Pontryagin Duality

(and Fourier inversion and the Plancherel theorem, oh my!)

Siyan Daniel Li-Huerta

October 8, 2020

Let G be an abelian locally compact topological group. Let m be a Haar measure on G. For any measurable $f : G \to \mathbb{C}$ and $1 \leq p < \infty$, define the L^p-norm

$$\|f\|_p = \left(\int_G \mathrm{d}x \, |f(x)|^p \right)^{1/p} \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

This gives $C_c(G)$ the structure of a pre-Banach space over $\mathbb{C}$.

Write $L^p(G)$ for the completion of $C_c(G)$ with respect to $\|\cdot\|_p$. Then $L^p(G)$ is a Banach space over $\mathbb{C}$, and recall we can identify it with

$$\{f : G \to \mathbb{C} \mid f \text{ is measurable, and } \|f\|_p < \infty\}/\sim,$$

where $f \sim g$ if and only if $f = g$ outside a subset of measure zero, i.e. *almost everywhere*.

Definition

Let $f : G \to \mathbb{C}$ be in $L^1(G)$. Its *Fourier transform* is the function $\widehat{f} : \widehat{G} \to \mathbb{C}$ given by $\chi \mapsto \int_G \mathrm{d}x \, f(x)\chi^{-1}(x)$.

Note that the triangle inequality yields

$$|\widehat{f}(\chi)| = \left| \int_G \mathrm{d}x\, f(x)\chi^{-1}(x) \right| \leq \int_G \mathrm{d}x\, |f(x)| = \|f\|_1.$$

Example

- Let $G = \mathbb{Z}/n\mathbb{Z}$ with the discrete topology, and let m be the counting measure. Then every function $f : G \to \mathbb{C}$ lies in $L^1(G)$, and $\widehat{f}(\zeta) = \sum_{k=1}^{n} f(k)\zeta^{-k}$ for any n-th root of unity ζ.
- Let $G = \mathbb{Z}$ with the discrete topology, and let m be the counting measure. Then $f : G \to \mathbb{C}$ lies in $L^1(G)$ if and only if $\sum_{k=-\infty}^{\infty} |f(k)|$ is finite, and in that case $\widehat{f}(z) = \sum_{k=-\infty}^{\infty} f(k)z^{-k}$ for any z in S^1.
- Let $G = S^1$, and let m be the pushforward of the Lebesgue measure via $\varphi : [0,1] \to S^1$. Let $f : G \to \mathbb{C}$ be in $L^1(G)$. Then

$$\widehat{f}(k) = \int_{S^1} \mathrm{d}z\, f(z)z^{-k} = \int_0^1 \mathrm{d}x\, f(\varphi(x))e^{-2\pi kix},$$

i.e. $\widehat{f}(k)$ is the k-th Fourier coefficient of the periodic function $f \circ \varphi$.

Theorem (Plancherel)

There exists a Haar measure $\widehat{m}$ on $\widehat{G}$ such that, for all f in $L^1(G) \cap L^2(G)$, its Fourier transform $\widehat{f}$ lies in $L^2(\widehat{G})$ and satisfies $\|f\|_2 = \|\widehat{f}\|_2$. Furthermore, the set of all such $\widehat{f}$ is dense in $L^2(\widehat{G})$.

We call $\widehat{m}$ the *dual* measure on $\widehat{G}$. Note this implies that $f \mapsto \widehat{f}$ extends to an isometry $L^2(G) \xrightarrow{\sim} L^2(\widehat{G})$, which we also denote using $\widehat{\cdot}$.

Next, let x be in G. Consider the group homomorphism $\mathrm{ev}_x : \widehat{G} \to S^1$ given by $\chi \mapsto \chi(x)$.

Proposition

The homomorphism ev_x is continuous.

Proof.

