Adelic Poisson Summation

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Recall F is a number field. We had a continuous group homomorphism $\psi_{\mathbb{Q}} : \mathbb{A}_{\mathbb{Q}}/\mathbb{Q} \to S^1$, and then we set $\psi_F = \psi_{\mathbb{Q}} \circ \mathrm{tr}_{\mathbb{A}_F/\mathbb{A}_{\mathbb{Q}}}$. Let's now finish computing self-dual Haar measures for ψ_F.

Example

If $p \neq \infty$, then $\psi_{F,v}$ equals the composition

$$F_v \overset{\mathrm{tr}_{F_v/\mathbb{Q}_p}}{\to} \mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p \subset \mathbb{Q}/\mathbb{Z} \subseteq \mathbb{R}/\mathbb{Z} \overset{\varphi}{\to} S^1.$$

Here, we take $f = \mathbf{1}_{\mathcal{O}_v}$. The Fourier transform of f with respect to the Lebesgue measure on F_v is

$$\widehat{f}(a) = \int_{F_v} \mathrm{d}x\, f(x)\psi_{F,v}(ax)^{-1} = \int_{\mathcal{O}_v} \mathrm{d}x\, \psi_{F,v}(ax).$$

This integral doesn't vanish if and only if $\psi_{F,v}(ax) = 1$ for all x in $\mathcal{O}_v$, in which case it equals 1. In turn, this occurs if and only if $\mathrm{tr}_{F_v/\mathbb{Q}_p}(a\mathcal{O}_v)$ lies in $\mathbb{Z}_p$, i.e. if and only if a lies in $\mathfrak{d}^{-1}_{F_v/\mathbb{Q}_p}$. Therefore $\widehat{f} = \mathbf{1}_{\mathfrak{d}^{-1}_{F_v/\mathbb{Q}_p}}$.

Write $\mathfrak{d}_{F_v/\mathbb{Q}_p} = \pi_v^d \mathcal{O}_v$, where $d \geq 0$. So $\mathfrak{d}^{-1}_{F_v/\mathbb{Q}_p} = \pi_v^{-d}\mathcal{O}_v$.

Example (continued)

The Fourier transform of $\widehat{f}$ with respect to the Lebesgue measure on F_v is

$$\begin{aligned}\widehat{\widehat{f}}(x) &= \int_{F_v} \mathrm{d}a\, \widehat{f}(a)\psi_{F,v}(ax)^{-1} = \int_{\mathfrak{d}_{F_v/\mathbb{Q}_p}^{-1}} \mathrm{d}a\, \psi_{F,v}(ax) \\ &= \|\pi_v\|_v^{-d} \int_{\mathcal{O}_v} \mathrm{d}a'\, \psi_{F,v}(\pi_v^{-d}a'x) \\ &= \#(\mathcal{O}_v/\mathfrak{d}_{F_v/\mathbb{Q}_p}) \int_{\mathcal{O}_v} \mathrm{d}a'\, \psi_{F,v}(\pi_v^{-d}a'x),\end{aligned}$$

where $a = \pi_v^{-d}a'$. This integral doesn't vanish if and only if $\psi_{F,v}(\pi_v^{-d}a'x) = 1$ for all a' in $\mathcal{O}_v$, which occurs if and only if

$$\mathrm{tr}_{F_v/\mathbb{Q}_p}(\pi_v^{-d}a'\mathcal{O}_v) \subseteq \mathbb{Z}_p \iff \pi_v^{-d}a' \in \mathfrak{d}_{F_v/\mathbb{Q}_p}^{-1} = \pi_v^{-d}\mathcal{O}_v \iff a' \in \mathcal{O}_v.$$

Therefore $\widehat{\widehat{f}} = \#(\mathcal{O}_v/\mathfrak{d}_{F_v/\mathbb{Q}_p})\mathbf{1}_{\mathcal{O}_v}$, so $\#(\mathcal{O}_v/\mathfrak{d}_{F_v/\mathbb{Q}_p})^{-1/2}$ times the Lebesgue measure on F_v is self-dual.

