Hecke L-functions and Zeta Integrals

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The Riemann zeta function is our prototype. Recall $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$ absolutely converges for $\operatorname{Re} s > 1$, and here we have the *Euler product*

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$

Write $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ for the *completed* Riemann zeta function.

Theorem (Riemann)

Our ξ has meromorphic continuation to all s in $\mathbb{C}$, and its only poles are at $s = 0$ and $s = 1$. Furthermore, we have $\xi(s) = \xi(1-s)$.

We want to know:

1. Where does $\pi^{-s/2}\Gamma(s/2)$ come from, conceptually?
2. Can we prove the theorem for more general "zeta functions"?

We explain (2) first. Let F be a number field, and let $\chi : \mathbb{A}_F^\times / F^\times \to S^1$ be a continuous group homomorphism. We'll take the Euler product as a definition for generalizations of $\zeta(s)$.

Let v be in M_F. Let's define local factors for the Euler product:

- If $F_v = \mathbb{R}$, then $\chi_v(x) = (x/|x|)^\varepsilon \|x\|_v^\nu$ for ε in $\{0,1\}$ and ν in $i\mathbb{R}$. Set $L_v(s,\chi_v) = \pi^{-(s+\varepsilon)/2}\Gamma((s+\varepsilon)/2)$.
- If $F_v = \mathbb{C}$, then $\chi_v(z) = (z/|z|)^k \|z\|_v^\nu$ for k in $\mathbb{Z}$ and ν in $i\mathbb{R}$. Set $L_v(s,\chi_v) = 2(2\pi)^{-s-\nu+|k|/2}\Gamma(s+\nu+|k|/2)$.
- If F_v is nonarchimedean, then χ_v is either ramified or unramified. In the former case, set $L_v(s,\chi_v) = 1$, and in the latter case, set $L_v(s,\chi_v) = (1-\chi_v(\varpi_v)q_v^{-s})^{-1}$.

For any finite subset $S \subseteq M_F$, write $L^S(s,\chi) = \prod_{v\notin S} L_v(s,\chi_v)$. We write $L(s,\chi)$ for $L^\varnothing(s,\chi)$. We call this the *Hecke L-function* of χ.

Examples

- Let $\chi = 1$. Then $L^{M_{F,\infty}}(s,\chi) = \prod_{\mathfrak{p}}(1-\mathrm{Nm}(\mathfrak{p})^{-s})^{-1} = \sum_I \mathrm{Nm}(I)^{-s}$ by unique factorization, where I runs through nonzero ideals of $\mathcal{O}_F$. Therefore this yields the Dedekind zeta function $\zeta_F(s)$ of F.
- Let (I, S_0) be a modulus for F. Then we get a χ from any group homomorphism $\mathcal{C}\ell_{(I,S_0)}(F) \to S^1$ via precomposition with $\mathbb{A}_F^\times/F^\times \to K_{(I,S_0)}\backslash\mathbb{A}_F^\times/F^\times = \mathcal{C}\ell_{(I,S_0)}(F)$.

Examples (continued)

- In particular, let $F = \mathbb{Q}$, and let χ be thusly obtained from a primitive Dirichlet character $\chi : (\mathbb{Z}/m\mathbb{Z})^\times \to S^1$, which corresponds to $(I, S_0) = (m\mathbb{Z}, \{\mathbb{Q} \to \mathbb{R}\})$. Primitivity implies χ_p is unramified if and only if $p \nmid m$, so unique factorization gives us

$$L^{\{\infty\}}(s, \chi) = \prod_{p \nmid m} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} = \prod_{\substack{n=1 \\ (n,m)=1}}^{\infty} \frac{\chi(n)}{n^s}.$$

Therefore this yields the Dirichlet L-function of χ.

Theorem (Hecke, Tate)

Our $L(s, \chi)$ has meromorphic continuation to all s in $\mathbb{C}$. It is entire unless $\chi = \|\cdot\|^\nu$ for some ν in $i\mathbb{R}$, in which case its only poles are at $s = -\nu$ and $s = 1 - \nu$. Furthermore, there exists a function $\epsilon(s, \chi) = ab^s$, where a lies in $\mathbb{C}^\times$ and b lies in $\mathbb{R}$, such that $L(s, \chi) = \epsilon(s, \chi)L(1 - s, \chi^{-1})$.

Proving the above result is the ultimate goal of this course. Our strategy will be to rewrite every local factor as a certain integral and then put them together using adeles and ideles.

We choose our Haar measure on $\mathbb{A}_F^\times$ as follows. For all v in M_F, take the Haar measure on $F_v^\times$ given by integrating $\mathrm{d}^\times x_v = \mathrm{d}x_v / \|x_v\|_v$ with respect to our self-dual measure on F_v, and then multiply by $(1 - q_v^{-1})$ for nonarchimedean v.