We have to show $\mathrm{ev}_x^{-1}(N(1)) = \{\chi \in \widehat{G} \mid \chi(x) \subseteq N(1)\}$ is open. But this equals $W(\{x\}, 1, \sqrt{3})$, so it's open. □

Hence we get a map $\mathrm{ev} : G \to \widehat{\widehat{G}}$, which we see is a group homomorphism.

Theorem (Pontryagin duality)

The map ev *is an isomorphism of topological groups.*

Example

- Let $G = \mathbb{Z}/n\mathbb{Z}$ with the discrete topology. Recall we identified $\widehat{G} \xrightarrow{\sim} \{\zeta \in \mathbb{C} \mid \zeta^n = 1\}$ via $\chi \mapsto \chi(1)$. By choosing a primitive n-th root of unity, we see that $\mathbb{Z}/n\mathbb{Z} \xrightarrow{\sim} \widehat{\widehat{G}}$ under this identification via $k \mapsto (\zeta \mapsto \zeta^k)$. As $\chi(1)^k = \chi(k)$, this shows ev is an isomorphism of groups. Since $\widehat{\widehat{G}}$ is discrete, it's a homeomorphism.
- Let $G = \mathbb{Z}$ with the discrete topology. Similarly, we have $\widehat{G} \xrightarrow{\sim} S^1$ via $\chi \mapsto \chi(1)$. Recalling also that $\mathbb{Z} \xrightarrow{\sim} \widehat{\widehat{G}}$ via $k \mapsto (z \mapsto z^k)$, the observation $\chi(1)^k = \chi(k)$ shows that ev is an isomorphism of topological groups here too.

We use ev to identify $\widehat{\widehat{G}}$ with G. Thus m yields a Haar measure on $\widehat{\widehat{G}}$.

Theorem (Fourier inversion)

Let f be in $L^2(G)$. Then $f(x) = \widehat{\widehat{f}}(x^{-1})$ almost everywhere on G.

Example

Let $G = S^1$ with the usual measure m, and let f be in $L^2(G)$. Since $\widehat{m}$ is a Haar measure on $\widehat{G} = \mathbb{Z}$, it equals c times the counting measure for some $c > 0$. Taking $f = 1$ in the Fourier inversion formula yields

$$1 = c \sum_{k=-\infty}^{\infty} \widehat{f}(k)(z^{-1})^{-k} = c,$$

since $\widehat{f}$ equals the indicator function on 0. Thus $\widehat{m}$ equals the counting measure. For general f in $L^2(G)$, Fourier inversion then becomes

$$f(\varphi(x)) = f(z) = \sum_{k=-\infty}^{\infty} \widehat{f}(k)(z^{-1})^{-k} = \sum_{k=-\infty}^{\infty} \widehat{f}(k)e^{2\pi kix},$$

where we set $z = \varphi(x)$. This is precisely the Fourier expansion of $f \circ \varphi$.

Next, let's discuss Pontryagin duality relates to closed subgroups.

Proposition

Let H be a closed subgroup of G.

1. H is an abelian locally compact topological group.
2. G/H is an abelian locally compact topological group.

Proof.

1. Now H is immediately an abelian Hausdorff topological group. For any open subset U of G with compact closure, $W = U \cap H$ is open in H, and $\mathrm{cl}_H W = \mathrm{cl}_G U \cap H$ is a closed subset of $\mathrm{cl}_G U$, thus compact.
2. Since H is closed, we see G/H is an abelian Hausdorff topological group. Let U be a neighborhood of 1 in G with compact closure. Because the quotient map $\pi : G \to G/H$ is open, we see $\pi(U)$ is a neighborhood of 1 in G/H. Now $\pi(\overline{U})$ is compact and hence closed, as G/H is Hausdorff. Thus $\overline{\pi(U)} \subseteq \pi(\overline{U})$ is also compact. □

Note that we can identify $\widehat{G/H}$ with $\{\chi \in \widehat{G} \mid \chi(H) = 1\}$ as groups.

