More on Pontryagin Duals

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October 6, 2020

Let G be an abelian topological group. Write $\varphi : \mathbb{R} \to S^1$ for the map $x \mapsto \exp(2\pi i x)$. Note φ realizes $\mathbb{R}$ as the universal cover of S^1, sending the base-point 0 to 1.

Example

Let $G = S^1$. Let $\chi : S^1 \to S^1$ be in $\widehat{G}$. Because $\mathbb{R}$ is simply connected, we can lift $\chi \circ \varphi : \mathbb{R} \to S^1$ uniquely to a continuous map $\widetilde{\chi} : \mathbb{R} \to \mathbb{R}$ satisfying $\widetilde{\chi}(0) = 0$. Thus we have a commutative diagram

$$\begin{array}{ccc} (\mathbb{R}, 0) & \xrightarrow{\widetilde{\chi}} & (\mathbb{R}, 0) \\ \downarrow \varphi & & \downarrow \varphi \\ (S^1, 1) & \xrightarrow{\chi} & (S^1, 1). \end{array}$$

One can use the fact that $\chi \circ \varphi$ is a homomorphism to show $\widetilde{\chi}$ is too, and $\widetilde{\chi}$ preserves $\varphi^{-1}(0) = \mathbb{Z}$. Thus $\widetilde{\chi}$ equals multiplication-by-k for some k in $\mathbb{Z}$, so χ equals $z \mapsto z^k$. This identifies $\widehat{G}$ with $\mathbb{Z}$ as a group. The neighborhoods $W(G, 1, \epsilon)$ show that $\widehat{G}$ is discrete.

Let t be in $(0,1]$, and write $N(t)$ for $\varphi((-\frac{t}{3},\frac{t}{3}))$. As $t \to 0$, note the $N(t)$ form a basis of neighborhoods of 1. Write $U^{(m)}$ for $\underbrace{U \cdots U}_{m \text{ times}}$, where $U \subseteq G$.

Lemma

Let z be in S^1, and suppose $z, z^2, \ldots, z^m$ lie in $N(1)$. Then z lies in $N(\frac{1}{m})$.

In particular, let $\chi : G \to S^1$ be a group homomorphism, and let U be a subset of G containing 1. If $\chi(U^{(m)}) \subseteq N(1)$, then $\chi(U) \subseteq N(\frac{1}{m})$.

Proof.

We induct on m, where the $m=1$ case is immediate. Assume now that $z, \ldots, z^{m+1}$ lie in $N(1)$. By induction, we know z lies in $N(\frac{1}{m})$. Since z^{m+1} lies in $N(1)$, there exists y in $N(\frac{1}{m+1})$ such that $y^{m+1} = z^{m+1}$. This that implies z/y is an $(m+1)$-th root of unity, so $z = y \cdot (z/y)$ lies in $N(\frac{1}{m+1})\varphi(\frac{q}{m+1})$ for an integer $0 \le q \le m$.

I claim that $N(\frac{1}{m})$ and $N(\frac{1}{m+1})\varphi(\frac{q}{m+1})$ intersect if and only if $q = 0$. To see this, note that $N(\frac{1}{m})$ and $N(\frac{1}{m+1})\varphi(\frac{q}{m+1})$ are the homeomorphic images of $(-\frac{1}{3m}, \frac{1}{3m})$ and $(\frac{3q-1}{3(m+1)}, \frac{3q+1}{3(m+1)})$, respectively.

Lemma

Let z be in S^1, and suppose $z, z^2, \ldots, z^m$ lie in $N(1)$. Then z lies in $N(\frac{1}{m})$.

Proof (continued).

These images intersect if and only if

$$\tfrac{1}{3m} > \tfrac{3q-1}{3(m+1)} \iff m+1 > 3qm - m \iff 2r+1 > 3qr \iff q = 0.$$

Because z lies in $N(\frac{1}{m})$, we have $q = 0$ and hence z lies in $N(\frac{1}{m+1})$. □

Drawing a picture and using the law of cosines shows that

$$N(t) = \{z \in S^1 \mid |z-1| < \sqrt{2 - 2\cos(2\pi t/3)}\}.$$

Therefore $W(K, 1, \sqrt{2 - 2\cos(2\pi t/3)})$ equals the set of χ in $\widehat{G}$ such that $\chi(K) \subseteq N(t)$. As $t \to 0$, we see $\sqrt{2 - 2\cos(2\pi t/3)} \to 0$, so these form a basis of neighborhoods of 1.

Proposition

Let G be an abelian topological group.

1. Let $\chi : G \to S^1$ be a group homomorphism. Then χ is continuous if and only if $\chi^{-1}(N(1))$ is open.
2. As K ranges over compact subsets of G, the $W(K, 1, \sqrt{3})$ form a basis of neighborhoods of 1.
3. If G is discrete, then $\widehat{G}$ is compact.
4. If G is compact, then $\widehat{G}$ is discrete.

