Volume of $\mathbb{A}_F^{\times,1}/F^\times$

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November 10, 2020

Let's find a fundamental domain for $\mathbb{A}_F^\times/F^\times$. First, choose a v_0 in $M_{F,\infty}$.

- If $F_v = \mathbb{R}$, consider the inclusion $\mathbb{R}_{>0} \hookrightarrow F_v^\times$.
- If $F_v = \mathbb{C}$, consider $\mathbb{R}_{>0} \hookrightarrow F_v^\times$ given by $x \mapsto \sqrt{x}$.

The composition $\mathbb{R}_{>0} \hookrightarrow F_v^\times \hookrightarrow \mathbb{A}_F^\times$ yields a section to $\|\cdot\| : \mathbb{A}_F^\times \to \mathbb{R}_{>0}$, so this identifies $\mathbb{A}_F^\times = \mathbb{A}_F^{\times,1} \times \mathbb{R}_{>0}$ as topological groups. Now we can just find a fundamental domain for $\mathbb{A}_F^{\times,1}/F^\times$.

Write $\mathbb{A}_{F,\infty}^{\times,1} = \mathbb{A}_F^{\times,1} \cap (\mathbb{A}_F^\times)_{M_{F,\infty}}$. Then we get a continuous group homomorphism $\mathcal{L} : \mathbb{A}_{F,\infty}^{\times,1} \to \prod_{v \in M_{F,\infty} \smallsetminus \{v_0\}} \mathbb{R}$ via $(x_v)_v \mapsto (\log\|x_v\|_v)_{v \neq v_0}$. Using flexibility in the v_0-factor, we see $\mathcal{L}$ is surjective.

Note that $\mathbb{A}_{F,\infty}^{\times,1} \cap F^\times = \mathcal{O}_F^\times$, and $(\ker \mathcal{L}) \cap F^\times$ is the set of roots of unity in F. The proof of Dirichlet's unit theorem shows that $\mathcal{L}(\mathcal{O}_F^\times)$ is a free $\mathbb{Z}$-module of rank $r_1 + r_2 - 1$ in $\prod_{v \in M_{F,\infty} \smallsetminus \{v_0\}} \mathbb{R} = \mathbb{R}^{r_1+r_2-1}$. Let $\mathcal{P}$ be a parallelepiped fundamental domain for $\mathcal{L}(\mathcal{O}_F^\times)$ in $\mathbb{R}^{r_1+r_1-1}$.

Definition

The *regulator* of F, denoted by $\mathcal{R}_F$, is $m(\mathcal{P})$, where m is the Lebesgue measure on $\mathbb{R}^{r_1+r_2-1}$.

Examples

- Suppose $r_1 + r_2 - 1 = 0$, i.e. F is either $\mathbb{Q}$ or an imaginary quadratic field. Then $\mathcal{R}_F$ equals 1.
- Let F be a real quadratic field. Then $\mathcal{L}(\mathcal{O}_F^\times)$ is free of rank 1, so $\mathcal{R}_F$ equals the absolute value of the logarithm of a *fundamental unit*.

Write w_F for the number of roots of unity in F. Since $K_{(\mathcal{O}_F,\varnothing),v_0}$ contains our $\mathbb{R}_{>0} \subset \mathbb{A}_F^\times$, we see that

$$(\mathbb{A}_F^{\times,1} \cap K_{(\mathcal{O}_F,\varnothing)})\backslash\mathbb{A}_F^{\times,1}/F^\times = K_{(\mathcal{O}_F,\varnothing)}\backslash\mathbb{A}_F^\times/F^\times = \mathcal{C}\ell(F).$$

Write $h_F = \#\mathcal{C}\ell(F)$, and let $b_1, \ldots, b_{h_F}$ be representatives of $\mathcal{C}\ell(F)$ in $\mathbb{A}_F^{\times,1}$. Let $D_0 = \{(x_v)_v \in \mathcal{L}^{-1}(\mathcal{P}) \mid 0 \leq \arg x_{v_0} < 1/w_F\}$, and set $D = D_0 b_1 \cup \cdots \cup D_0 b_{h_F}$, where $\arg : \mathbb{C}^\times \to [0,1)$ is the normalized argument.

Proposition

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Proof.

