Pontryagin Duality on the Adeles

(featuring differents)

Siyan Daniel Li-Huerta

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Let F be a number field, and let $\psi : \mathbb{A}_F \to S^1$ be a continuous homomorphism such that $\psi_v : F_v \to S^1$ is nontrivial for all v in M_F. Our previous work implies the map $\psi_\cdot : \mathbb{A}_F \to \widehat{\mathbb{A}_F}$ given by $a \mapsto (x \mapsto \psi(ax))$ is an isomorphism of topological groups.

In addition, suppose that $\psi|_F = 1$. Then we have $\psi_a|_F = 1$ for all a in F, so $\psi_\cdot$ induces a morphism $\psi_\cdot : F \to \widehat{\mathbb{A}_F/F}$ of topological groups.

Example

Let $F = \mathbb{Q}$, let $\psi_\infty : \mathbb{R} \to S^1$ be $x \mapsto \varphi(-x)$, and for all prime numbers p, let ψ_p be $\mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p \subset \mathbb{Q}/\mathbb{Z} \subset \mathbb{R}/\mathbb{Z} \xrightarrow{\varphi} S^1$. Then $\psi = \prod_{v \in M_\mathbb{Q}} \psi_v$ works.

Proposition

This yields an isomorphism $\psi_\cdot : F \xrightarrow{\sim} \widehat{\mathbb{A}_F/F}$ of topological groups.

Proof.

Homework problem. □

Example

Recall that $\mathbb{A}_\mathbb{Q} \otimes_\mathbb{Q} F \xrightarrow{\sim} \mathbb{A}_F$. Hence $\mathbb{A}_F$ is a free $\mathbb{A}_\mathbb{Q}$-module of finite rank, so we have a trace map $\mathrm{tr}_{\mathbb{A}_F/\mathbb{A}_\mathbb{Q}} : \mathbb{A}_F \to \mathbb{A}_\mathbb{Q}$. This is evidently a continuous group homomorphism, and we see its restriction to F equals $\mathrm{tr}_{F/\mathbb{Q}}$. Hence it sends F to $\mathbb{Q}$, so the composition $\psi_F = \psi \circ \mathrm{tr}_{\mathbb{A}_F/\mathbb{A}_\mathbb{Q}}$ yields a continuous group homomorphism $\mathbb{A}_F \to S^1$ that is trivial on F.

For any p (including $p = \infty$) in $M_\mathbb{Q}$, recall that $\mathbb{Q}_p \otimes_\mathbb{Q} F \xrightarrow{\sim} \prod_{v|p} F_v$ in the above isomorphism. Thus $\mathrm{tr}_{\mathbb{A}_F/\mathbb{A}_\mathbb{Q}}$ restricted to F_v equals $\mathrm{tr}_{F_v/\mathbb{Q}_p}$, which makes $\psi_{F,v} = \psi_p \circ \mathrm{tr}_{F_v/\mathbb{Q}_p}$. This is evidently nontrivial, so altogether ψ_F works.

We use this ψ_F in computations, for which we want to know what the corresponding self-dual Haar measure is.

Example

If $p = \infty$ and $F_v = \mathbb{R}$, then $\psi_{F,v}(x) = \varphi(-x)$ for all x in $\mathbb{R}$. Integrating the Gaussian $f(x) = e^{-\pi x^2}$ again, since the change of variables $x \mapsto -x$ preserves $f(x)$, shows the Lebesgue measure is self-dual.

Example

If $p = \infty$ and $F_v = \mathbb{C}$, then $\psi_{F,v}(z) = \varphi(-2\operatorname{Re} z)$ for all z in $\mathbb{C}$. Here, we take $f(z) = e^{-2\pi|z|^2}$. Writing $\xi = a + bi$ for a variable valued in $\widehat{F_v} = \mathbb{C}$, the Fourier transform of f with respect to the Lebesgue measure on $\mathbb{C}$ is