We henceforth use the self-dual Haar measure on $\mathbb{A}_F$.

Classical Poisson summation relates functions on $\mathbb{R}$ with their Fourier transforms via summing on the discrete subgroup $\mathbb{Z}$. Adelic Poisson summation does the same with $\mathbb{A}_F$ and F instead.

For all v in M_F, let f_v be in $\mathcal{S}(F_v^n)$, and suppose $f_v = \mathbf{1}_{\mathcal{O}_v^n}$ for cofinitely many v. Then we can form $f = \prod_{v \in M_F} f_v$. Since the f_v are continuous and integrable, we see f is as well.

Definition

A *Bruhat–Schwartz* function on $\mathbb{A}_F^n$ is a finite sum of functions of the above form.

Write $\mathcal{S}(\mathbb{A}_F^n)$ for the set of Bruhat–Schwartz functions on $\mathbb{A}_F^n$. Note it is preserved under addition, multiplication, and scaling by $\mathbb{C}$.

Remark

Because $F_v/\mathbb{Q}_p$ is ramified only for finitely many v, we see the Fourier transform of $\mathbf{1}_{\mathcal{O}_v}$ equals itself for cofinitely many v. As the Fourier transform on F_v yields a $\mathbb{C}$-linear isomorphism $\mathcal{S}(F_v) \xrightarrow{\sim} \mathcal{S}(F_v)$ for all v in M_F, this implies the Fourier transform on $\mathbb{A}_F$ yields a $\mathbb{C}$-linear isomorphism $\mathcal{S}(\mathbb{A}_F) \xrightarrow{\sim} \mathcal{S}(\mathbb{A}_F)$ too.

Proposition

Let f be in $\mathcal{S}(\mathbb{A}_F)$. Then $\mathcal{F}(x) = \sum_{\gamma \in F} f(x+\gamma)$ converges uniformly on compact subsets of $\mathbb{A}_F$ and defines a continuous function $\mathcal{F} : \mathbb{A}_F/F \to \mathbb{C}$.

Proof.

Let $S \supseteq M_{F,\infty}$ be finite. It suffices to consider convergence on $\prod_{v \in M_F} C_v$, where C_v is a compact subset of F_v such that $C_v = \mathcal{O}_v$ for all v not in S and $C_v = \mathfrak{m}_v^{a_v}$ for all v in $S \smallsetminus M_{F,\infty}$. By enlarging S, we can assume it contains all v for which $f_v \neq \mathbf{1}_{\mathcal{O}_v}$. For v in $S \smallsetminus M_{F,\infty}$, by scaling f_v and enlarging its support, we can assume $f_v = \mathbf{1}_{\mathfrak{m}_v^{b_v}}$.

Form the fractional ideal $I = \prod_{v \in S \smallsetminus M_{F,\infty}} v^{\min\{a_v, b_v\}}$ of $\mathcal{O}_F$. For all x in $\prod_{v \in M_F} C_v$, if $f(x+\gamma) \neq 0$, we see $x_v + \gamma$ lies in $\mathfrak{m}_v^{b_v}$ for all v in $S \smallsetminus M_{F,\infty}$ and $\mathcal{O}_v$ for all the other v. Thus γ lies in I, so we get $|\mathcal{F}(x)| \leq \sum_{\gamma \in I} |\prod_{v \in M_{F,\infty}} f_v(x_v+\gamma)|$. Recall that I is a lattice in $\prod_{v \in M_{F,\infty}} F_v$, and note $(x_v)_{v \in M_{F,\infty}}$ lies in the compact subset $\prod_{v \in M_{F,\infty}} C_v$ of $\prod_{v \in M_{F,\infty}} F_v$, so uniform convergence follows from the case of $\mathbb{R}^n$. This also implies $\mathcal{F}$ descends to a continuous function $\mathbb{A}_F/F \to \mathbb{C}$. □

Lemma

Let G be an abelian locally compact topological group, let m be a Haar measure on G, let H be a countable closed subgroup of G, and let D be a Borel subset of G. If D has compact closure, nonempty interior, and maps bijectively to G/H, then the pushforward of m via $D \to G/H$ yields a Haar measure on G/H.

