Absolute Values on Number Fields

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Let A be a Dedekind domain, and write $\mathcal{I}_A$ for the group of its nonzero fractional ideals. We have a group homomorphism $(\cdot) : (\operatorname{Frac} A)^\times \to \mathcal{I}_A$ given by $x \mapsto (x)$. Since the identity in $\mathcal{I}_A$ is A, we see $\ker(\cdot) = A^\times$. We call an element of $\operatorname{im}(\cdot)$ a *principal ideal*.

Definition

The *class group* of A, denoted by $\mathcal{C}\ell(A)$, is the quotient group $\mathcal{I}_A / \operatorname{im}(\cdot)$.

When $A = \mathcal{O}_F$, we often index everything with F instead of A.

Example

- We see A is a principal ideal domain if and only if $\mathcal{C}\ell(A) = 0$,
- Let $F = \mathbb{Q}(\sqrt{-5})$. Then $\mathcal{C}\ell(F) = \mathbb{Z}/2\mathbb{Z}$ and is generated by the class of $(2, 1+\sqrt{-5})$,
- Let κ be an algebraically closed field of characteristic $\neq 2$. One can show that the class group of $\kappa[x,y]/(y^2 - x^3 + x)$ bijects with

$$\{(x,y) \in \kappa^2 \mid y^2 - x^3 + x = 0\} \cup \{\infty\}.$$

In particular, this class group is infinite.

Let F be a number field. Recall Minkowski's convex body theorem is used to prove the following.

Theorem

The abelian group $\mathcal{C}\ell(F)$ is finite.

Another application of Minkowski's convex body theorem is the following. Write d for $[F:\mathbb{Q}]$. Let $\sigma_1, \ldots, \sigma_{r_1}$ denote the field homomorphisms $F \to \mathbb{C}$ whose image lies in $\mathbb{R}$, and index the other field homomorphisms $F \to \mathbb{C}$ as $\sigma_{r_1+1}, \ldots, \sigma_d$ such that $\overline{\sigma}_k = \sigma_{k+r_2}$ for all $r_1 + 1 \leq k \leq r_1 + r_2$. So $r_1 + 2r_2 = d$.

Theorem (Dirichlet unit)

The abelian group $\mathcal{O}_F^\times$ is finitely generated of rank $r_1 + r_2 - 1$.

For any $1 \leq k \leq r_1 + r_2$, the composite $|\cdot| \circ \sigma_k$ yields an archimedean norm $|\cdot|_k$ on F. If $1 \leq k \leq r_1$, we see the completion of F with respect to $|\cdot|_k$ is identified with $\mathbb{R}$ via σ_k. If $r_1 + 1 \leq k \leq r_1 + r_2$, we see the completion of F with respect to $|\cdot|_k$ is identified with $\mathbb{C}$ via σ_k.

Let F be a field, and let $|\cdot|_v$ be a norm on F that is discretely valued or archimedean. We write F_v for the completion of F with respect to $|\cdot|_v$.

Remark

Let E_w/F_v be a finite extension of degree d_v. Now $|\cdot|_w = |\operatorname{Nm}_{E_w/F_v} \cdot|^{1/d_v}$ yields an extension of $|\cdot|_v$ to an absolute value on E_w, and it is the unique extension up to isomorphism. We proved this for discretely valued $|\cdot|_v$ using Hensel's lemma, and for archimedean $|\cdot|_v$ it follows from Ostrowski's theorem, because then F_v is either $\mathbb{R}$ or $\mathbb{C}$.

Let E/F be a finite separable extension, and suppose $E = F[t]/f$ for some irreducible f in $F[t]$. Write $f = f_1 \cdots f_r$ for the irreducible factorization of f in $F_v[t]$. For $1 \leq j \leq r$, we get a field homomorphism $F[t]/f \to F_v[t]/f_j$. Now $|\cdot|_v$ on F_v extends uniquely to $|\cdot|_j$ on $F_v[t]/f_j$, and precomposing with this homomorphism yields an extension $|\cdot|_j$ on E of $|\cdot|_v$ on F.

Proposition

Up to isomorphism, every extension of $|\cdot|_v$ from F to E arises in this way. Furthermore, for $j_1 \neq j_2$, the norms $|\cdot|_{j_1}$ and $|\cdot|_{j_2}$ on E are not isomorphic.

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Proof.

Note that F is dense in F_v, and t in E generates $F_v[t]/f_j$ over F_v. Thus E is dense in $F_v[t]/f_j$, so $F_v[t]/f_j$ is the completion of E with respect to $|\cdot|_j$. If $|\cdot|_{j_1}$ and $|\cdot|_{j_2}$ on E are isomorphic, then the completions $F_v[t]/f_{j_1}$ and $F_v[t]/f_{j_2}$ are isomorphic over F_v. Hence the image of t in them is a root of f_{j_1} and f_{j_2}, so $j_1 = j_2$.

