Poisson Summation
(with number fields at the end)

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Example

Let p be a prime number, and let $G = \mathbb{Q}_p$. Let m be the Lebesgue measure on G. By using p-adic expansions, we can identify $\mathbb{Q}_p/\mathbb{Z}_p$ with $\{z \in \mathbb{Q}/\mathbb{Z} \mid p^n z = 0 \text{ for some } n \geq 0\}$ as groups. We take ψ to be the composition

$$\mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p \subset \mathbb{Q}/\mathbb{Z} \subset \mathbb{R}/\mathbb{Z} \xrightarrow{\varphi} S^1.$$

Since $\widehat{m}$ is a Haar measure on $\widehat{G} \cong G$, it equals c times m for some $c > 0$. Next, let f be the indicator function on $\mathbb{Z}_p$. Its Fourier transform is

$$\widehat{f}(a) = \int_{\mathbb{Q}_p} \mathrm{d}x\, f(x)\psi_a(x)^{-1} = \int_{\mathbb{Z}_p} \mathrm{d}x\, \psi(ax)^{-1}.$$

If a lies in $\mathbb{Z}_p$, then ax does too for all x in $\mathbb{Z}_p$. So $\psi_a|_{\mathbb{Z}_p}$ is trivial, making $\widehat{f}(a) = 1$. If a does not lie in $\mathbb{Z}_p$, then $v_p(a) < 0$, and $\psi_a|_{\mathbb{Z}_p}$ factors through a nontrivial group homomorphism $\mathbb{Z}_p/p^{-v_p(a)}\mathbb{Z}_p \to S^1$.

Example (continued)

Therefore the integral becomes

$$\sum_{x \in \mathbb{Z}_p / p^{-v_p(a)} \mathbb{Z}_p} m(x) \psi_a(x)^{-1} = p^{v_p(a)} \sum_{x \in \mathbb{Z} / p^{-v_p(a)} \mathbb{Z}} \psi_a(x)^{-1} = 0.$$

Altogether $\widehat{f}$ is also the indicator function on $\mathbb{Z}_p$. Thus $c = 1$, i.e. the Lebesgue measure on $\mathbb{R}$ is self-dual with respect to this choice of ψ.

We want a handy class of functions we can apply *Poisson summation* to.

Definition

Let $f : \mathbb{R}^n \to \mathbb{C}$ be a smooth function. We say it is *Schwartz* if, for all non-negative integers $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$, the function

$$x_1^{\alpha_1} \cdots x_n^{\alpha_n} \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} \cdots \frac{\partial^{\beta_n}}{\partial x_n^{\beta_n}} f$$

approaches 0 as $|x| \to \infty$, where $x = (x_1, \ldots, x_n)$.

Write $\mathcal{S}(\mathbb{R}^n)$ for the set of Schwartz functions on $\mathbb{R}^n$.

Note $\mathcal{S}(\mathbb{R}^n)$ is preserved under addition, multiplication, and scaling by $\mathbb{C}$.

Example

- $f(x) = \frac{1}{1+x}$ does not lie in $\mathcal{S}(\mathbb{R})$, since $xf(x)$ does not approach 0 as $|x| \to \infty$,
- $f(x) = e^{-x^2} \sin(e^{x^2})$ does not lie in $\mathcal{S}(\mathbb{R})$, since $f'(x)$ does not approach 0 as $x \to \infty$,
- $f(x) = e^{-x^2}$ lies in $\mathcal{S}(\mathbb{R})$.

For nonarchimedean local fields, we use the following functions. Let X be a totally disconnected topological space.

Definition

Let $f : X \to \mathbb{C}$ be a function. We say that f is

- *smooth* if it is locally constant,
- *Bruhat–Schwartz* if it is smooth and compactly supported.

Write $\mathcal{S}(X)$ for the set of Bruhat–Schwartz functions on X. Note that $\mathcal{S}(X)$ is preserved under addition, multiplication, and scaling by $\mathbb{C}$.

Proposition

Let f be in $\mathcal{S}(\mathbb{R})$. Then $F(x) = \sum_{k=-\infty}^{\infty} f(x+k)$ converges uniformly on compact subsets of $\mathbb{R}$ and defines a continuous function $F : S^1 \to \mathbb{C}$.

Proof.

It suffices to consider convergence on $[-r, r]$ for $r > 0$. There exists $B > 0$ such that $y^2|f(y)| < 1$ for all $|y| > B$. Thus for all $|k| > B + r$, we have $|f(x+k)| < 1/(k-r)^2$ for all x in $[-r, r]$. The convergence of this series yields our desired uniform convergence, so F yields a continuous function $F : \mathbb{R} \to \mathbb{C}$. For c in $\mathbb{Z}$, uniform convergence allows us to re-index to see

$$F(x+c) = \sum_{k=-\infty}^{\infty} f(x+c+k) = \sum_{j=-\infty}^{\infty} f(x+j) = F(x),$$

where $j = c + k$. So F descends to a continuous function $S^1 \to \mathbb{C}$. □

Remark (Fourier inversion)

Let G be an abelian locally compact topological group, and let m be a Haar measure on G. If $f : G \to \mathbb{C}$ is continuous, in $L^1(G)$, and has $\widehat{f}$ in $L^1(G)$, then $f(x) = \widehat{\widehat{f}}(x^{-1})$ for *all* x in G.

Remark

Let F be a local field, and fix a continuous homomorphism $\psi : F \to S^1$. One can show that the Fourier transform yields a $\mathbb{C}$-linear isomorphism $\mathcal{S}(F) \xrightarrow{\sim} \mathcal{S}(\widehat{F}) = \mathcal{S}(F)$ (where we identify $\mathbb{R}^2 = \mathbb{C}$ when $F = \mathbb{C}$) and exclusively work with f in $\mathcal{S}(F)$ instead of $L^2(F)$.

