Haar Measures and Pontryagin Duals

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Let X be a topological space, and write $\mathcal{B}$ for its Borel σ-algebra. Let m be a measure on $\mathcal{B}$.

Definition

Let E be in $\mathcal{B}$. We say that m is

- *outer regular* on E if $m(E) = \inf\{m(O) \mid O \supseteq E \text{ is open}\}$,
- *inner regular* on E if $m(U) = \sup\{m(K) \mid K \subseteq E \text{ is compact}\}$.

Example

Let X be $\mathbb{R}$ or a nonarchimedean local field, and let m be the Lebesgue measure. For all E in $\mathcal{B}$ and $\epsilon > 0$, we have an open subset O and a closed subset C such that $m(O \setminus E) < \epsilon/2$ and $m(E \setminus C) < \epsilon/2$. Taking $\epsilon \to 0$ shows m is outer regular on E.

For inner regularity, note that $C = \bigcup_{i=1}^{\infty} C \cap B_c(0, i)$, and the $C \cap B_c(0, i)$ are compact. Thus $m(C) \lim_{i\to\infty} m(C \cap B_c(0, i))$, so using these compact subsets shows m is inner regular on E too.

Next, suppose X is a topological group.

Definition

We say m is *left invariant* if, for all x in X and E in $\mathcal{B}$, we have $m(xE) = m(E)$. We say m is *right invariant* if, for all x in X and E in $\mathcal{B}$, we have $m(Ex) = m(E)$.

Example

In our previous example (with $X = \mathbb{R}$ or a nonarchimedean field), m is left (and right) invariant, where we view X as a group under addition as usual.

Definition

We say m is *left Haar* if it is nonzero, left invariant, outer regular on Borel subsets, inner regular on open subsets, and finite on compact subsets.

Theorem (Haar)

Assume X is locally compact. Then X has a left Haar measure, and any two left Haar measures m_1 and m_2 on X satisfy $m_1 = cm_2$ for some $c > 0$.

Example

- Consider any group G with the discrete topology. Then the counting measure on G is left Haar,
- The pushforward of the Lebesgue measure via $\exp(2\pi i\cdot) : [0,1] \to S^1$ yields a left Haar measure on S^1,
- Let F be a local field. Then the Lebesgue measure on F (where we use the product measure when $F = \mathbb{C} = \mathbb{R} \times \mathbb{R}$) is left Haar,
- Let F be a local field, and consider $X = F^\times$. The measure given by

  $$E \mapsto \int_E \frac{\mathrm{d}x}{\|x\|}$$

  for E in $\mathcal{B}$ is left Haar, where the integral is taken with the Lebesgue measure, and $\|x\| = |x|$ (except when $F = \mathbb{C}$, where $\|x\| = |x|^2$). This reduces to the fact $m(xE) = \|x\| m(E)$, which follows from the case when E is a closed ball.
  We often denote $\mathrm{d}x\,/\|x\|$ as $\mathrm{d}^\times x$.

Example

- Let F be a local field, and consider the group $X = \mathrm{GL}_n(F) = \{A \in \mathrm{Mat}_{n\times n}(F) \mid \det\{A\} \neq 0\}$ with the subspace topology of the product topology. The measure given by

$$E \mapsto \int_E \frac{\mathrm{d}x}{\|\det x^n\|}$$

for E in $\mathcal{B}$ is left Haar, where the integral is taken with the Lebesgue product measure.

Definition

Let G be an abelian topological group. Its *Pontryagin dual*, denoted by $\widehat{G}$, is the set of continuous group homomorphisms $\chi : G \to S^1$. We give $\widehat{G}$ a group structure by setting $(\chi_1\chi_2)(x) = \chi_1(x)\chi_2(x)$ for all χ_1 and χ_2 in $\widehat{G}$.

We give $\widehat{G}$ a topology by using $W(K, \xi, \epsilon) = \{\chi \in \widehat{G} \mid |\chi - \xi| < \epsilon \text{ on } K\}$ for a basis, where K runs over compact subsets of G, ξ runs over elements of $\widehat{G}$, and ϵ runs over positive reals.

Remark

Say χ lies in $W(K, \xi, \epsilon)$. The compactness of K implies that $|\chi - \xi|$ attains a maximum $M < \epsilon$ on K. The triangle inequality shows that $W(K, \chi, \epsilon - M)$ is a neighborhood of χ in $W(K, \xi, \epsilon)$, so altogether subsets of the form $W(K, \chi, \epsilon)$ form a basis of neighborhoods of χ.

Proposition

Let G be an abelian topological group. Then $\widehat{G}$ is too.

Proof.

Since elements of $\widehat{G}$ are valued in S^1, we have $|\chi^{-1} - \xi| = |\xi^{-1} - \chi|$. Thus the inverse in $\widehat{G}$ of $W(K, \xi, \epsilon)$ is $W(K, \xi^{-1}, \epsilon)$, making inversion continuous. Next, let $\{(\chi_\alpha, \rho_\alpha)\}_{\alpha \in A}$ be a net in $\widehat{G} \times \widehat{G}$ converging to (χ, ρ), and suppose $W(K, \chi, \epsilon)$ is a neighborhood of $\chi\rho$. As $\{\chi_\alpha\}_{\alpha \in A}$ converges to χ, we see that χ_α lies in $W(K, \chi, \epsilon/2)$ for sufficiently large α. The same holds for ρ_α and ρ. Hence $|\chi_\alpha \rho_\alpha - \chi\rho|$ is bounded by $|\chi_\alpha(\rho_\alpha - \rho)| + |(\chi_\alpha - \chi)\rho| = |\rho_\alpha - \rho| + |\chi_\alpha - \chi| < \epsilon$ on K, so $\{\chi_\alpha \rho_\alpha\}_{\alpha \in A}$ converges to $\chi\rho$. This shows that multiplication on $\widehat{G}$ is continuous.

Remark

Suppose G is discrete. Then every compact subset K is finite, so the topology on $\widehat{G}$ is precisely the subspace topology from $\widehat{G} \subseteq \prod_{x \in G} S^1$.

Example

- Let $G = \mathbb{Z}/n\mathbb{Z}$ with the discrete topology. Then $\widehat{G} = \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, S^1)$ is identified with $\{\zeta \in \mathbb{C} | \zeta^n = 1\}$ via $\chi \mapsto \chi(1)$. We see $\widehat{G}$ is discrete too, and choosing a primitive n-th root of unity yields $G \overset{\sim}{\to} \widehat{G}$.
- Let $G = \mathbb{Z}$ with the discrete topology. Then $\widehat{G} = \mathrm{Hom}(\mathbb{Z}, S^1)$ is similarly identified with S^1 via $\chi \mapsto \chi(1)$. Under this identification, $W(\{n\}, \xi, \epsilon)$ becomes $\{z \in S^1 \mid |z^n - \xi(1)^n| < \epsilon\}$. Taking all n-th roots shows this is a union of n open intervals in S^1. Using $n = 1$ indicates that the resulting topology is the Euclidean topology on S^1.