Adeles and Ideles

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October 22, 2020

Mentally recall our restricted product setup from last time.

Corollary

For all v in M, let $f_v : G_v \to \mathbb{C}$ be a continuous function in $L^1(G_v)$. If $f_v = \mathbf{1}_{K_v}$ for cofinitely many v, then the continuous map $f = \prod_{v \in M} f_v$ lies in $L^1(G)$. Furthermore, if each of the G_v is abelian, then $\widehat{f} = \prod_{v \in M} \widehat{f_v}$.

Proof.

Let $S \supseteq M_\infty$ be a finite subset of M such that $f_v = \mathbf{1}_{K_v}$ for v not in S. Then $\int_G \mathrm{d}x\, f(x) = \prod_{v \in S} \int_{G_v} \mathrm{d}x_v\, f_v(x_v)$ converges, so f lies in $L^1(G)$. If each of the G_v is abelian, for all $\chi = \prod_{v \in M} \chi_v$ in $\widehat{G}$ we can thus form

$$\widehat{f}(\chi) = \int_G \mathrm{d}x\, f(x)\chi(x)^{-1} = \prod_{v \in M} \int_{G_v} \mathrm{d}x_v\, f_v(x_v)\chi_v(x_v)^{-1} = \prod_{v \in M} \widehat{f_v}(\chi_v).$$

□

Assume now that each of the G_v is abelian, and write $\widehat{m_v}$ for the dual measure on $\widehat{G_v}$. How are the $\widehat{m_v}$ related to the dual measure on $\widehat{G}$?

Lemma

Let v not be in M_∞. If $f_v = \mathbf{1}_{K_v}$, then $\widehat{f_v} = m_v(K_v)\mathbf{1}_{W_v}$.

Proof.

If χ_v lies in W_v, then $\chi_v|_{K_v} = 1$. Thus $\widehat{f_v}(\chi_v) = \int_{K_v} \mathrm{d}x_v = m_v(K_v)$. Otherwise, there exists g in K_v such that $\chi_v(g) \neq 1$. Left invariance gives $\widehat{f_v}(\chi_v) = \int_{K_v} \mathrm{d}x_v\, \chi_v(x_v)^{-1} = \int_{K_v} \mathrm{d}x_v\, \chi_v(g^{-1}x_v)^{-1} = \chi_v(g)\widehat{f_v}(\chi_v)$, so we have $\widehat{f_v}(\chi_v) = 0$. □

Applying Fourier inversion to f_v shows that $m_v(K_v)\widehat{m_v}(W_v) = 1$. Thus $\widehat{m_v}(W_v) = 1$ for cofinitely many v, so we can form the Haar measure $\widehat{m} = \prod_{v \in M} \widehat{m_v}$ on $\widehat{G} = \prod'_{v \in M} \widehat{G_v}$.

For all v in M, let $f_v : G_v \to \mathbb{C}$ be a compactly supported continuous function with $f_v(1) \neq 0$ that $f_v = \mathbf{1}_{K_v}$ for cofinitely many v. Applying Fourier inversion to $f = \prod_{v \in M} f_v$ shows that $\widehat{m}$ is the dual measure on $\widehat{G}$.

Now let's focus on our cases of interest. Recall that F is a number field.

Definition

We write $\mathbb{A}_F = \prod'_{v \in M_F} F_v$ for the *adeles* of F, and we write $\mathbb{A}_F^\times = \prod'_{v \in M_F} F_v^\times$ for the *ideles* of F.

Remark

The adeles $\mathbb{A}_F$ naturally form a topological ring, and we can identify its invertible elements with $\mathbb{A}_F^\times$ as a group. However, they do not have the same topology!

For any nonzero x in F, we have $v_\mathfrak{p}(x) = 0$ for cofinitely many nonzero prime ideals $\mathfrak{p}$ of $\mathcal{O}_F$. Hence we get an injective ring homomorphism $F \to \mathbb{A}_F$ given by $x \mapsto (x)_v$, and it induces a group homomorphism $F^\times \to \mathbb{A}_F^\times$.