$$\begin{aligned}\widehat{f}(\xi) &= \int_{\mathbb{C}} \mathrm{d}z\, f(z)\psi_{F,v}(\xi z)^{-1} = \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y\, e^{-2\pi(x^2+y^2)} e^{4\pi i(ax-by)} \\ &= \int_{-\infty}^{\infty} \mathrm{d}x\, e^{-2\pi x^2} e^{4\pi aix} \int_{-\infty}^{\infty} \mathrm{d}y\, e^{-2\pi y^2} e^{-4\pi biy} = \frac{e^{-\pi(\sqrt{2}a)^2}}{\sqrt{2}} \cdot \frac{e^{-\pi(\sqrt{2}b)^2}}{\sqrt{2}} \\ &= \frac{1}{2} f(\xi).\end{aligned}$$

Thus 2 times the Lebesgue measure on $\mathbb{C}$ is self-dual.

Before proceeding to nonarchimedean v, we need to introduce the *different*. Let A be a Dedekind domain, write $F = \operatorname{Frac} A$, let E/F be a finite separable extension, and write B for the integral closure of A in E. Recall that $\operatorname{Frac} B = E$.

Examples

- For a finite extension E/F of number fields, taking $A = \mathcal{O}_F$ yields $B = \mathcal{O}_E$.
- For a finite separable extension E_w/F_v of nonarchimedean local fields, taking $A = \mathcal{O}_v$ yields $B = \mathcal{O}_w$.

Since E/F is separable, the F-bilinear pairing $E \times E \to F$ given by $(x, y) \mapsto \mathrm{tr}_{E/F}(xy)$ is non-degenerate. We call this the *trace pairing*.

Definition

Let M be an A-submodule of E. The *dual* of M with respect to the trace pairing is the A-submodule $M^* = \{x \in E \mid \mathrm{tr}_{E/F}(xM) \subseteq A\}$ of E.

Note that if $M_1 \subseteq M_2$ are A-submodules of E, then $M_2^* \subseteq M_1^*$.

Let M be a B-submodule of E. For all b in B and x in M^*, we have $\mathrm{tr}_{E/F}(bxM) = \mathrm{tr}_{E/F}(xbM) \subseteq \mathrm{tr}_{E/F}(xM) \subseteq A$, so M^* is a B-submodule of E.

Proposition

Let M be a nonzero fractional ideal of B. Then M^* is also a nonzero fractional ideal of B.

Proof.

Suppose $bM \subseteq B$ for some b in B. Then $\mathrm{tr}_{E/F}(bM) \subseteq \mathrm{tr}_{E/F}(B) = A$, so b lies in B. Since M is nonzero, we can take $b \neq 0$, making M^* also nonzero.

Next, we show M^* is a finitely generated B-module. Let $x_1, \ldots, x_n$ be an F-basis of E, which we may assume to lie in B via scaling. For any nonzero m in M, the $x_1 m, \ldots, x_n m$ are also an F-basis of E, and they lie in M. Hence the free A-module N they generate lies in M, so $M^* \subseteq N^*$. Now N^* is a free A-module of rank n, so the noetherianity of B implies that M^* is finitely generated over A. So M^* is finitely generated over B too. □

Definition

The *different* of B over A, denoted by $\mathfrak{d}_{B/A}$, is the nonzero fractional ideal $(B^*)^{-1}$.

Note that the B-submodule B^* contains 1, so $B \subseteq B^*$. Hence $\mathfrak{d}_{B/A} \subseteq B^{-1} = B$, so $\mathfrak{d}_{B/A}$ is actually an ideal of B.

Proposition

Let S be a multiplicative subset of A. Then $\mathfrak{d}_{S^{-1}B/S^{-1}B} = S^{-1}\mathfrak{d}_{B/A}$.

Proof.

I claim that $S^{-1}(B^*) = (S^{-1}B)^*$, where the latter is taken with respect to $S^{-1}A$. For $\frac{x}{s}$ in $S^{-1}(B^*)$, where s lies S and x lies in B^*, and $\frac{b}{t}$ in $S^{-1}B$, where t lies in S and b lies in B, we see $\mathrm{tr}_{E/F}(\frac{x}{s} \cdot \frac{b}{t}) = \frac{1}{st}\,\mathrm{tr}_{E/F}(xb)$ lies in $S^{-1}B$. Therefore $S^{-1}(B^*) \subseteq (S^{-1}B)^*$.

