$$

Proposition

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Proof (continued).

Because χ and $\|\cdot\|$ are trivial on $F^{\times}$, this new left term equals

$$\int_{D^{-1}\times(1,\infty)} \mathrm{d}^{\times}x' \sum_{\gamma \in F^{\times}} \widehat{f}(x'\gamma)\chi(x'\gamma)^{-1}\|x'\gamma\|^{1-s},$$

which is the integral for $Z(1 - s, \chi^{-1}, \widehat{f})$ over $\{x \in \mathbb{A}_F^{\times} \mid \|x\| > 1\}$. This similarly converges for all s. Adding all these terms together yields the desired result. □

We ignored the subset $\{x \in \mathbb{A}_F^{\times} \mid \|x\| = 1\}$ throughout. It'll be a homework problem to show it has measure zero.

Here's another reason why it's nice to have flexibility in choosing f_v:

Proposition

Let s_0 be in $\mathbb{C}$. We can choose f_v such that $Z_v(s, \chi_v, f_v)$ has no zeroes nor poles at $s = s_0$. Furthermore, if v is nonarchimedean, we can choose f_v such that $Z_v(s, \chi_v, f_v) = 1$.

Proof.

Recall that if f_v vanishes in a neighborhood of 0, then $Z_v(s, \chi_v, f_v) = \int_{F_v^\times} \mathrm{d}^\times x_v f_v(x_v) \chi_v(x_v) \|x_v\|_v^s$ is holomorphic for all s in $\mathbb{C}$. In particular, it has no poles. By choosing nonzero f_v supported in a sufficiently small neighborhood of 1, we can make this integral at $s = s_0$ arbitrarily close to $\chi_v(1)\|1\|_v^{s_0} = 1$. Finally, if v is nonarchimedean, we can choose such an f_v supported in $\mathcal{O}_v^\times$. Since $\|\mathcal{O}_v^\times\|_v = 1$, this makes the integral independent of s, and we can rescale f_v such that this constant integral equals 1. □

Let $S \supseteq M_{F,\infty}$ be a finite subset of M_F such that v is unramified, χ_v is unramified, and $f_v = \mathbf{1}_{\mathcal{O}_v}$ for all v not in S.

Theorem

Our $L^S(s,\chi)$ *has meromorphic continuation to all* s *in* $\mathbb{C}$. *It is entire unless* $\chi = \|\cdot\|^\nu$ *for some* ν *in* $i\mathbb{R}$, *in which case its only poles are at* $s = -\nu$ *and* $s = 1-\nu$. *Furthermore, we have*
$L^S(s,\chi) = \left(\prod_{v\in S}\gamma_v(s,\chi_v)\right) L^S(1-s,\chi^{-1})$.

Proof.

Recall that $L_v(s,\chi_v) = Z_v(s,\chi_v,f_v)$ for v not in S. Therefore we see that $L^S(s,\chi)\prod_{v\in S} Z_v(s,\chi_v,f_v) = Z(s,\chi,f)$. Now the $Z_v(s,\chi_v,f_v)$ and $Z(s,\chi,f)$ have meromorphic continuation to all s in $\mathbb{C}$, so $L^S(s,\chi)$ does as well. Next, let s_0 be in $\mathbb{C}$. For v in S, we can choose f_v such that $Z_v(s,\chi_v,f_v)$ has no zeroes nor poles at $s = s_0$. Hence $L^S(s,\chi)$ has a pole at $s = s_0$ if and only if $Z(s,\chi,f)$ does, which yields the desired statement. Finally, we have

$$\begin{aligned} L^S(s,\chi) &= Z(s,\chi,f)\prod_{v\in S} Z_v(s,\chi_v,f_v)^{-1} \\ &= Z(1-s,\chi^{-1},\widehat{f})\prod_{v\in S}\left(\gamma_v(s,\chi_v)Z_v(1-s,\chi_v^{-1},\widehat{f_v})^{-1}\right) \\ &= \left(\prod_{v\in S}\gamma_v(s,\chi_v)\right) L^S(1-s,\chi^{-1}). \end{aligned}$$

□

The above theorem suffices for many applications, but we'd like to reach our ultimate result for the completed Hecke L-function. Let's conclude today with a complex-analytic lemma.

Lemma

Let $\mathfrak{S}$ be a discrete subset of $\mathbb{R}$ that's bounded below, let $a : \mathfrak{S} \to \mathbb{C}$ be a function, and suppose that $f : (0,1] \to \mathbb{C}$ is a continuous function such that $f(x) = \sum_{p \in \mathfrak{S}} a(p) x^p$ for small enough x. Then $\int_0^1 \mathrm{d}^\times x \, f(x) x^s$ has meromorphic continuation to all s in $\mathbb{C}$. Its only poles are at $s = -p$ for p in $\mathfrak{S}$, with residue $a(p)$. Furthermore, for any $\epsilon > 0$ and real N, this integral is bounded on $\{s \in \mathbb{C} \mid \operatorname{Re} s > -N \text{ and } d(s, \mathfrak{S}) > \epsilon\}$.

Proof.

Set $R(x) = f(x) - \sum_{\substack{p \in \mathfrak{S} \\ p < N}} a(p) x^p$. Then $R(x) = \sum_{\substack{p \in \mathfrak{S} \\ p > N}} a(p) x^p$ for small enough x, so we see that $\int_0^1 \mathrm{d}^\times x \, R(x) x^s$ converges for $\operatorname{Re} s > -N$. Thus $\int_0^1 \mathrm{d}^\times x \, f(x) x^s = \sum_{\substack{p \in \mathfrak{S} \\ p < N}} a(p)/(s+p) + \int_0^1 \mathrm{d}^\times x \, R(x) x^s$ has the desired poles and residues. Taking absolute values also yields the desired bound. □