Let E/F be a finite extension, and let v be a nonarchimedean nonarchimedean norm on F. We write v for the nonzero prime ideal corresponding to v. Recall that the natural map $E \otimes_F F_v \to \prod_{w|v} E_w$ is an isomorphism.

Proposition

The natural map $\mathcal{O}_E \otimes_{\mathcal{O}_F} \mathcal{O}_v \to \prod_{w|v} \mathcal{O}_w$ is an isomorphism of $\mathcal{O}_v$-algebras.

Proof (sketch).

Both sides are free $\mathcal{O}_v$-modules of the same rank, so it suffices to show the map is surjective. By Nakayama's lemma, it suffices to check this modulo $\mathfrak{m}_v$. The nonzero ideal $v\mathcal{O}_E$ factorizes as $\prod_{w|v} w^{e_w}$ in $\mathcal{O}_E$, and we see that $\mathfrak{m}_v\mathcal{O}_w = \mathfrak{m}_w^{e_w}$. The left-hand side modulo $\mathfrak{m}_v$ is

$$\begin{aligned}(\mathcal{O}_E \otimes_{\mathcal{O}_F} \mathcal{O}_v)/\mathfrak{m}_v &= \mathcal{O}_E/v\mathcal{O}_E = \mathcal{O}_E/\prod_{w|v} w^{e_w} \\ &= \prod_{w|v} \mathcal{O}_E/w^{e_w} = \prod_{w|v} \mathcal{O}_w/\mathfrak{m}_v\mathcal{O}_w = (\prod_{w|v} \mathcal{O}_w)/\mathfrak{m}_v\end{aligned}$$

by the Chinese remainder theorem. This identification is our map modulo $\mathfrak{m}_v$, so we get an isomorphism as desired. □

Write n for the degree of E/F.

By choosing an F-basis of E, we can identify $E = F^n$ as F-vector spaces. This identifies $\mathbb{A}_F \otimes_F E = \mathbb{A}_F^n$, which we give the product topology. Because $\mathbb{A}_F$ is a topological ring, this topology on $\mathbb{A}_F \otimes_F E$ is independent of our F-basis of E.

Proposition

The natural map $\mathbb{A}_F \otimes_F E \to \mathbb{A}_E$ is an isomorphism of topological rings.

Proof.

By looking at $(\mathbb{A}_F)_S \otimes_F E$ for finite subsets $S \supseteq M_{F,\infty}$ of M_F, we can identify $\mathbb{A}_F \otimes_F E$ with the restricted product of the $F_v \otimes_F E$ with respect to the $\mathcal{O}_v \otimes_{\mathcal{O}_F} \mathcal{O}_E$. By the Proposition, this equals the restricted product of the $\prod_{w|v} E_w$ with respect to the $\prod_{w|v} \mathcal{O}_w$. As the $\{w \in M_E \mid w \text{ divides } v\}$ are finite and their union is M_E, this restricted product is precisely $\mathbb{A}_E$. □

This lets us reduce statements to the case of $F = \mathbb{Q}$, as we did with Ostrowski's theorem.

I probably should've proved this earlier:

Proposition

Let A be a Dedekind domain. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_d$ be distinct nonzero prime ideals of A, let $x_1, \ldots, x_d$ be in $\operatorname{Frac} A$, and let $n \geq 0$ be an integer. Then there exists x in $\operatorname{Frac} A$ such that $v_{\mathfrak{p}_i}(x - x_i) \geq n$ for all $1 \leq i \leq d$ and $v_{\mathfrak{p}}(x) \geq 0$ for nonzero prime ideals $\mathfrak{p}$ not in $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_d\}$.

Proof.

First, suppose $x_2 = \cdots = x_d = 0$. Now $\mathfrak{p}_1^n + \mathfrak{p}_2^n \cdots \mathfrak{p}_d^n = A$, so $x_1 = y + x$ for y in $\mathfrak{p}_1^n$ and x in $\mathfrak{p}_2^n \cdots \mathfrak{p}_d^n$. This x works.

Secondly, suppose the $x_1, \ldots, x_d$ lie in A. Apply the above to each of the x_i in turn to obtain a_i in A, and take $x = a_1 + \cdots + a_d$.