Our D_0 and hence D is evidently Borel, has compact closure, and has nonempty interior. To show $D \to \mathbb{A}_F^{\times,1}/F^\times$ is bijective, let x be in $\mathbb{A}_F^{\times,1}$. There exists a unique $1 \le i \le h_F$ such that xb_i^{-1} yields a principal fractional ideal $\alpha\mathcal{O}_F$, where α lies in $F^\times$. Hence $xb_i^{-1}\alpha^{-1}$ yields the ideal $\mathcal{O}_F$, i.e. it lies in $\mathbb{A}_{F,\infty}^{\times,1}$. Finally, the definition of $\mathcal{P}$ and our argument bound in D_0 imply there exists a unique element β of $\mathcal{O}_F^\times$ such that $xb_i^{-1}\alpha^{-1}\beta^{-1}$ lies in D_0. Altogether, this process uniquely decomposes x as $(xb_i^{-1}\alpha^{-1}\beta^{-1})b_i \cdot \alpha\beta$, concluding the proof. $\square$

While we're here, let's compute the volume of $\mathbb{A}_F^{\times,1}/F^\times$.

Proposition

We have $m(\mathbb{A}_F^{\times,1}/F^\times) = 2^{r_1}(2\pi)^{r_2} h_F \mathcal{R}_F / |\mathcal{D}_{F/\mathbb{Q}}|^{1/2} w_F$.

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Proof.

Because $D_0 \subseteq \mathcal{L}^{-1}(\mathcal{P}) \subseteq \mathbb{A}_{F,\infty}^{\times,1} = (\mathbb{A}_F^{\times,1} \cap K_{(\mathcal{O}_F,\varnothing)})$, we see the $D_0 b_i$ are disjoint. Thus it suffices to show $m(D_0) = 2^{r_1}(2\pi)^{r_2}\mathcal{R}_F/|\mathcal{D}_{F/\mathbb{Q}}|^{1/2} w_F$. Now $w_F \geq 2$, and if $w_F \geq 3$ then F has no real embeddings. Thus in every case it suffices to show $m(\mathcal{L}^{-1}(\mathcal{P})) = 2^{r_1}(2\pi)^{r_2}\mathcal{R}_F/|\mathcal{D}_{F/\mathbb{Q}}|^{1/2}$. By an $\mathbb{R}$-linear change of variables, it suffices to show $m(\mathcal{L}^{-1}([0,1)^{r_1+r_2-1})) = 2^{r_1}(2\pi)^{r_2}/|\mathcal{D}_{F/\mathbb{Q}}|^{1/2}$. Next, note that

$$\mathcal{L}^{-1}([0,1)^{r_1+r_2-1}) = \{(x_v)_v \in \mathbb{A}_{F,\infty}^{\times,1} \mid 1 \leq \|x_v\|_v < e \text{ for } v \in M_{F,\infty} \setminus \{v_0\}\}.$$

Write $\mathcal{N} = \{(x_v)_v \in (\mathbb{A}_F^\times)_{M_{F,\infty}} \mid 1 \leq \|x_v\|_v < e \text{ for } v \in M_{F,\infty}\}$. For x in $\mathbb{A}_F^\times$, write y for its $\mathbb{A}_F^{\times,1}$-component and t for its $\mathbb{R}_{>0}$-component. Then x lies in $\mathcal{N}$ if and only if y lies in $\mathcal{L}^{-1}([0,1)^{r_1+r_2-1})$ and $1 \leq \|x_{v_0}\|_{v_0} < e$.

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Proof (continued).

Note that $\mathcal{N}$ is an $\mathbb{A}_F^{\times,1}$-coset. Therefore

$$\int_{\mathcal{N}} \mathrm{d}^\times x = \int_{\mathbb{A}_F^{\times,1}} \mathrm{d}^\times y \int_{\mathcal{N} y^{-1} \cap \mathbb{R}_{>0}} \mathrm{d}^\times t = \int_{\mathcal{L}^{-1}([0,1)^{r_1+r_2-1})} \mathrm{d}^\times y \int_{\|y\|^{-1}}^{e\|y\|^{-1}} \mathrm{d}^\times t$$
$$= \int_{\mathcal{L}^{-1}([0,1)^{r_1+r_2-1})} \mathrm{d}^\times y \int_1^e \mathrm{d}^\times t = m(\mathcal{L}^{-1}([0,1)^{r_1+r_2-1})),$$

so it suffices to show $m(\mathcal{N}) = 2^{r_1}(2\pi)^{r_2}/|\mathcal{D}_{F/\mathbb{Q}}|^{1/2}$. The nonarchimedean factors amount to $|\mathcal{D}_{F/\mathbb{Q}}|^{-1/2}$. The archimedean factors amount to $\int_{-e}^{-1} \mathrm{d}^\times x + \int_1^e \mathrm{d}^\times x = 2$ for each real embedding and $4\pi \int_0^1 \mathrm{d}\theta \int_1^{\sqrt{e}} \mathrm{d}r\,/r = 2\pi$ for each complex embedding. □