More on Adeles and Ideles

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October 27, 2020

Let F be a number field. Here's a generalization of class groups.

Definition

A *modulus* for F is a pair (I, S_0), where I is a nonzero ideal of $\mathcal{O}_F$, and S_0 is a subset of the real embeddings $\{\sigma_1, \ldots, \sigma_{r_1}\}$.

For any modulus (I, S_0), write $J(I, S_0)$ for the subgroup

$$J(I, S_0) = \left\{ \ x \in F^\times \ \middle| \ \begin{array}{l} \text{for all } v|I, \text{ we have } v(x-1) \geq v(I), \\ \text{for all } \sigma \in S_0, \text{ we have } \sigma(x) > 0, \end{array} \right\}.$$

Write $\mathcal{I}_{F,I}$ for the subgroup of $\mathcal{I}_F$ of nonzero fractional ideals that are coprime to I. For x in $J(I, S_0)$, we see that (x) lies in $\mathcal{I}_{F,I}$.

Definition

The *ray class group* of conductor (I, S_0), denoted by $\mathcal{C}\ell_{(I,S_0)}(F)$, is the quotient group $\mathcal{I}_{F,I} / \operatorname{im} J(I, S_0)$.

Examples

- Let $(I, S_0) = (\mathcal{O}_F, \varnothing)$. We see that $J(I, S_0) = F^\times$ and $\mathcal{I}_{F,I} = \mathcal{I}_F$, so here $\mathcal{C}\ell_{(I,S_0)}(F)$ is the usual class group.

Examples (continued)

- Let $F = \mathbb{Q}$, and let $(I, S_0) = (m\mathbb{Z}, \varnothing)$. Form the homomorphism $\mathcal{I}_{F,I} \to (\mathbb{Z}/m\mathbb{Z})^\times/\{\pm 1\}$ given by $\frac{a}{b}\mathbb{Z} \mapsto ab^{-1}$ for integers a and b not divisible by m. This homomorphism is well-defined and evidently surjective. Furthermore, its kernel consists of $\frac{a}{b}\mathbb{Z}$ such that $a \equiv \pm b \pmod{m}$, and such $\frac{a}{b}$ are precisely those in $J(I, S_0)$. So here $\mathcal{C}\ell_{(I,S_0)}(F) = (\mathbb{Z}/m\mathbb{Z})^\times/\{\pm 1\}$,
- Let $F = \mathbb{Q}$, and let $(I, S_0) = (m\mathbb{Z}, \{\mathbb{Q} \to \mathbb{R}\})$. Form the homomorphism $\mathcal{I}_{F,I} \to (\mathbb{Z}/m\mathbb{Z})^\times$ given by $\frac{a}{b}\mathbb{Z} \mapsto ab^{-1}$ for integers a and b not divisible by m, where we choose $a > 0$ and $b > 0$. This homomorphism is well-defined and evidently surjective, and we see its kernel consists of $\frac{a}{b}\mathbb{Z}$ such that $a \equiv b \pmod{m}$, where $\frac{a}{b}$ must be positive. Such $\frac{a}{b}$ are precisely those in $J(I, S_0)$, so here $\mathcal{C}\ell_{(I,S_0)}(F) = (\mathbb{Z}/m\mathbb{Z})^\times$,
- Let F be a real quadratic field of discriminant D, and let $(I, S_0) = (\mathcal{O}_F, \{\sigma_1, \sigma_2\})$. One can show that $\mathcal{C}\ell_{(I,S_0)}(F)$ naturally bijects with $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of primitive integral binary quadratic forms of discriminant D.

For any modulus (I, S_0) and any v in M_F, write $K_{(I,S_0),v}$ for the subgroup

$$K_{(I,S_0),v} = \begin{cases} \mathcal{O}_v^\times & \text{if } v \text{ is nonarchimedean and } v \nmid I, \\ 1 + \mathfrak{m}_v^{v(I)} & \text{if } v \text{ is nonarchimedean and } v | I, \\ \mathbb{R}_{>0} & \text{if } v \text{ is archimedean and } v \in S_0, \\ F_v^\times & \text{otherwise,} \end{cases}$$

of $F_v^\times$. Then $K_{(I,S_0)} = \prod_{v \in M_F} K_{(I,S_0),v}$ is an open subgroup of $\mathbb{A}_F^\times$.

Proposition

The double quotient $K_{(I,S_0)} \backslash \mathbb{A}_F^\times / F^\times$ is naturally isomorphic to $\mathcal{C}\ell_{(I,S_0)}(F)$.

Proof.

