Restricted Products

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Let X be a set, and let X_i be a collection of subsets such that $X = \bigcup_i X_i$. Suppose each X_i has a topological space structure such that, if $X_i \subseteq X_j$, then X_i has the subspace topology from X_j.

Definition

The *coherent* (or *direct limit*) topology on X with respect to the X_i is given by defining $U \subseteq X$ to be open if and only if $U \cap X_i$ is open in X_i for all U.

Now let $\{G_v\}_{v \in M}$ be a collection of locally compact topological groups. Let M_∞ be a finite subset of M, and for all v in $M \smallsetminus M_\infty$, let K_v be a compact open subgroup of G_v.

Example

Let F be a number field, write M_F for the set of isomorphism classes of nontrivial norms on F, and write $M_{F,\infty}$ for the subset of archimedean norms. Take $M = M_F$ and $M_\infty = M_{F,\infty}$.

1. We can take $G_v = F_v$ for all v in M, and $K_v = \mathcal{O}_v$ for v not in M_∞.
2. We can take $G_v = F_v^\times$ for all v in M, and $K_v = \mathcal{O}_v^\times$ for v not in M_∞.

For any finite subset $S \supseteq M_\infty$ of M, set $G_S = \prod_{v \in S} G_v \times \prod_{v \notin S} K_v$. Note that G_S is a locally compact topological group. For $S \subseteq S'$, we see G_S is an open and closed subgroup of $G_{S'}$.

Definition

The *restricted product* of the G_v with respect to the K_v is the group

$$\prod_{v \in M}{}' G_v = \left\{ (x_v)_v \in \prod_{v \in M} G_v \,\middle|\, x_v \in K_v \text{ for cofinitely many } v \right\}.$$

We give $\prod'_{v \in M} G_v$ the coherent topology with respect to the G_S, as S runs through finite subsets of M containing M_∞.

Write $G = \prod'_{v \in M} G_v$. Because all the G_S are topological groups, we see G is also a topological group. And since G_v is a closed subgroup of G_S for all S containing v, we identify G_v with a closed subgroup of G.

Example

In our previous example, we call G from (1) the *adeles* of F, and we call G from (2) the *ideles* of F.

Proposition

1. The topological group G is locally compact.
2. Let Y be a closed subset of G. Then Y is compact if and only if $Y \subseteq \prod_{v \in M} C_v$, where the C_v are compact subsets of G_v such that $C_v = K_v$ for cofinitely many v.

Proof.

1. Let $x \neq y$ be in G. Then x and y both lie in G_S for some S, and G_S is both Hausdorff as well as open in G. Thus G is Hausdorff. As for compact neighborhoods of 1, they follow from the local compactness of G_S and the fact that G_S is open and closed in G.
2. Any such $Y \subseteq \prod_{v \in M} C_v$ is evidently compact, because $\prod_{v \in M} C_v$ is. Conversely, suppose Y is compact. Then the open cover $\{G_S\}_S$ of Y has a finite subcover $\{G_{S_i}\}_{i=1}^n$. In particular, Y lies in G_S for $S = \bigcup_{i=1}^n S_i$. For all v in S, the projection map $\pi_v : G_S \to G_v$ is continuous, so $\pi_v(Y)$ is compact. Thus we can take $C_v = \pi_v(Y)$ for v in S and $C_v = K_v$ otherwise. □

Let's study continuous homomorphisms $G \to \mathbb{C}^\times$.

Lemma

For all $v \in M$, let $\chi_v : G_v \to \mathbb{C}^\times$ be a continuous homomorphism. If $\chi_v|_{K_v} = 1$ for cofinitely many v, then the map $\chi : G \to \mathbb{C}^\times$ sending $(x_v)_v \mapsto \prod_{v \in M} \chi_v(x_v)$ is a continuous homomorphism.

Proof.

We immediately see χ is a well-defined homomorphism. Let $S \supseteq M_\infty$ be a finite subset of M such that $\chi_v|_{K_v} = 1$ for v not in S, and write $m = \#S$. By the homogeneity of G, it suffices to check the continuity of χ at 1. Let U be a neighborhood of 1 in $\mathbb{C}^\times$, and let V be a neighborhood of 1 in $\mathbb{C}^\times$ such that $V^{(m)} \subseteq U$. The continuity of the χ_v implies that $\prod_{v \in S} \chi_v^{-1}(V) \times \prod_{v \notin S} K_v$ is a neighborhood of 1 in $\chi^{-1}(U)$. □

Lemma

Every continuous homomorphism $\chi : G \to \mathbb{C}^\times$ is of the above form.

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Every continuous homomorphism $\chi : G \to \mathbb{C}^\times$ is of the above form.

Proof.

Let U be a neighborhood of 1 in $\mathbb{C}^\times$ containing no nontrivial subgroups of $\mathbb{C}^\times$. As $\chi^{-1}(U)$ is a neighborhood of 1 in G, it contains $\prod_{v \in M} N_v$, where the N_v are neighborhoods of 1 in G_v such that $N_v = K_v$ for v not in some finite subset $S \supseteq M_\infty$ of M. Now $\prod_{v \notin S} N_v$ and hence $\chi(\prod_{v \notin S} N_v)$ are subgroups, so we see $\prod_{v \notin S} N_v \subseteq \ker \chi$. By setting $\chi_v(x_v) = \chi(x_v)$ for all v and x_v in G_v, we obtain the desired result. □

Assume now that each of the G_v is abelian. Thus G is locally compact abelian, so we can form $\widehat{G}$. How is it related to the $\widehat{G_v}$?