Theorem (Poisson summation)

Let f be in $\mathcal{S}(\mathbb{R})$. Then $\sum_{k=-\infty}^{\infty} f(k) = \sum_{k=-\infty}^{\infty} \widehat{f}(k)$.

Proof.

Let $F(x) = \sum_{k=-\infty}^{\infty} f(x+k)$, considered as a function $S^1 \to \mathbb{C}$.

Theorem (Poisson summation)

Let f be in $\mathcal{S}(\mathbb{R})$. Then $\sum_{k=-\infty}^{\infty} f(k) = \sum_{k=-\infty}^{\infty} \widehat{f}(k)$.

Proof (continued).

Note that $F(1)$ equals the left-hand side above. First, I claim $\widehat{f}(c) = \widehat{F}(c)$ for all c in $\mathbb{Z}$, where we use the Lebesgue measure on $\mathbb{R}$ and the usual measure on S^1. To see this, note that

$$\widehat{F}(c) = \int_0^1 \mathrm{d}x\, F(x)e^{-2\pi cix} = \int_0^1 \mathrm{d}x \sum_{k=-\infty}^{\infty} f(x+k)e^{-2\pi cix}$$

$$= \int_0^1 \mathrm{d}x \sum_{k=-\infty}^{\infty} f(x+k)e^{-2\pi ci(x+k)} = \int_{-\infty}^{\infty} \mathrm{d}y\, f(y)e^{-2\pi ciy} = \widehat{f}(c),$$

where $y = x + k$. Now $\widehat{f}$ lies in $\mathcal{S}(\mathbb{R})$, so $\sum_{c=-\infty}^{\infty} |\widehat{f}(c)| = \sum_{c=-\infty}^{\infty} |\widehat{F}(c)|$ converges. In other words, $\widehat{F}$ lies in $L^1(\mathbb{Z})$, so Fourier inversion applies to F. Hence $F(1) = \sum_{k=-\infty}^{\infty} \widehat{F}(k)1^k = \sum_{k=-\infty}^{\infty} \widehat{f}(k)$, as desired. □

Remark

One can use Poisson summation to prove Minkowski's convex body theorem, which in turn implies fundamental facts on number fields!

Recall that number fields are finite extensions F of $\mathbb{Q}$. For any number field F, write $\mathcal{O}_F$ for the integral closure of $\mathbb{Z}$ in F. We call $\mathcal{O}_F$ the *ring of integers* of F.

Example

- For $F = \mathbb{Q}$, we have $\mathcal{O}_F = \mathbb{Z}$.
- Let D be a squarefree integer, and let $F = \mathbb{Q}(\sqrt{D})$. Then

$$\mathcal{O}_F = \begin{cases} \mathbb{Z}[\frac{1+\sqrt{D}}{2}] & \text{if } D \equiv 1 \pmod 4, \\ \mathbb{Z}[\sqrt{D}] & \text{otherwise.} \end{cases}$$

- Let N be a positive integer, choose an N-th root of unity ζ_N, and let $F = \mathbb{Q}(\zeta_N)$. Then $\mathcal{O}_F = \mathbb{Z}[\zeta_N]$.

Now $\mathcal{O}_F$ is always a free $\mathbb{Z}$-module of rank $[F : \mathbb{Q}]$, so $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Q} = F$.

Definition

Let A be a commutative ring. We say A is a *Dedekind domain* if it is integrally closed, noetherian, and every nonzero prime ideal is maximal.

Example

- Every principal ideal domain is also Dedekind,
- Let F be a number field. Then $\mathcal{O}_F$ is a Dedekind domain,
- Let κ be a field of characteristic $\neq 2$. One can show that $\kappa[x,y]/(y^2 - x^3 + x)$ is a Dedekind domain.

Definition

Let A be an integral domain. A *fractional ideal* of A is an A-submodule I of $\operatorname{Frac} A$ such that $aI \subseteq A$ for some a in A.

Example

Let $F = \mathbb{Q}(\sqrt{-5})$. Then $\mathfrak{p} = (2, 1 + \sqrt{-5})$ is a prime ideal of $\mathcal{O}_F$, and $\mathfrak{p}^{-1} = \{x \in F \mid x\mathfrak{p} \subseteq \mathcal{O}_F\} = (1, \frac{1}{2} + \frac{\sqrt{-5}}{2})$ is a fractional ideal of $\mathcal{O}_F$.

Uniquely factorizing elements plays a key role in $\mathbb{Z}$ and $\mathbb{Q}$. For general Dedekind domains A, we have a weaker version for ideals.

Proposition

Let I be a nonzero fractional ideal of A. Then I can be uniquely written as $\mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_r^{n_r}$ for nonzero prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ of A and integers $n_1, \dots, n_r$.

Thus the set $\mathcal{I}_A$ of nonzero fractional ideals of A forms a group, and it's isomorphic to $\bigoplus_{\mathfrak{p}} \mathbb{Z}$, where $\mathfrak{p}$ runs over nonzero prime ideals of A.

For any nonzero prime ideal $\mathfrak{p}$ of A and x in $(\operatorname{Frac} A)^\times$, write $v_{\mathfrak{p}}(x)$ for the exponent of $\mathfrak{p}$ in the unique factorization of (x). We see that $v_{\mathfrak{p}} : \operatorname{Frac} A \to \mathbb{Z} \cup \{\infty\}$ yields a discrete valuation on $\operatorname{Frac} A$.