Conversely, let x be in $(S^{-1}B)^*$. Write $x_1, \ldots, x_r$ for generators of B over A. Then every $\mathrm{tr}_{E/F}(xx_i)$ lies in $S^{-1}A$, so it equals $\frac{a_i}{s_i}$ for some s_i in S and a_i in A. Then $\mathrm{tr}_{E/F}(s_1 \cdots s_r xB) = (s_1 \cdots s_r)\,\mathrm{tr}_{E/F}(xB) \subseteq A$, so we have $s_1 \cdots s_r x$ lying in B^*. Hence x lies in $S^{-1}(B^*)$.

The desired result follows from taking ideal inverses. □

Let v be a nonzero prime ideal of A. We use $(\cdot)_v$ to denote completions with respect to the norm induced by v.

Proposition

Let w be a nonzero prime ideal of B dividing v. Then $\mathfrak{d}_{B_w/A_v} = \mathfrak{d}_{B/A} B_w$.

Proof.

Recall that $E_v \xrightarrow{\sim} \prod_{w|v} E_w$ as F_v-algebras, so $\mathrm{tr}_{E_v/F_v} = \sum_{w|v} \mathrm{tr}_{E_w/F_v}$. We have shown that $B_v = \prod_{w|v} B_w$ as A_v-algebras under this identification, so we see that $(B_v)^* = \prod_{w|v} (B_w)^*$. Now $(B_v)^* = (B^*)_v$, so it's generated by B^* over A_v. Looking at the w-component shows $(B_w)^*$ is generated by B^* over B_w, so taking ideal inverses yields the desired result. □

Combining this Proposition with unique factorization shows that we can compute $\mathfrak{d}_{B/A}$ by computing it after completing at each nonzero prime w of B.

Remark

The ideal $\mathfrak{d}_{B/A}$ contains fine information about ramification—for example, w is ramified over v if and only if w divides $\mathfrak{d}_{B/A}$, and the ramification degree can be bounded using $\mathfrak{d}_{B/A}$. In *tamely ramified* cases, the ramification degree can be computed exactly using $\mathfrak{d}_{B/A}$.

Instead of the above, we content ourselves with the following link to ramification. Write $\mathcal{D}_{B/A}$ for the discriminant ideal of B over A, which is a nonzero ideal of A.

Proposition

We have $\mathcal{D}_{B/A} = \mathrm{Nm}_{B/A}(\mathfrak{d}_{B/A})$.

Proof.

Because norms, differents, and discriminants commute with localization, unique factorization allows us to reduce to the case when A and hence B are local rings. Thus A is a principal ideal domain, so B is a free A-module of finite rank. Let $x_1, \ldots, x_n$ be an A-basis of B. Then B^* is also a free A-module, with an A-basis $x_1^*, \ldots, x_n^*$ characterized by $\mathrm{tr}_{E/F}(x_i x_j^*) = \delta_{ij}$.

Proposition

We have $\mathcal{D}_{B/A} = \mathrm{Nm}_{B/A}(\mathfrak{d}_{B/A})$.

Proof (continued).

Thus for any free A-submodule M of E with A-basis $b_1, \ldots, b_n$, we see that $m_i = \sum_{j=1}^{n} \mathrm{tr}_{E/F}(m_i x_j) x_j^*$. Applying this to B shows that

$$\mathcal{D}_{B/A} = \det\big(\mathrm{tr}_{E/F}(x_i x_j)\big)_{i,j=1}^{n}$$

is the ideal generated by the product of the elementary divisors of B^* over B, where both are considered as free A-modules of finite rank. This in turn equals the product of the elementary divisors of $B^{-1} = B$ over $(B^*)^{-1} = \mathfrak{d}_{B/A}$, which is precisely $\mathrm{Nm}_{B/A}(\mathfrak{d}_{B/A})$. □

As usual, when working with number fields or local fields, we often index everything with the field instead of the Dedekind domain.