For v in $M \smallsetminus M_\infty$, write W_v for the subgroup $\{\chi \in \widehat{G_v} \mid \chi(K_v) = 1\}$ of $\widehat{G_v}$. Since $N(1)$ contains no nontrivial subgroups of S^1, we see W_v equals $W(K_v, 1, \sqrt{3})$ and hence is open. As K_v is open, we see G_v/K_v is discrete, and the compactness of K_v implies that W_v is homeomorphic to $\widehat{G_v/K_v}$.

Now $\widehat{G_v/K_v}$ is compact, so altogether W_v is a compact open subgroup of G_v. Write $\prod'_{v\in M}\widehat{G_v}$ for the restricted product of the $\widehat{G_v}$ with respect to the W_v.

Proposition

The map $\chi \mapsto (\chi_v)_v$ yields an isomorphism $\widehat{G} \xrightarrow{\sim} \prod'_{v\in M}\widehat{G_v}$ of topological groups.

Proof.

Our previous two Lemmas imply this is an isomorphism of groups. By homogeneity, it suffices to check continuity and openness at 1. Let $\prod_{v\in M} W(C_v,1,\sqrt{3})$ be a neighborhood of 1 in $\prod'_{v\in M}\widehat{G_v}$, where the C_v are compact subsets of G_v such that $C_v = K_v$ for cofinitely many v. Then any χ in $W(\prod_{v\in M} C_v, 1, \sqrt{3})$ has $(\chi_v)_v$ in $\prod_{v\in M} W(C_v,1,\sqrt{3})$, which proves continuity. For openness, let Y be a compact subset of G. Then $Y \subseteq \prod_{v\in M} Q_v$, where the Q_v are compact subsets of G_v such that $Q_v = K_v$ for v not in some finite subset $S \supseteq M_\infty$ of M. Write $m = \#S$. We see any $(\chi_v)_v$ in $\prod_{v\in S} W(Q_v, 1, \sqrt{2-2\cos(2\pi/3m)}) \times \prod_{v\notin S} W_v$ has $\prod_{v\in M}\chi_v$ in $W(Y,1,\sqrt{3})$, which proves openness. □

Relax our assumption that the G_v are abelian. We want to relate left Haar measures on G with those on the G_v. We first need the following fact.

Lemma

Let G be a locally compact topological group, and let m be a left Haar measure on G. For any nonempty open subset U of G, we have $m(U) > 0$.

Thus if H is an open subgroup of G, then $m|_H$ is a left Haar measure on H.

Proof.

Homework problem. □

For all v in M, let m_v be a left Haar measure on G_v such that $m_v(K_v) = 1$ for cofinitely many v.

Example

In our previous example, in (1) we can take m_v to be the Lebesgue measure on F_v, and in (2) we can take m_v to be the $(1 - q_v^{-1})^{-1}$ times the $E \mapsto \int_E \mathrm{d}x \,/\|x\|$ measure on $F_v^\times$ for nonarchimedean v and any positive multiple of this measure for archimedean v.

Proposition

There is a unique left Haar measure m on G such that, for all finite subsets $S \supseteq M_\infty$ of M, the restriction $m|_{G_S}$ equals the product measure $\prod_{v \in S} m_v \times \prod_{v \notin S} m_v$ on G_S.

Proof.

Since $m(K_v) = 1$ for cofinitely many v, we see that $\prod_{v \notin S} m_v$ gives $\prod_{v \notin S} K_v$ nonzero finite measure, so $\prod_{v \notin S} m_v$ yields a left Haar measure on $\prod_{v \notin S} K_v$. Therefore $\prod_{v \in S} m_v \times \prod_{v \notin S} m_v$ is a left Haar measure on G_S, and for $S \subseteq S'$ we immediately see it is the restriction of $\prod_{v \in S'} m_v \times \prod_{v \notin S'} m_v$ to G_S. By Haar's theorem, G has a left Haar measure, and its restriction to G_S must be a positive multiple of $\prod_{v \in S} m_v \times \prod_{v \notin S} m_v$. The uniqueness in Haar's theorem provides the desired result. □

We write $m = \prod_{v \in M} m_v$, and when integrating we write $\mathrm{d}x = \prod_{v \in M} \mathrm{d}x_v$.

Lemma

Let $f : G \to \mathbb{C}$ be continuous. Then $\int_G \mathrm{d}x\, f(x) = \lim_{S\to\infty} \int_{G_S} \mathrm{d}x\, f(x)$.

Proof.

The inner regularity of m implies $\int_G \mathrm{d}x\, f(x) = \lim_{Y\to\infty} \int_Y \mathrm{d}x\, f(x)$, where Y runs over compact subsets of G. But every Y lies in G_S for some S. $\square$

Let $S \supseteq M_\infty \cup \{v \in M \setminus M_\infty \mid m_v(K_v) \neq 1\}$ be a finite subset of M.

Proposition

For all v in M, let $f_v : G_v \to \mathbb{C}$ be a continuous function in $L^1(G_v)$. If $f_v|_{K_v} = 1$ for v not in S, then the map $f : G \to \mathbb{C}$ sending $(x_v)_v \mapsto \prod_{v\in M} f_v(x_v)$ is continuous, and we have

$$\int_{G_S} \mathrm{d}x\, f(x) = \prod_{v\in S} \int_{G_v} \mathrm{d}x_v\, f_v(x_v).$$

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Proof.

We immediately see f is a well-defined function. As f is evidently continuous on G_S and the G_S form an open cover G, we see f is continuous. Finally, since $m_v(K_v) = 1$ and $f_v|_{K_v} = 1$ for v not in S, we obtain the equality of integrals. □

The Lemma and the Proposition imply $\int_G \mathrm{d}x\, f(x) = \prod_{v \in M} \int_{G_v} \mathrm{d}x_v\, f_v(x_v)$. Hence if the right-hand side converges, then f lies in $L^1(G)$.