Definition

Let f be in $\mathcal{S}(\mathbb{A}_F)$. Write $Z(s, \chi, f) = \int_{\mathbb{A}_F^\times} \mathrm{d}^\times x\, f(x)\chi(x)\|x\|^s$.

If $f = \prod_{v \in M_F} f_v$ for f_v in $\mathcal{S}(F_v)$ such that $f_v = \mathbf{1}_{\mathcal{O}_v}$ for cofinitely many v, we see that $Z(s, \chi, f) = \prod_{v \in M_F} Z_v(s, \chi_v, f_v)$, where

$$Z_v(s, \chi_v, f_v) = \int_{F_v^\times} \mathrm{d}^\times x_v\, f_v(x_v)\chi_v(x_v)\|x_v\|_v^s.$$

We call these *zeta integrals*. Since elements of $\mathcal{S}(\mathbb{A}_F)$ are finite sums of such f, we always reduce to this case.

Proposition

The integral $Z_v(s, \chi_v, f_v)$ converges for $\operatorname{Re} s > 0$.

Proof.

Since χ is valued in S^1, we need to show $\int_{F_v^\times} \mathrm{d}^\times x_v \, |f_v(x_v)| \|x_v\|_v^{\operatorname{Re} s}$ converges. As f_v is (Bruhat–)Schwartz, we see the integral over $\{x_v \in F_v^\times | \|x_v\|_v > 1\}$ converges. Hence it suffices to show the integral over $\{x_v \in F_v^\times \mid \|x_v\|_v \leq 1\}$ converges. Now f_v is continuous and hence bounded on $\{x_v \in F_v^\times \mid \|x_v\|_v \leq 1\}$, so we can reduce to showing $\int_{\{\|x_v\|_v \leq 1\}} \mathrm{d}^\times x \, \|x_v\|_v^{\operatorname{Re} s}$ converges.

If $F_v = \mathbb{R}$, this integral is $\int_{-1}^{1} \mathrm{d}x_v \, |x_v|^{\operatorname{Re} s - 1} < \infty$. If $F_v = \mathbb{C}$, using polar coordinates $x_v = re^{2\pi i\theta}$ turns this integral into $4\pi \int_0^1 \mathrm{d}\theta \int_0^1 \mathrm{d}r \, r^{2 \operatorname{Re} s - 1}$, which converges. If F_v is nonarchimedean, this integral equals $\#(\mathcal{O}_v/\mathfrak{d}_{F_v/\mathbb{Q}_p})^{-1/2}(1 - q_v^{\operatorname{Re} s})^{-1} < \infty$. □

It follows from the homework that, if v is nonarchimedean and unramified, χ_v is unramified, and $f_v = \mathbf{1}_{\mathcal{O}_v}$, then $Z_v(s, \chi_v, f_v) = (1 - \chi_v(\varpi_v) q_v^{-s})^{-1}$.

Proposition

The integral $Z(s,\chi,f)$ converges for $\operatorname{Re} s > 1$.

Proof.

Let $S \supseteq M_{F,\infty}$ be a finite subset of M_F such that v is unramified, χ_v is unramified, and $f_v = \mathbf{1}_{\mathcal{O}_v}$ for all v not in S. As the $Z_v(s,\chi_v,f_v)$ and hence $\prod_{v\in S} Z_v(s,\chi_v,f_v)$ converge, it suffices to consider $\prod_{v\notin S} Z_v(s,\chi_v,f_v)$. For v not in S, we have $|Z_v(s,\chi_v,f_v)| \le \sum_{k=0}^{\infty} |\chi_v(\varpi_v) q_v^{-s}| = \sum_{k=0}^{\infty} q_v^{-\operatorname{Re} s}$. Hence we get

$$\prod_{v\notin S} |Z_v(s,\chi_v,f_v)| \le \prod_{v\notin S} \left(1 - \frac{1}{q_v^{\operatorname{Re} s}}\right)^{-1}.$$

The same method used to show that the Dedekind zeta function of F absolutely converges for $\operatorname{Re} s > 1$ shows the same for this too. □

Next, we work to prove a local version of our ultimate result.

Proposition

Our $Z_v(s, \chi_v, f_v)$ has meromorphic continuation to all s in $\mathbb{C}$, with no poles for $\operatorname{Re} s > 0$. Furthermore, there exists a meromorphic function $\gamma_v(s, \chi_v)$ such that $\gamma_v(s, \chi_v) Z_v(s, \chi_v, f_v) = Z_v(1 - s, \chi_v^{-1}, \widehat{f_v})$ for all f_v in $\mathcal{S}(F_v)$.