Set $\mathbb{A}_F^{\times,(I,S_0)} = \{(x_v)_v \in \mathbb{A}_F^\times \mid \text{for } v|I \text{ and } v \text{ archimedean, } x_v \in K_{(I,S_0),v}\}$. Evidently $\mathbb{A}_F^{\times,(I,S_0)}$ contains $K_{(I,S_0)}$, and the valuation maps induce an isomorphism $K_{(I,S_0)} \backslash \mathbb{A}_F^{\times,(I,S_0)} \xrightarrow{\sim} \mathcal{I}_{F,I}$. Writing $F_{(I,S_0)}^\times$ for $F^\times \cap \mathbb{A}_F^{\times,(I,S_0)}$, we see that $F_{(I,S_0)}^\times = J(I, S_0)$. Hence $K_{(I,S_0)} \backslash \mathbb{A}_F^{\times,(I,S_0)} / F_{(I,S_0)}^\times \xrightarrow{\sim} \mathcal{C}\ell_{(I,S_0)}(F)$.

Proposition

The double quotient $K_{(I,S_0)} \backslash \mathbb{A}_F^\times / F^\times$ is naturally isomorphic to $\mathcal{C}\ell_{(I,S_0)}(F)$.

Proof (continued).

Applying weak approximation to $\{v \in M_F \mid v|I \text{ or } v \in S_0\}$ shows that $\mathbb{A}_F^\times = F^\times \mathbb{A}_F^{\times,(I,S_0)}$. Therefore inclusion yields an isomorphism $\mathbb{A}_F^{\times,(I,S_0)} / F_{(I,S_0)}^\times \xrightarrow{\sim} \mathbb{A}_F^\times / F^\times$. Quotienting by $K_{(I,S_0)}$ and combining this with the above yields the desired result. □

Next, let's turn towards *strong approximation*. First, recall that for any Borel subset U_v of F_v and x_v in F_v, we have $m_v(x_v U_v) = \|x_v\|_v m_v(U_v)$. As the measure m on $\mathbb{A}_F^\times$ is given by the product of every m_v, for any Borel subset U of $\mathbb{A}_F$ and x in $\mathbb{A}_F$, we similarly have $m(xU) = \|x\| m(U)$.

Recall from the proof of the compactness of $\mathbb{A}_F / F$ that we had a compact subset K of $\mathbb{A}_F$ with nonempty interior such that $K + F = \mathbb{A}_F$. Thus $m(K)$ is positive and finite.

Lemma

Let G be an abelian locally compact topological group, let m be a Haar measure on G, let H be a countable subgroup of G, and let K be a Borel subset of G such that $K + H = G$. If Y is a Borel subset of G such that $m(Y) > m(K)$, then there exist distinct y_1 and y_2 in Y such that $y_1 - y_2$ lies in H.

Lemma (Blichfeldt–Minkowski)

There exists a positive $C > 0$ such that, for any z in $\mathbb{A}_F$ with $\|z\| > C$, there exists b in $F^\times$ such that $\|b\|_v \leq \|z_v\|_v$ for all v in M_F.

Proof.

Write c_0 for $m(K)$, where K is as above. Next, write c_1 for the measure of $X = \{(x_v)_v \in \mathbb{A}_F \mid \|x_v\|_v \leq 1 \text{ for } v \nmid \infty \text{ and } \|x_v\|_v \leq \frac{1}{4} \text{ for } v \mid \infty\}$, which is positive and finite. Set $C = \frac{c_0}{c_1}$, and suppose z in $\mathbb{A}_F$ has $\|z\| > C$. Now form the compact subset

$$Y = \{(y_v)_v \in \mathbb{A}_F \mid \|y_v\|_v \leq \|z_v\|_v \text{ for } v \nmid \infty \text{ and } \|y_v\|_v \leq \tfrac{1}{4}\|z_v\|_v \text{ for } v \mid \infty\}$$

Lemma (Blichfeldt–Minkowski)

There exists a positive $C > 0$ *such that, for any* x *in* $\mathbb{A}_F$ *with* $\|z\| > C$*, there exists* b *in* $F^\times$ *such that* $\|b\|_v \le \|z_v\|_v$ *for all* v *in* M_F*.*

Proof (continued).

We see $Y = zX$, so $m(Y) = \|z\| c_1 > c_0 = m(D)$. Thus there exist distinct y_1 and y_2 in Y such that $b = y_1 - y_2$ lies in F.