Proof.

Homework problem. □

We call this the *quotient measure* on G/H, and we call D a *fundamental domain* for G/H.

Examples

- Let $G = \mathbb{R}$, with m being the Lebesgue measure, and $H = \mathbb{Z}$. We can take $D = [0, 1)$, which results in the usual measure on $\mathbb{R}/\mathbb{Z} = S^1$.
- Let $G = \mathbb{A}_{\mathbb{Q}}$ and $H = \mathbb{Q}$. It's a homework problem to show we can take $D = \{(x_v)_v \in \mathbb{A}_{\mathbb{Q}} \mid \|x_v\|_v \leq 1 \text{ for } v \neq \infty \text{ and } 0 \leq x_\infty < 1\}$.

Examples (continued)

- Let $G = \mathbb{A}_F$ and $H = F$. By choosing a $\mathbb{Q}$-basis of F, we can identify $F = \mathbb{Q}^n$ and hence $\mathbb{A}_F = \mathbb{A}_{\mathbb{Q}}^n$. Thus we can take D to be the n-th power of the fundamental domain on $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}$.

Lemma

Write m for the quotient measure on $\mathbb{A}_F/F$. Then the dual measure on $\widehat{\mathbb{A}_F/F} = F$ equals $m(\mathbb{A}_F/F)^{-1}$ times the counting measure.

Proof.

As F is discrete and the dual measure is a Haar measure, we see it equals c times the counting measure for some $c > 0$. Taking $f = 1$ in the Fourier inversion formula yields

$$1 = c \sum_{\gamma \in F} \widehat{f}(\gamma)\psi_F(\gamma x)^{-1} = cm(\mathbb{A}_F/F),$$

since $\widehat{f}$ equals $m(\mathbb{A}_F/F)$ times the indicator function on 0. □

Theorem (adelic Poisson summation)

Let f be in $\mathcal{S}(\mathbb{A}_F)$. Then $\sum_{\gamma\in F} f(\gamma) = \sum_{\gamma\in F} \widehat{f}(\gamma)$.

Proof.

Let $\mathcal{F}(x) = \sum_{\gamma\in F} f(x+\gamma)$, considered as a function $\mathbb{A}_F/F \to \mathbb{C}$. Note that $\mathcal{F}(0)$ equals the left-hand side above. Let $D \subseteq \mathbb{A}_F$ be a fundamental domain for G/H. First, I claim $\widehat{f}(c) = \widehat{\mathcal{F}}(c)$ for all c in F, where we use the self-dual measure on $\mathbb{A}_F$ and the quotient measure on $\mathbb{A}_F/F$. To see this, note that

$$\widehat{\mathcal{F}}(c) = \int_D \mathrm{d}x\, \mathcal{F}(x)\psi_F(cx)^{-1} = \int_D \mathrm{d}x \sum_{\gamma\in F} f(x+\gamma)\psi_F(cx)^{-1}$$

$$= \int_D \mathrm{d}x \sum_{\gamma\in F} f(x+\gamma)\psi_F(c(x+\gamma))^{-1} = \int_{\mathbb{A}_F} \mathrm{d}y\, f(y)\psi_F(cy)^{-1} = \widehat{f}(c),$$

where $y = x+\gamma$. Now $\widehat{f}$ lies in $\mathcal{S}(\mathbb{A}_F)$, so $\sum_{\gamma\in F} |\widehat{f}(\gamma)| = \sum_{\gamma\in F} |\widehat{\mathcal{F}}(\gamma)|$ converges. In other words, $\widehat{\mathcal{F}}$ lies in $L^1(F)$.

Theorem (adelic Poisson summation)

Let f be in $\mathcal{S}(\mathbb{A}_F)$. Then $\sum_{\gamma \in F} f(\gamma) = \sum_{\gamma \in F} \widehat{f}(\gamma)$.

Proof (continued).