Proof.

1. If χ is continuous, then $\chi^{-1}(N(1))$ is open since $N(1)$ is. Conversely, suppose $\chi^{-1}(N(1))$ is open. Let x be in G, and consider the neighborhood $N(t)\chi(x)$ of $\chi(x)$. There exists an integer $m \geq 1$ such that $\frac{1}{m} \leq t$, and $\chi^{-1}(N(1))$ contains a neighborhood V of 1 such that $V^{(m)} \subseteq \chi^{-1}(N(1))$. Therefore $\chi(V)^{(m)} \subseteq N(1)$, so $\chi(V) \subseteq N(\frac{1}{m})$. Hence $V \subseteq \chi^{-1}(N(\frac{1}{m}))$, so the image of Vx under χ lies in $N(t)\chi(x)$.

Proposition

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Proof (continued).

2. Consider a neighborhood $W(K', 1, \sqrt{2 - 2\cos(2\pi/3m)})$ of 1. Let $K = K'^{(m)}$, which is compact. If χ lies in $W(K, 1, \sqrt{3})$, then $\chi(K')^{(m)} \subseteq N(1)$ and thus $\chi(K') \subseteq N(\frac{1}{m})$. Thus we see $W(K, 1, \sqrt{3})$ is a neighborhood of 1 contained in $W(K', 1, \sqrt{2 - 2\cos(2\pi/3m)})$.
3. Homework problem.
4. Let χ be in $\widehat{G}$. Then $\chi(G)$ is a subgroup of S^1, but $N(1)$ contains no nontrivial subgroups. Thus $W(G, 1, \sqrt{3}) = \{1\}$.

Proposition

If G is locally compact, then $\widehat{G}$ is too.

Proof.

Suppose χ in $\widehat{G}$ is nontrivial. Then $\chi(g) \neq 1$ for some g in G, so the open subset $W(\{g\}, \chi, \frac{1}{2}|\chi(g) - 1|)$ does not contain 1. Taking unions over all such χ shows that $\{1\}$ is closed in $\widehat{G}$.

Next, we have a neighborhood O of 1 whose closure is compact. I claim that $W(\overline{O}, 1, \sqrt{2 - 2\cos(2\pi/12)})$ has compact closure. To see this, note it suffices to prove

$$W = \{\chi \in \widehat{G} \mid \chi(\overline{O}) \subseteq \overline{N(\tfrac{1}{4})}\}$$

is compact. Write G_0 for the group G with the discrete topology. Then $\widehat{G_0}$ is compact, and we view $\widehat{G}$ as a subgroup of $\mathrm{Hom}(G, S^1) = \widehat{G_0}$.

Proposition

If G is locally compact, then $\widehat{G}$ is too.

Proof (continued).

Write $W_0 = \{\chi \in \widehat{G_0} \mid \chi(\overline{O}) \subseteq \overline{N(\frac{1}{4})}\}$. Then W_0 is an intersection of closed subsets of $\widehat{G_0}$ and hence is closed in $\widehat{G_0}$. Thus W_0 is compact. Next, we immediately have $W \subseteq W_0$, and because O is a neighborhood of 1 and $\overline{N(\frac{1}{4})} \subseteq N(1)$, we have $W_0 \subseteq W$.

So we just have to show the topology on W_0 from $\widehat{G_0}$ is finer than the topology on W from $\widehat{G}$. Let χ be in W, and consider $U = W \cap W(K, \chi, \sqrt{2 - 2\cos(2\pi/3m)})$. Now O contains a neighborhood V of 1 such that $V^{(2m)} \subseteq O$. As K is compact, we have $K \subseteq FV$ for some finite subset F of G.

Form $U_0 = W_0 \cap W_0(F, \chi, \sqrt{2 - 2\cos(2\pi/6m)})$, and suppose ρ lies in U_0. Since $\overline{N(\frac{1}{4})}^{-1} = \overline{N(\frac{1}{4})}$, we see $\xi = \chi^{-1}\rho$ sends $\overline{O}$ to $\overline{N(\frac{1}{2})} \subseteq N(1)$.

Proposition

If G is locally compact, then $\widehat{G}$ is too.

Proof (continued).

Therefore ξ is continuous, and since $V^{(2m)} \subseteq O$, we get $\xi(V) \subseteq N(\frac{1}{2m})$. Because we translated by χ^{-1}, we also see that ξ lies in $W_0(F, 1, \sqrt{2 - 2\cos(2\pi/6m)})$, so $\xi(F) \subseteq N(\frac{1}{2m})$. Hence $\xi(K) \subseteq \xi(F)\xi(O) \subseteq N(\frac{1}{m})$, so ξ lies in $W(K, 1, \sqrt{2 - 2\cos(2\pi/3m)})$. Multiplying by χ shows ρ lies in U, so altogether U_0 is a neighborhood of χ in W_0 contained in U. Hence we obtain the desired fineness statement. □