For $v \nmid \infty$, the strong triangle inequality yields $\|b\|_v \le \|z_v\|_v$. For real v, we have $\|b\|_v \le \|y_{1,v}\|_v + \|y_{2,v}\|_v \le \frac{1}{2}\|z_v\|_v$, and for complex v we get

$$(\|b\|_v^{1/2})^2 \le (\|y_{1,v}\|_v^{1/2} + \|y_{2,v}\|_v^{1/2})^2 \le (\tfrac{1}{2}\|z_v\|_v^{1/2} + \tfrac{1}{2}\|z_v\|_v^{1/2})^2 \le \|z_v\|_v.$$

□

Note that the above strongly resembles the usual proof of Minkowski's convex body theorem.

Theorem (Strong approximation)

Let w be in M_F, let S be a finite subset of M_F not containing w. For all v in S, let x_v be in K_v, and let $\epsilon > 0$. Then there exists x in F such that $\|x - x_v\|_v < \epsilon$ for all v in S, and $\|x\|_v \leq 1$ for all v not in $S \cup \{w\}$.

Proof.

Write $K = \{(x_v)_v \in \mathbb{A}_F \mid \|x_v\|_v \leq 1 \text{ for all } v\}$. Evidently K is compact, and it'll be a homework problem to show $K + F = \mathbb{A}_F$. For any b in $F^\times$, we see that $bK + F = b(K + F) = b\mathbb{A}_F = \mathbb{A}_F$.

Choose z in $\mathbb{A}_K$ such that $0 < \|z_v\|_v < \epsilon$ for v in S, $0 < \|z_v\|_v \leq 1$ for v not in $S \cup \{w\}$, and $\|z\| > C$. Blichfeldt–Minkowski yields b in $F^\times$ such that $\|b\|_v \leq \|z_v\|_v$ for all v in M_F.

Let $a = (a_v)_v$ be given by $a_v = x_v$ for all v in S and $a_v = 0$ otherwise. Then $a = bk + x$ for some k in K and x in F. Thus $x - a = bk$ satisfies $\|x - a_v\|_v = \|b\|_v \leq \|z_v\|_v$ for all v in M_F, and this concludes the proof by our choice of z. □

Upshot: we get great approximations by sacrificing one w.

Corollary

Let w be in M_F. Then the image of F in $\mathbb{A}_F/F_w$ is dense.

So even though F is discrete in all of $\mathbb{A}_F$, if you get rid of any one factor in $\mathbb{A}_F$, it becomes dense!

Recall that the continuous surjective group homomorphism $\|\cdot\| : \mathbb{A}_F^\times/F^\times \to \mathbb{R}_{>0}$ prevents $\mathbb{A}^\times/F^\times$ from being compact. We'll see that this is the only obstruction to compactness. Write $\mathbb{A}_F^{\times,1}$ for the closed subgroup of $\mathbb{A}_F^\times$ given by x with $\|x\| = 1$.

Proposition

The quotient $\mathbb{A}_F^{\times,1}/F^\times$ is compact.

Proof.

Let K be our compact subset such that $K + F = \mathbb{A}_F$. Choose any compact subset Y such that $m(Y) > m(K)$. Then $Y_1 = Y - Y$ and $Y_2 = Y_1 Y_1$ are also compact, and $Y_1 \cap F^\times$ and hence $Y_2 \cap F^\times$ is nonempty.

Proposition

The quotient $\mathbb{A}_F^{\times,1}/F^\times$ is compact.

Proof (continued).

As $Y_2 \cap F^\times$ is discrete and compact, it is finite. Write $x_1, \ldots, x_r$ for its elements. Write $\Delta : \mathbb{A}_F^\times \to \mathbb{A}_F^2$ for the map $x \mapsto (x, x^{-1})$, and set

$$K^* = \bigcup_{i=1}^{r} \Delta^{-1}(\{(y, x_i^{-1}y') \in \mathbb{A}_F^2 \mid y, y' \in Y_1\}).$$

Since Δ is a homeomorphism onto its image, which is closed in $\mathbb{A}_F^2$, we see K^* is compact. I claim that $\mathbb{A}_F^{\times,1} \subseteq K^* F^\times$.

To see this, let x be in $\mathbb{A}_F^{\times,1}$. Then $m(xY) = m(x^{-1}Y) = m(Y) > m(K)$, so there exist y_1, y_2, y_3, y_4 in Y with $y_1 \neq y_2$ and $y_3 \neq y_4$ such that $\alpha = x(y_1 - y_2)$ and $\beta = x^{-1}(y_1 - y_2)$ lie in F. Therefore $\alpha\beta = (y_1 - y_2)(y_3 - y_4)$ lies in $Y_2 \cap F^\times$, so $\alpha\beta$ equals some x_i. Thus $\Delta(x\beta) = \Delta(y_1 - y_2) = (y_1 - y_2, x_i^{-1}(y_3 - y_4))$, making $x\beta$ lie in K^*. □