Finally, suppose the $x_1, \ldots, x_d$ lie in $\operatorname{Frac} A$. Write $x_i = a_i/b$ for a_i and b in A. Apply the above to obtain a in A satisfying, for $1 \leq i \leq d$, $v_{\mathfrak{p}_i}(a - a_i) \geq n + \max\{v_{\mathfrak{p}_1}(b), \ldots, v_{\mathfrak{p}_d}(b)\}$, and $v_{\mathfrak{q}}(a) \geq v_{\mathfrak{q}}(b)$ for nonzero prime ideals $\mathfrak{q}$ with $v_{\mathfrak{q}}(b) \geq 1$. Then $x = a/b$ works. □

Proposition

The field F is discrete in $\mathbb{A}_F$, and the quotient $\mathbb{A}_F/F$ is compact.

Proof.

Write $n = [F : \mathbb{Q}]$. By choosing a $\mathbb{Q}$-basis of F, we can identify $F = \mathbb{Q}^n$ as $\mathbb{Q}$-vector spaces. This identifies $\mathbb{A}_F = \mathbb{A}_{\mathbb{Q}}^n$ by the Proposition, so we get $\mathbb{A}_F/F = (\mathbb{A}_{\mathbb{Q}}/\mathbb{Q})^n$ as topological groups. This it suffices to prove this for $F = \mathbb{Q}$.

Consider $U = \{(x_v)_v \in \mathbb{A}_{\mathbb{Q}} \mid \|x_v\|_v \leq 1 \text{ for } v \neq \infty \text{ and } \|x_v\|_\infty < 1\}$ in $\mathbb{A}_{\mathbb{Q}}$. It's evidently open, and any x in $\mathbb{Q}$ that lies in U must be an integer with $|x|_\infty < 1$. Thus $x = 0$, so $\mathbb{Q}$ is discrete in $\mathbb{A}_{\mathbb{Q}}$.

Next, form $K = \{(x_v)_v \in \mathbb{A}_{\mathbb{Q}} \mid \|x_v\|_v \leq 1 \text{ for all } v\}$. It's evidently compact, and it suffices to show that $K + \mathbb{Q} = \mathbb{A}_{\mathbb{Q}}$. Let $(a_v)_v$ be in $\mathbb{A}_{\mathbb{Q}}$. For $v \neq \infty$, if $\|a_v\|_v \leq 1$, set $b_v = 0$. If $\|a_v\|_v > 1$, choose a'_v in $\mathbb{Q}$ with $\|a_v - a'_v\|_v \leq 1$, and use the Proposition to choose b_v in $\mathbb{Q}$ such that $\|b_v - a'_v\|_v \leq 1$ and and $\|b_v\|_{v'} \leq 1$ for nontrivial nonarchimedean $v' \neq v$. Finally, set $b' = \sum_{v \neq \infty} b_v$, and let b_∞ be the integer closest to $a_\infty - b'$.

Proposition

The field F is discrete in $\mathbb{A}_F$, and the quotient $\mathbb{A}_F/F$ is compact.

Proof (continued).

Set $b = b' + b_\infty$. Then b lies in $\mathbb{Q}$, and we see that $a_v - b$ lies in K for all v in $M_{\mathbb{Q}}$. □

For any $x = (x_v)_v$ in $\mathbb{A}_F$, write $\|x\| = \prod_{v \in M_F} \|x_v\|_v$. This converges since $\|x_v\|_v \leq 1$ for cofinitely many v. We see $\|\cdot\|$ yields a continuous group homomorphism $\mathbb{A}_F^\times \to \mathbb{R}_{>0}$, and evaluating $\|\cdot\|$ on $F_v^\times$ for $v \mid \infty$ shows it is surjective.

The product formula implies $\|x\| = 1$ for x in $F^\times$, so we see this induces a continuous surjective group homomorphism $\mathbb{A}_F^\times / F^\times \to \mathbb{R}_{>0}$. In particular, $\mathbb{A}_F^\times / F^\times$ is not compact.