Next, let $|\cdot|_w$ be a norm on E that extends $|\cdot|_v$, and identify F_v with the closure of F in E_w. Then t generates a finite and hence complete extension of F_v. But this extension contains the dense subfield E, so E_w is generated by t over F_v. As t is a zero of f, we see it must be a zero of f_j for some $1 \leq j \leq r$, and this identifies E_w with $F_v[t]/f_j$. □

Let $|\cdot|_w$ be a norm on E. We write $w|v$ if E extends $|\cdot|_v$.

Corollary

We have an isomorphism $E \otimes_F F_v \xrightarrow{\sim} \prod_{w|v} E_w$ *of* F_v*-algebras.*

Proof.

The Chinese remainder theorem gives

$$E \otimes_F F_v = (F[t]/f) \otimes_F F_v = F_v[t]/(f_1 \cdots f_r) = \prod_{j=1}^{r} F_v[t]/f_j = \prod_{w|v} E_w. \quad \square$$

Corollary

Let F *be a number field. Then our* $|\cdot|_1, \ldots, |\cdot|_{r_1+r_2}$ *are all the archimedean norms on* F*.*

Proof.

Let $|\cdot|$ be an archimedean norm on F. Its restriction to $\mathbb{Q}$ is archimedean and thus isomorphic to $|\cdot|_\infty$ by Ostrowski's theorem. Applying the proposition to the extension $F/\mathbb{Q}$ and the norm $|\cdot|_\infty$ yields the result. $\square$

Let F be a number field, and let $|\cdot|_v$ be a nontrivial nonarchimedean norm on F. Write $\mathcal{O}_{(v)}$ for the valuation ring of F with respect to $|\cdot|_v$, and write $\mathfrak{m}_{(v)}$ for its maximal ideal.

Proposition

Our $|\cdot|_v$ is isomorphic to the absolute value associated with $v_{\mathfrak{p}}$, where $\mathfrak{p}$ is a nonzero prime ideal of $\mathcal{O}_F$.

Proof.

The restriction of $|\cdot|_v$ to $\mathbb{Q}$ is nontrivial nonarchimedean and thus isomorphic to $|\cdot|_p$ for a prime number p by Ostrowski's theorem. Now $\mathfrak{m}_{(v)} \cap \mathcal{O}_F$ is a prime ideal of $\mathcal{O}_F$, and it contains p. Thus $\mathfrak{p} = \mathfrak{m}_{(v)} \cap \mathcal{O}_F$ is a nonzero prime ideal of $\mathcal{O}_F$.

Because $|\cdot|_v$ is nonarchimedean, we see that $\mathcal{O}_F \subseteq \mathcal{O}_{(v)}$. Since $\mathcal{O}_F \smallsetminus \mathfrak{p}$ lies in $\mathcal{O}_{(v)}^\times$, we see $\mathcal{O}_{F,\mathfrak{p}} \subseteq \mathcal{O}_{(v)}$. Writing $|\cdot|_{\mathfrak{p}}$ for the norm associated with $v_{\mathfrak{p}}$, note that its valuation ring equals $\mathcal{O}_{F,\mathfrak{p}}$. Hence $|x|_{\mathfrak{p}} \leq 1$ implies $|x|_v \leq 1$, so $|\cdot|_{\mathfrak{p}}$ and $|\cdot|_v$ are isomorphic. $\square$

Let F be a number field, and let $|\cdot|_v$ be a nontrivial norm on F. Altogether, we see that F_v is a local field.

- If $F_v = \mathbb{R}$, normalize $|\cdot|_v$ to be the classic absolute value, and set $\|\cdot\|_v = |\cdot|_v$,
- If $F_v = \mathbb{C}$, normalize $|\cdot|_v$ to be the classic absolute value, and set $\|\cdot\|_v = |\cdot|_v^2$,
- If F_v is nonarchimedean, write $\mathcal{O}_v$ for its ring of integers and $\mathfrak{m}_v$ for its maximal ideal. Write q_v for the cardinality of its residue field, normalize $|\cdot|_v$ such that $|\pi|_v = \frac{1}{q_v}$ for uniformizers π in F_v, and set $\|\cdot\|_v = |\cdot|_v$.

Proposition (product formula)

Let x be in $F^\times$. Then we have $\prod_v \|x\|_v = 1$, where v runs over all nontrivial norms on F.

Proof.

As $F = \operatorname{Frac} \mathcal{O}_F$, it suffices to take x in $\mathcal{O}_F$. Chinese remainder implies $\prod_{v \nmid \infty} \|x\|_v = \#(\mathcal{O}/x)^{-1}$, while we know $\prod_{v \mid \infty} \|x\|_v = \#(\mathcal{O}/x)$. □