Proposition

This identifies $\widehat{G/H}$ as a closed subgroup of $\widehat{G}$, and we have a short exact sequence of topological groups $1 \to \widehat{G/H} \to \widehat{G} \to \widehat{H} \to 1$, where $\widehat{G} \to \widehat{H}$ is given by restriction.

Example

Let $G = F$ be a local field, and let $\psi : G \to S^1$ be a nontrivial continuous homomorphism. For any a in G, the homomorphism $\psi_a : G \to S^1$ given by $x \mapsto \psi(ax)$ is continuous, since multiplication by a is continuous. I claim this yields an isomorphism $\psi_\cdot : G \to \widehat{G}$ of topological groups.

It is injective because if $\psi(ax) = 1$ for all x in G, the nontriviality of ψ implies that $a = 0$. Next, consider the closed subgroup $H = \overline{\psi_\cdot(G)}$ of $\widehat{G}$. We can identify $\widehat{\widehat{G}/H}$ with the group $\{\chi \in \widehat{\widehat{G}} \mid \chi(H) = 1\}$. This group is trivial, since $H \supseteq \psi_\cdot(G)$, and if $\psi(ax) = 1$ for all a in G, then $x = 0$ as before. Thus the proposition shows $\widehat{\widehat{G}} \xrightarrow{\sim} \widehat{H}$, and Pontryagin duality gives $H = \widehat{G}$.

Example (continued)

If we could show $\psi.$ is a homeomorphism onto its image, we'd be done, because $\psi.(G)$ would be locally compact and hence closed. For continuity, let a be in G, and consider the neighborhood $W(B_c(0,r),1,\sqrt{3})\psi_a$ of ψ_a. As ψ is continuous, we see $\psi(VB_c(0,r))$ lies in $N(1)$ for a small enough neighborhood V of 1. Thus $\psi.(V)$ lies in $W(B_c(0,r),1,\sqrt{3})$, implying that $\psi.$ sends $V+a$ to $W(B_c(0,r),1,\sqrt{3})\psi_a$.

For openness, let $x_0 \neq 0$ in G satisfy $\psi(x_0) \neq 1$, and consider the neighborhood $B_o(a,\epsilon)$ of a. Any ψ_b in $W(B_c(0,|x_0|/\epsilon),1,|\psi(x_0)-1|)$ must not have x_0 in $bB_c(0,|x_0|/\epsilon)$. Therefore $|x_0| > |b|(|x_0|/\epsilon)$ and hence $\epsilon > |b|$, implying that $\psi.^{-1}$ sends $W(B_c(0,|x_0|/\epsilon),1,|\psi(x_0)-1|)$ to $B_o(a,\epsilon)$.

Our flexibility in choosing ψ for this isomorphism is convenient for making calculations.

Example

Let $G = \mathbb{R}$, and let m be the Lebesgue measure on G. Choose $\psi = \varphi$, and let f be in $L^1(G)$. Under the above identification, the Fourier transform of f is given by

$$\widehat{f}(a) = \int_{\mathbb{R}} \mathrm{d}x\, f(x)\psi_a(x)^{-1} = \int_{-\infty}^{\infty} \mathrm{d}x\, f(x)e^{-2\pi aix},$$

i.e. it's the usual Fourier transform. Since $\widehat{m}$ is a Haar measure on $\widehat{G} \cong G$, it equals c times m for some $c > 0$. Taking $f(x) = e^{-\pi x^2}$ in the above yields $\widehat{f}(a) = e^{-\pi a^2}$. Thus $c = 1$, i.e. the Lebesgue measure on $\mathbb{R}$ is *self-dual* with respect to this choice of ψ.

Suppose now that f lies in $L^1(G) \cap L^2(G)$. Fourier inversion then becomes

$$f(x) = \int_{\mathbb{R}} \mathrm{d}a\, \widehat{f}(a)\psi_a(-x)^{-1} = \int_{-\infty}^{\infty} \mathrm{d}a\, \widehat{f}(a)e^{2\pi aix},$$

i.e. it's the classic Fourier inversion formula.