Proof.

The convergence and thus holomorphy of $Z_v(s, \chi_v, f_v)$ and $Z_v(1 - s, \chi_v^{-1}, \widehat{f_v})$ for $\operatorname{Re} s > 0$ imply that $Z_v(1 - s, \chi_v^{-1}, \widehat{f_v})/Z_v(s, \chi_v, f_v)$ is meromorphic for $0 < \operatorname{Re} s < 1$.

First, I claim this quotient is independent of f_v, i.e. for all f_v' in $\mathcal{S}(F_v)$

$$Z_v(1 - s, \chi_v^{-1}, \widehat{f_v}) Z_v(s, \chi_v, f_v') = Z_v(1 - s, \chi_v^{-1}, \widehat{f_v'}) Z_v(s, \chi_v, f_v).$$

To see this, expand the left-hand side to get

$$\int_{F_v^\times} \mathrm{d}^\times x_v \int_{F_v} \mathrm{d}y_v \, f_v(y_v) \psi_{F,v}(x_v y_v)^{-1} \chi_v^{-1}(x_v) \|x_v\|_v^{1-s} \cdot \int_{F_v^\times} \mathrm{d}^\times z_v \, f_v'(z_v) \chi_v(z_v) \|z_v\|_v^s.$$

Proposition

Our $Z_v(s, \chi_v, f_v)$ has meromorphic continuation to all s in $\mathbb{C}$, with no poles for $\operatorname{Re} s > 0$. Furthermore, there exists a meromorphic function $\gamma_v(s, \chi_v)$ such that $\gamma_v(s, \chi_v) Z_v(s, \chi_v, f_v) = Z_v(1 - s, \chi_v^{-1}, \widehat{f_v})$ for all f_v in $\mathcal{S}(F_v)$.

Proof (continued).

Since $F_v^\times$ has full measure in F_v, setting $x_v' = x_v y_v$ turns our integral into

$$(1 - q_v^{-1})^{-1} \int_{F_v^\times} \mathrm{d}^\times x_v' \int_{F_v^\times} \mathrm{d}^\times y_v \int_{F_v^\times} \mathrm{d}^\times z_v f_v(y_v) f_v'(z_v) \cdot \psi_{F,v}(x_v')^{-1} \chi_v(x_v')^{-1} \chi_v(y_v z_v) \|x_v'\|_v^{1-s} \|y_v z_v\|_v^s,$$

where we omit $(1 - q_v^{-1})^{-1}$ for archimedean v. Similarly expanding the right-hand side reverses the role of y_v and z_v. As the above integral is evidently invariant under this reversal, we see the two sides are equal.

So we can set $\gamma_v(s, \chi_v) = Z_v(1 - s, \chi_v^{-1}, \widehat{f_v}) / Z_v(s, \chi_v, f_v)$. By choosing f_v with $\widehat{f_v}$ vanishing in a neighborhood of 0, our previous proof shows $Z_v(1 - s, \chi_v^{-1}, \widehat{f_v})$ is holomorphic for all s in $\mathbb{C}$.

Proposition

Our $Z_v(s, \chi_v, f_v)$ has meromorphic continuation to all s in $\mathbb{C}$, with no poles for $\operatorname{Re} s > 0$. Furthermore, there exists a meromorphic function $\gamma_v(s, \chi_v)$ such that $\gamma_v(s, \chi_v) Z_v(s, \chi_v, f_v) = Z_v(1 - s, \chi_v^{-1}, \widehat{f_v})$ for all f_v in $\mathcal{S}(F_v)$.

Proof (continued).

As $Z_v(s, \chi_v, f_v)$ is holomorphic for $\operatorname{Re} s > 0$, we see $\gamma_v(s, \chi_v)$ meromorphically extends for $\operatorname{Re} s > 0$. Choosing f_v that vanishes in a neighborhood of 0 similarly shows that $\gamma_v(s, \chi_v)$ meromorphically extends for $\operatorname{Re} s < 1$.

Finally, we turn to $Z_v(s, \chi_v, f_v)$. We already showed it's holomorphic for $\operatorname{Re} s > 0$, and the meromorphy of $\gamma_v(s, \chi_v)$ indicates that $Z_v(1 - s, \chi_v^{-1}, \widehat{f_v}) = \gamma_v(s, \chi_v) Z_v(s, \chi_v, f_v)$ meromorphically extends for $\operatorname{Re} s > 0$. Replacing s with $1 - s$, χ_v with χ_v^{-1}, and f_v with $\widehat{f_v}$ shows that $Z_v(s, \chi_v, f_v)$ meromorphically extends for $\operatorname{Re} s < 1$. □