Hence Fourier inversion applies to $\mathcal{F}$, so

$$\sum_{\gamma \in F} f(\gamma) = \mathcal{F}(0) = m(\mathbb{A}_F/F)^{-1} \sum_{\gamma \in F} \widehat{\mathcal{F}}(\gamma) \psi_F(0)^{-1} = m(\mathbb{A}_F/F)^{-1} \sum_{\gamma \in F} \widehat{f}(\gamma).$$

Replacing f with $\widehat{f}$ in this formula and applying Fourier inversion to f yields

$$\sum_{\gamma \in F} f(\gamma) = m(\mathbb{A}_F/F)^{-2} \sum_{\gamma \in F} f(-\gamma) = m(\mathbb{A}_F/F)^{-2} \sum_{\gamma \in F} f(\gamma).$$

Taking any f with $\sum_{\gamma \in F} f(\gamma) \neq 0$ indicates $m(\mathbb{A}_F/F)^{-2} = 1$, so we see $m(\mathbb{A}_F/F) = 1$. Thus the original formula yields the desired result. □

Remark

This shows $m(\mathbb{A}_F/F) = 1$ when m is the quotient measure of the self-dual measure on $\mathbb{A}_F$ with respect to ψ_F. We won't use it, but it's often useful take the measure on $\mathbb{A}_F$ that takes the usual Lebesgue measure for $v \nmid \infty$. With the quotient measure of this, the volume of $\mathbb{A}_F/F$ is

$$\prod_{v \notin M_{F,\infty}} \#(\mathcal{O}_v/\mathfrak{d}_{F_v/\mathbb{Q}_p})^{1/2} = \#(\mathcal{O}_F/\mathfrak{d}_{F/\mathbb{Q}})^{1/2}$$

$$= |\operatorname{Nm}_{F/\mathbb{Q}}(\mathfrak{d}_{F/\mathbb{Q}})|^{1/2} = |\mathcal{D}_{F/\mathbb{Q}}|^{1/2}.$$

We conclude by relating idelic and ray class group characters as follows.

Proposition

Let $\chi : \mathbb{A}_F^\times/F^\times \to \mathbb{C}^\times$ be a continuous group homomorphism such that $\chi^m = 1$ for some m. Then $\ker \chi$ contains the image of $K_{(I,S_0)}$ for some modulus (I, S_0) for F.

Hence χ induces a continuous group homomorphism from the ray class group $\mathcal{C}\ell_{(I,S_0)}(F) = K_{(I,S_0)} \backslash \mathbb{A}_F^\times/F^\times \to \mathbb{C}^\times$.

Proposition

Let $\chi : \mathbb{A}_F^\times / F^\times \to \mathbb{C}^\times$ be a continuous group homomorphism such that $\chi^m = 1$ for some m. Then $\ker \chi$ contains the image of $K_{(I,S_0)}$ for some modulus (I, S_0) for F.

Proof.

As $\chi^m = 1$, the image of χ lies in $\{\zeta \in \mathbb{C} \mid \zeta^m = 1\}$, which is discrete. Now $\mathbb{R}_{>0}$ and $\mathbb{C}^\times$ are connected, so their images under χ must be trivial by continuity. Thus if we take S to be all the real embeddings, $\ker \chi$ contains $\prod_{v \in M_{F,\infty}} K_{(I,S_0),v}$.

Let U be a neighborhood of 1 in $\mathbb{C}^\times$ containing no nontrivial subgroups of $\mathbb{C}^\times$. As the preimage of $\chi^{-1}(U)$ in $\mathbb{A}_F^\times$ is a neighborhood of 1, we see it contains $\prod_{v \notin M_{F,\infty}} N_v$, where the N_v are open subsets of $F_v^\times$ such that $N_v = \mathcal{O}_v^\times$ for all v not in some finite subset $S \supseteq M_{F,\infty}$, and $N_v = 1 + \mathfrak{m}_v^{a_v}$ for all v in $S \setminus M_{F,\infty}$. Now the image of $\prod_{v \notin M_{F,\infty}} N_v$ in $\mathbb{C}^\times$ is a subgroup and thus trivial, so we can take $I = \prod_{v \in S \setminus M_{F,\infty}} v^{a_v}$. □